STELLAR STABILITY

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Introduction

Stellar evolution

- Generally slow, driven by the change of chemical composition.
- Stars $\approx$ dynamical and thermal equilibrium structures

Stellar stability

- Complementary to stellar evolution
- Primarily concerned with stellar behaviour on shorter time scales (variable stars, helio- and asteroseismology)
- Secular stability will not be considered in these lectures

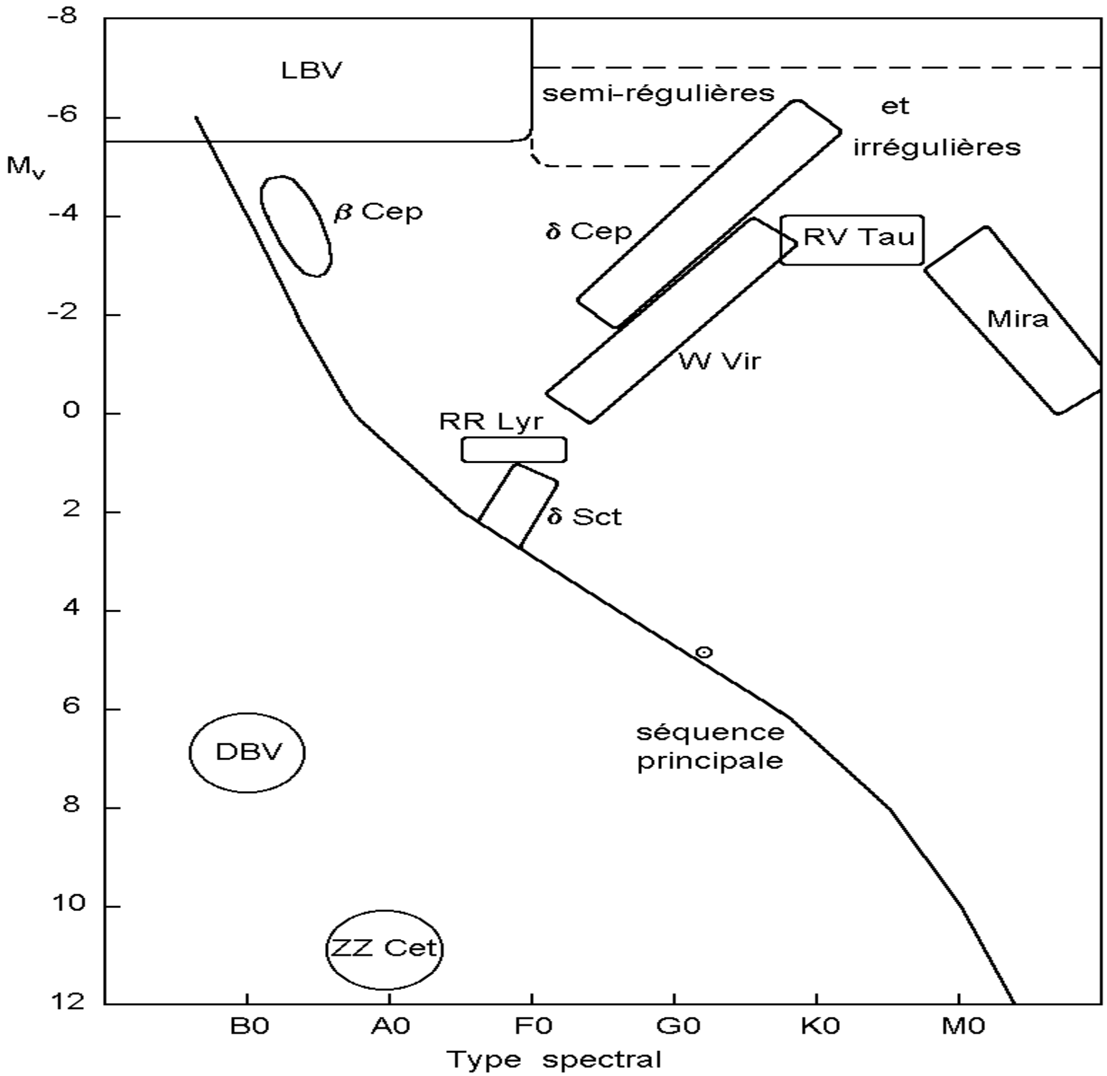
Introduction

Hypotheses

- Gaseous stars
- Non relativistic mechanics and newtonian gravitation

Method

- Small perturbations $\longrightarrow$ linearized equations
- No information on amplitudes, no interactions between modes
- Spherical symmetry : no rotation, no magnetic field, except considered as a small correction.



A few classes of variable stars in the HR diagram

Characteristic timescales

Dynamical timescale

$$\frac{d^2r}{dt^2} = -\frac{Gm}{r^2} - \frac{1}{\rho}\frac{dP}{dr}$$

$$\frac{R}{\tau_{ff}^2} \approx \frac{GM}{R^2} \Rightarrow \tau_{ff} \approx \sqrt{R^3/GM} \approx 1/\sqrt{G\rho}$$

$$\frac{R}{\tau_{expl}^2} \approx \frac{P}{\rho R} \Rightarrow \tau_{expl} \approx R/c \quad \text{with} \quad c = \sqrt{\frac{\Gamma_1 P}{\rho}}$$

$$\tau_{dyn} \approx \tau_{expl} \approx \tau_{ff} \Rightarrow c \approx \sqrt{GM/R}$$

Stars	ρ (g cm^{-3})	$\tau_{dyn} = 1/\sqrt{G\rho}$
neutron star	10^{15}	0.12 ms
white dwarf	10^{6}	3.9 s
Sun	1.41	54 min
red supergiant	10^{-9}	3.9 y

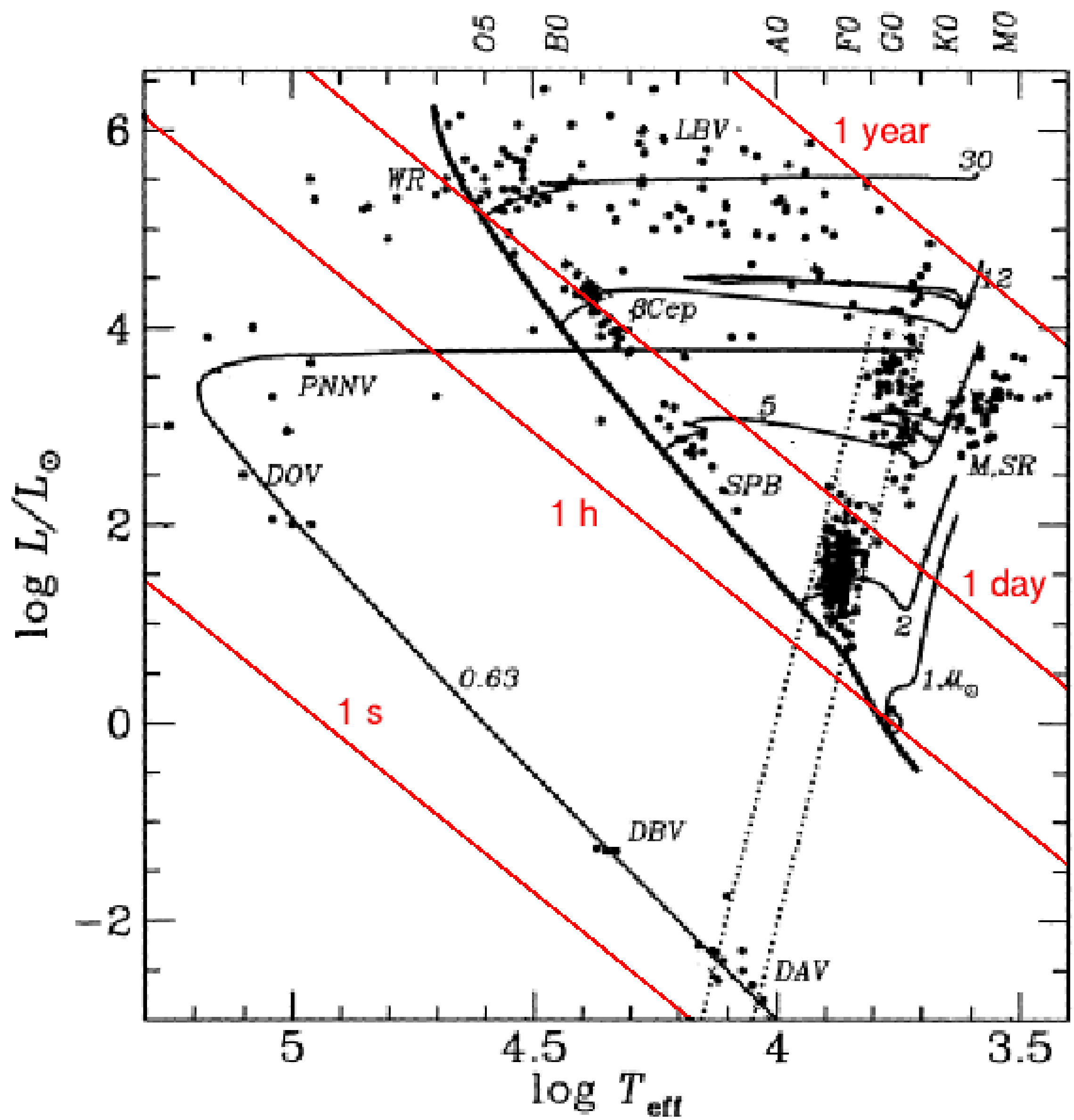
$$\text{Period} \approx \tau_{dyn} \Rightarrow Q = \text{Period} \times \sqrt{\frac{\rho}{\rho_\odot}} \approx \text{constant}$$

$$0.03 \text{ days} \leq Q \leq 0.08 \text{ days}$$

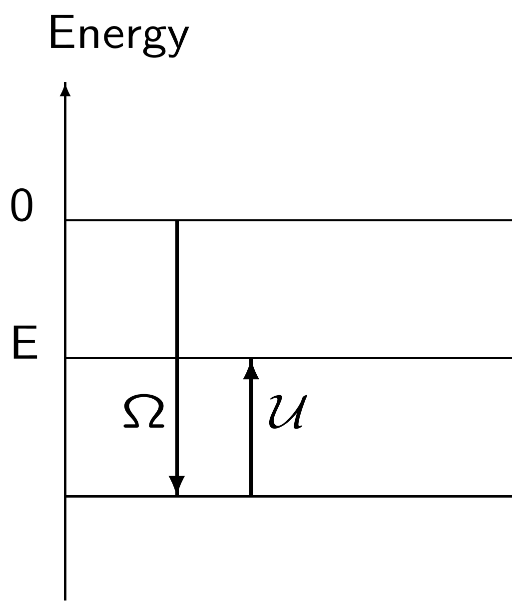
With $\tau_{dyn} = 1/\sqrt{G\rho}$, $M = 4\pi R^3\rho/3$ and $L = 4\pi R^2 \sigma T_e^4$,

$$\log \tau_{dyn} = 14.8 - \frac{1}{2}\log\frac{M}{M_\odot} + \frac{3}{4}\log\frac{L}{L_\odot} - 3\log T_e$$

τ_{dyn} for 1 $M_\odot$



Kelvin-Helmholtz timescale



$$E = \mathcal{U} + \Omega \qquad \Omega = -2\mathcal{U}$$

$$\tau_{KH} \approx |E|/L$$

$$\Omega = -\int \frac{Gm}{r} dm \approx -\frac{GM^2}{R}$$

$$\tau_{KH} \approx \frac{GM^2}{LR}$$

For the Sun, $\tau_{KH} \approx 3.1 \times 10^7$ years

$\tau_{dyn}/\tau_{KH} \approx 1.6 \times 10^{-12}$

Globally, transfer phenomena are much slower than dynamical phenomena.
Locally, this is not always true.

Nuclear timescale

In the fusion of 1 g of ^{1}H into ^{4}He, 0.007 g is converted into energy

$$0.007c^2 \approx 6 \times 10^{18} \text{ erg}$$

If a star burns 1/10 of its hydrogen on the main sequence, its life-time may be estimated to be

$$\tau_{nuc} \approx 6 \times 10^{17} M/L \text{ (CGS)}$$

For the Sun, $\tau_{nuc} \approx 9.8 \times 10^9$ years

$$\tau_{KH}/\tau_{nuc} \approx 3.2 \times 10^{-3}$$

The chemical evolution of a normal star is too slow to interact with its pulsation.

General equations

Differential equations with boundary conditions

- hydrodynamics
- gravitational field
- conservation and transport of energy

Algebraic equations (material equations)

- equation of state
- opacity
- nuclear energy generation

Differential equations

Two complementary descriptions of the hydrodynamics are possible: Eulerian (generally described in textbooks) and Lagrangian.

Eulerian description

Independent variables: $\vec{r}$, t .

Functions: $\rho(\vec{r}, t)$, $\vec{v}(\vec{r}, t)$,...

$\vec{r}$ is not a function of t.

$\partial\vec{r}/\partial t = 0$ and generally $\vec{\gamma} \neq \partial\vec{v}/\partial t$.

Lagrangian description

Same point of view as in particles mechanics.

Independent variables: $\vec{a}$, t (*e.g.* $\vec{a} = \vec{r}_0$).

Functions: $\rho(\vec{a}, t)$, $\vec{r}(\vec{a}, t)$,...

$\vec{r}$ is a function $\vec{r}(\vec{a}, t)$.

$\vec{v} = \partial\vec{r}/\partial t$ and $\vec{\gamma} = \partial\vec{v}/\partial t$.

Eulerian vs Lagrangian

A mathematician would have used distinct notations, $\rho_{Euler}(\vec{r},t)$ and $\rho_{Lagrange}(\vec{a},t)$.
He would have written

$$\rho_{Lagrange}(\vec{a},t) = \rho_{Euler}(\vec{r}(\vec{a},t),t)$$

$\Rightarrow$ it is very simple to deduce a relation between the time derivatives of the two functions.

$$\frac{\partial \rho_{Lagrange}(\vec{a},t)}{\partial t} = \frac{\partial \rho_{Euler}(\vec{a},t)}{\partial t} + \vec{v}\cdot \mathrm{grad}\,\rho\,.$$

But a physicist uses the same notation ρ for both functions $\Rightarrow$ problems to distinguish the derivatives $\Rightarrow$ different notations for the time derivative operator.

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v}\cdot \mathrm{grad}\,\rho\,.$$

$\partial/\partial t$: *local* time derivative

d/dt: time derivative *following the motion*

Equation of continuity

Conservation of mass

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \vec{v}) = 0 \quad \text{or} \quad \frac{d\rho}{dt} + \rho\,\mathrm{div}\,\vec{v} = 0$$

In the Lagrangian formalism, this equation can be written in an integrated form

$$\rho \left| \frac{\partial(x)}{\partial(a)} \right| = \mathrm{const} \quad \text{or} \quad \rho \left| \frac{\partial(x)}{\partial(x_0)} \right| = \rho_0$$

Equation of motion

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \mathrm{grad})\vec{v} \;=\; -\,\mathrm{grad}\,\Phi - \frac{1}{\rho}\,\mathrm{grad}\,P$$

$$\text{or} \quad \frac{d\vec{v}}{dt} \;=\; -\,\mathrm{grad}\,\Phi - \frac{1}{\rho}\,\mathrm{grad}\,P$$

No molecular viscosity terms.

Turbulent viscosity: no satisfactory theory of non stationnary convection $\Rightarrow$ we do not discuss this problem.

Poisson equation

$$\Delta\Phi = 4\pi G\rho \qquad \Phi(P) = -G\int \frac{\rho_Q\, dV_Q}{|PQ|}$$

Energy conservation

$$T\left(\frac{\partial S}{\partial t} + \vec{v}\cdot \mathrm{grad}\, S\right) = \epsilon - \frac{1}{\rho}\,\mathrm{div}\,\vec{F}$$

$$\text{or} \quad T\frac{dS}{dt} = \epsilon - \frac{1}{\rho}\,\mathrm{div}\,\vec{F}$$

where $\vec{F}\cdot\vec{n}\, dS =$ energy flowing through dS per unit time.

Transfer equation

We limit ourselves to the star interior and we do not discuss convective transfer $\Rightarrow$ diffusion equation

$$\vec{F} = -\lambda\,\mathrm{grad}\, T \quad \text{with} \quad \lambda = \frac{4acT^3}{3\kappa\rho}$$

Boundary conditions

At the center: regularity conditions.

At the surface: continuity with an atmospheric model.

Material equations

Equation of state

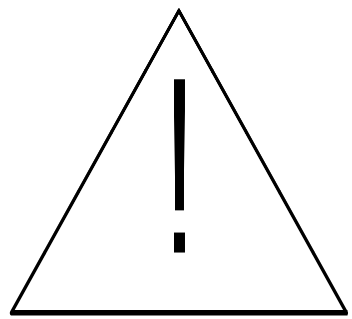
$$P = P(\rho, T, \chi),\ U = U(\rho, T, \chi),\ \ldots$$

Opacity

$$\kappa = \kappa(\rho, T, \chi)$$

Nuclear energy production rate

$$\epsilon = \epsilon(\rho, T, \chi)$$



However ϵ can be considered as a function of ρ, T and χ only if the minor constituents (^{2}H, ^{3}He, ^{7}Li,...) have reached their equilibrium abundances !

Equilibrium configuration

Usual definitions: $m(r) = \int_0^r 4\pi r^2 \rho \, dr, \quad L(r) = 4\pi r^2 F.$

Poisson equation integrates once, $\dfrac{d\Phi}{dr} = \dfrac{Gm}{r^2}.$

Differential equations

$$
\begin{aligned}
\frac{dP}{dr} &= -\frac{Gm\rho}{r^2} \\
\frac{dm}{dr} &= 4\pi r^2 \rho \\
\frac{dL}{dr} &= 4\pi r^2 \rho(\epsilon + \epsilon_G) \quad \text{where } \epsilon_G = -T\, dS/dt \\
\frac{dT}{dr} &= \begin{cases} -\frac{3\kappa\rho L}{16\pi r^2 acT^3} & \text{(rad. zone)} \\ \frac{\Gamma_2 - 1}{\Gamma_2} \frac{T}{P} \frac{dp}{dr} & \end{cases}
\end{aligned}
$$

Boundary conditions

• At the center ($r = 0$)

$m = 0$ and $L = 0$.

• At the surface ($r = R$)

- A very crude approximation: $P = 0$ and $T = 0$

- A better approximation: smooth fit to a grey atmosphere with Eddington approximation

$$P = \frac{2GM}{3\bar{\kappa}R^2}$$
$$T = T_e \quad \text{with } L = 4\pi R^2 \sigma T_e^4$$

- Smooth fit to a model atmosphere.

Small perturbation methods

Principle

$$X = X_0 + \delta X \Rightarrow \text{linearized equations}$$

No information on: amplitude, stability towards finite perturbations, metastable states, limit cycles

a mode depends on t by a factor e^{st}
general solution $= \sum$ modes
stability of a mode $\Leftrightarrow \Re s < 0$
stability of a model $\Leftrightarrow$ all $\Re s < 0$

Eulerian and Lagrangian perturbations

$$X(\vec{r},t) = X_0(\vec{r},t) + X'(\vec{r},t)$$
$$X(\vec{a},t) = X_0(\vec{a},t) + \delta X(\vec{a},t)$$

$$\delta X = X' + \overrightarrow{\delta r} \cdot \mathrm{grad}\, X$$

$$\frac{\partial X'}{\partial t} = \left(\frac{\partial X}{\partial t}\right)'$$
$$\frac{\partial X'}{\partial x_i} = \left(\frac{\partial X}{\partial x_i}\right)'$$
$$\frac{\delta X}{dt} = \delta \frac{dX}{dt}$$

But generally, $\dfrac{\partial \delta X}{\partial x_i} \neq \delta \dfrac{\partial X}{\partial x_i}$

Linearized differential equations

In general, it is simpler to linearize the differential equations in the Eulerian formalism.

Continuity equation

$$\frac{\partial \rho'}{\partial t} + \mathrm{div}\left(\rho \frac{\partial \overrightarrow{\delta r}}{\partial t}\right) = 0 \quad \text{or} \quad \rho' + \mathrm{div}(\rho \overrightarrow{\delta r}) = 0$$

Equation of motion

$$\frac{\partial^2 \overrightarrow{\delta r}}{\partial t^2} = -\,\mathrm{grad}\,\Phi' + \frac{\rho'}{\rho^2}\mathrm{grad}\,P - \frac{1}{\rho}\mathrm{grad}\,P'$$

Poisson equation

$$\Delta\Phi' = 4\pi G\rho' \Rightarrow \Phi'(P) = -G\int \frac{\rho'_Q\, dV_Q}{|PQ|}$$

Energy conservation

$$T\left(\frac{\partial S'}{\partial t} + \vec{v}\cdot\text{grad}\, S\right) = \epsilon' + \frac{\rho'}{\rho^2}\,\text{div}\,\vec{F} - \frac{1}{\rho}\,\text{div}\,\vec{F}'$$

Radiative transfer

$$\vec{F}' = -\lambda'\,\text{grad}\, T - \lambda\,\text{grad}\, T'$$

Linearized material equations

Easier to write with lagangian perturbations.

Equation of state

$$\frac{\delta P}{P} = P_\rho \frac{\delta \rho}{\rho} + P_T \frac{\delta T}{T}$$

$$\text{with} \quad P_\rho = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_T \quad \text{and} \quad P_T = \left(\frac{\partial \ln P}{\partial \ln T}\right)_\rho$$

A lot of other useful quantities.

$$\begin{aligned} \delta U(V,S) &= T\delta S - P\delta V \\ &\Rightarrow P = -\left(\frac{\partial U}{\partial V}\right)_S \quad \text{and} \quad T = \left(\frac{\partial U}{\partial S}\right)_V \end{aligned}$$

Other useful quantities can be expressed in terms of the 2nd order derivatives

$$\left(\frac{\partial^2 U}{\partial V^2}\right)_S, \quad \frac{\partial^2 U}{\partial V \partial S}, \quad \left(\frac{\partial^2 U}{\partial S^2}\right)_V$$

or in terms of equivalent quantities

$$\Gamma_1 = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S , \quad \Gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_S ,$$

$$c_v = \left(\frac{\partial U}{\partial T}\right)_\rho = T\left(\frac{\partial S}{\partial T}\right)_\rho$$

$1/\Gamma_1 =$ compressibility coefficient

$$\frac{\partial^2 U}{\partial S\,\partial V} = \frac{\partial^2 U}{\partial V\,\partial S} \Rightarrow -\left(\frac{\partial P}{\partial S}\right)_V = \left(\frac{\partial T}{\partial V}\right)_S$$

$$\begin{aligned}\frac{\delta P}{P} &= \Gamma_1 \frac{\delta \rho}{\rho} + \frac{(\Gamma_3 - 1) c_v \rho T}{P}\frac{\delta S}{c_v} \\ \frac{\delta T}{T} &= (\Gamma_3 - 1)\frac{\delta \rho}{\rho} + \frac{\delta S}{c_v}\end{aligned}$$

Elimination of δS

$$\frac{\delta P}{P} = \underbrace{\left[\Gamma_1 - \frac{(\Gamma_3 - 1)^2 c_v \rho T}{P}\right]}_{P_\rho} \frac{\delta \rho}{\rho} + \underbrace{\frac{(\Gamma_3 - 1) c_v \rho T}{P}}_{P_T} \frac{\delta T}{T}$$

$$\begin{cases} P_\rho = \Gamma_1 - \dfrac{(\Gamma_3 - 1)^2 c_v \rho T}{P} \\ P_T = \dfrac{(\Gamma_3 - 1) c_v \rho T}{P} \end{cases}$$

$$\Rightarrow \begin{cases} \Gamma_3 - 1 = \dfrac{P_T P}{c_v \rho T} \\ \\ \Gamma_1 = P_\rho + (\Gamma_3 - 1) P_T \end{cases}$$

Normal stellar matter

$$P = \frac{\mathcal{R}\rho T}{\mu} \qquad U = \frac{3}{2}\frac{\mathcal{R}T}{\mu}$$

$$P_\rho = P_T = 1 \qquad c_v = \frac{3}{2}\frac{\mathcal{R}}{\mu} \qquad \Gamma_1 = \Gamma_3 = \frac{5}{3}$$

Radiation and the behaviour of the Γ_i

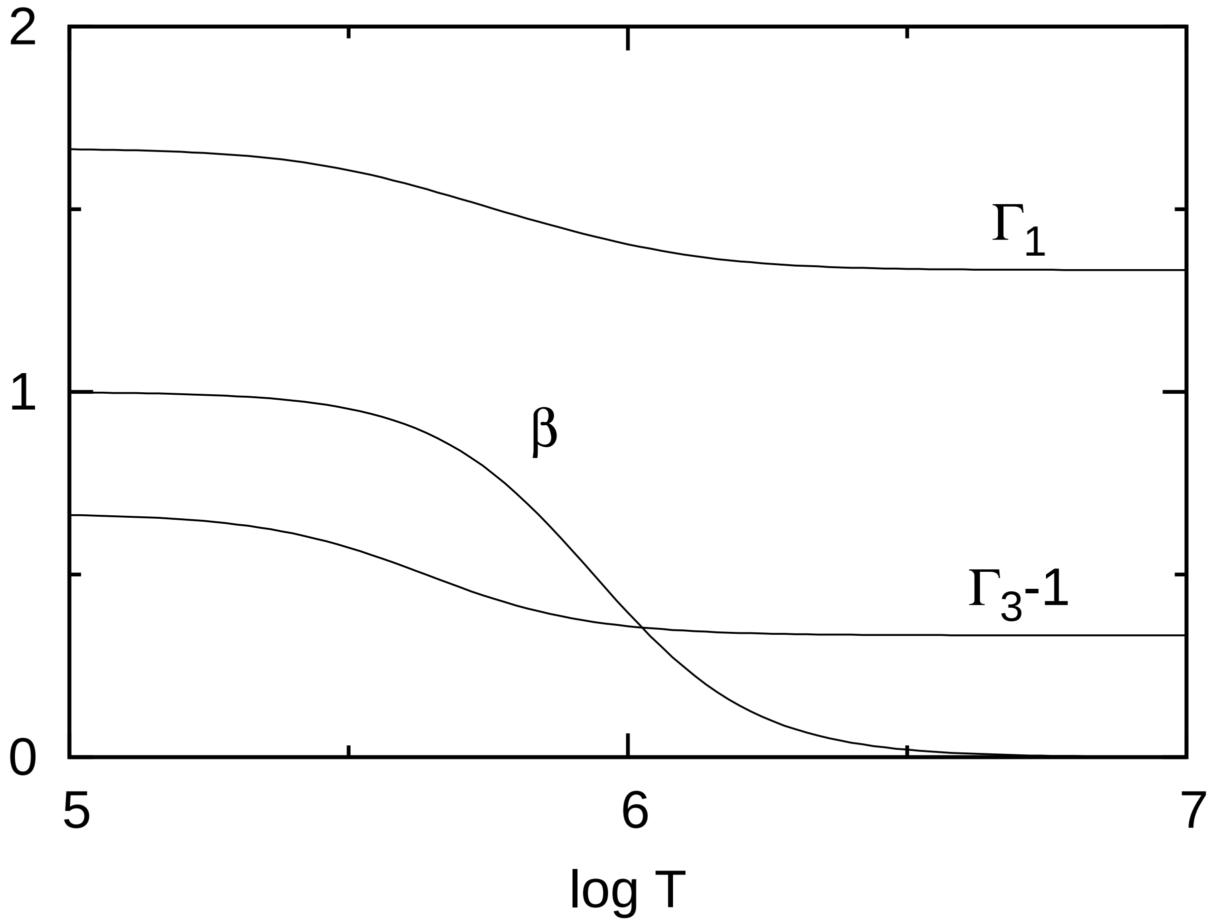
$$P = \frac{\mathcal{R}\rho T}{\mu} + \frac{1}{3}aT^4 = P_g + P_R \qquad \rho U = \frac{3}{2}\frac{\mathcal{R}\rho T}{\mu} + aT^4 \qquad \beta = P_g/P$$

$$P_\rho = \beta \qquad P_T = 4 - 3\beta \qquad c_v = \frac{\mathcal{R}}{\mu}\left[\frac{3}{2} + \frac{12(1-\beta)}{\beta}\right]$$

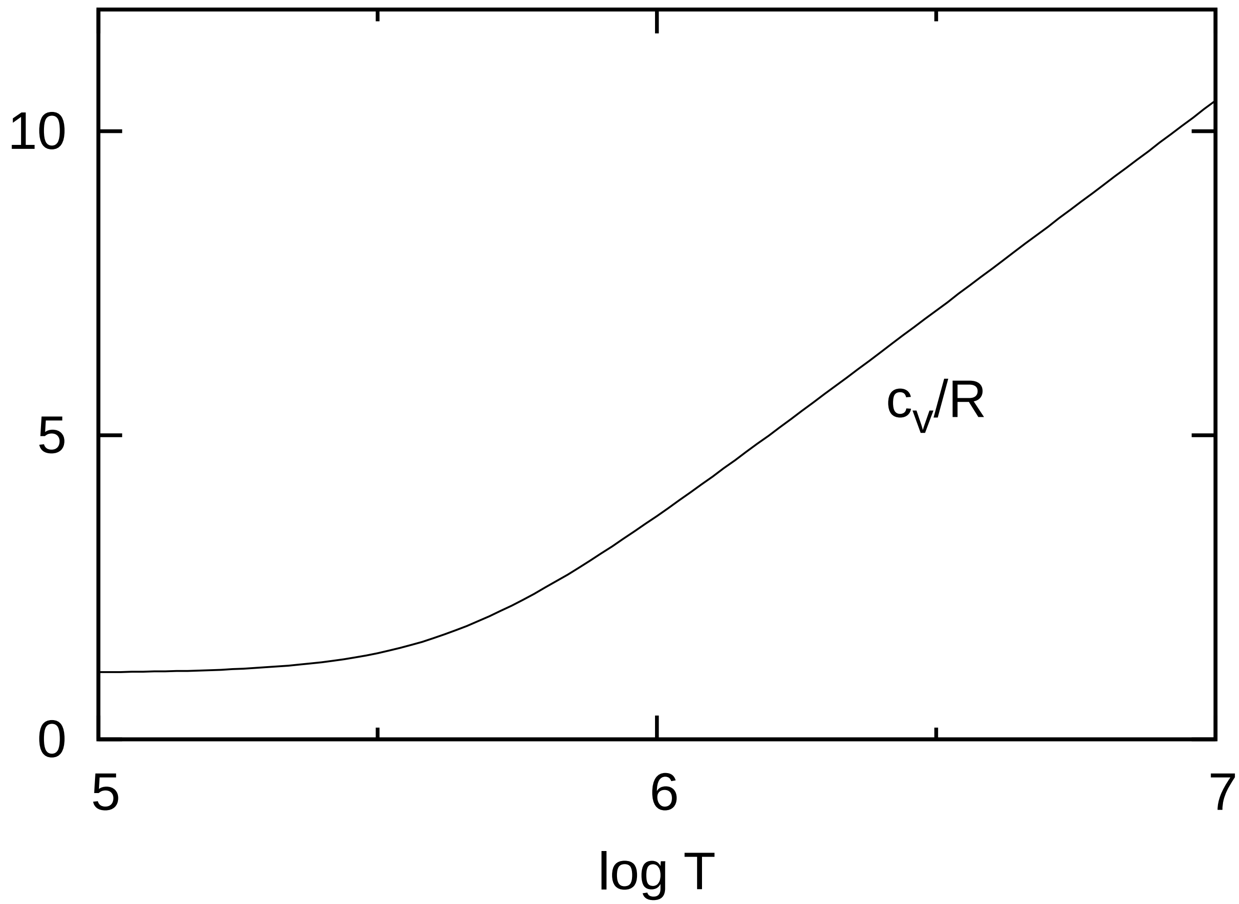
$$\Gamma_1 = \beta + \frac{2(4-3\beta)^2}{3(8-7\beta)} \qquad \Gamma_3 - 1 = \frac{2(4-3\beta)}{3(8-7\beta)}$$

When $\beta \to 0$, Γ_1 and $\Gamma_3 \to 4/3$ and $c_V \to \infty$.

Pure hydrogen, $\rho = 10^{-5}$ g cm^{-3}



Pure hydrogen, $\rho = 10^{-5}$ g cm^{-3}



Ionization and the behaviour of the Γ_i

Example: pure hydrogen
Particles: H, H^+ and e^-

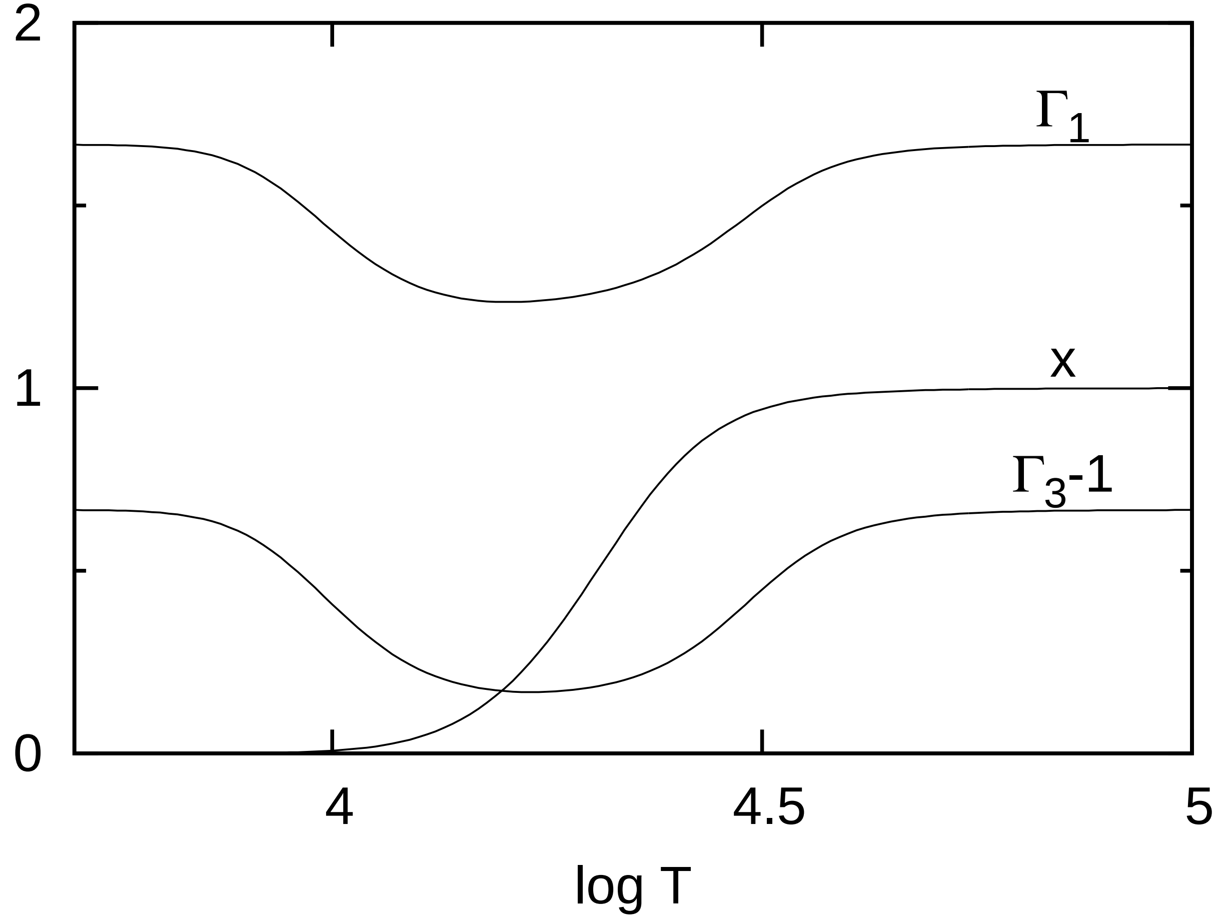
$$
\begin{aligned}
P &= nkT \quad \text{where} \quad n = n_0 + n_1 + n_e \\
\rho &= n_{at} m_H \quad \text{where} \quad n_{at} = n_0 + n_1 \\
\rho U &= \frac{3}{2} nkT + n_e \chi \quad \text{where} \quad \chi = 13.6 \text{ eV} \\
\frac{n_1 n_e}{n_0} &= \frac{2Z_1}{Z_0} \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi/kT}
\end{aligned}
$$

Define $x = n_1/n_{at}$, then

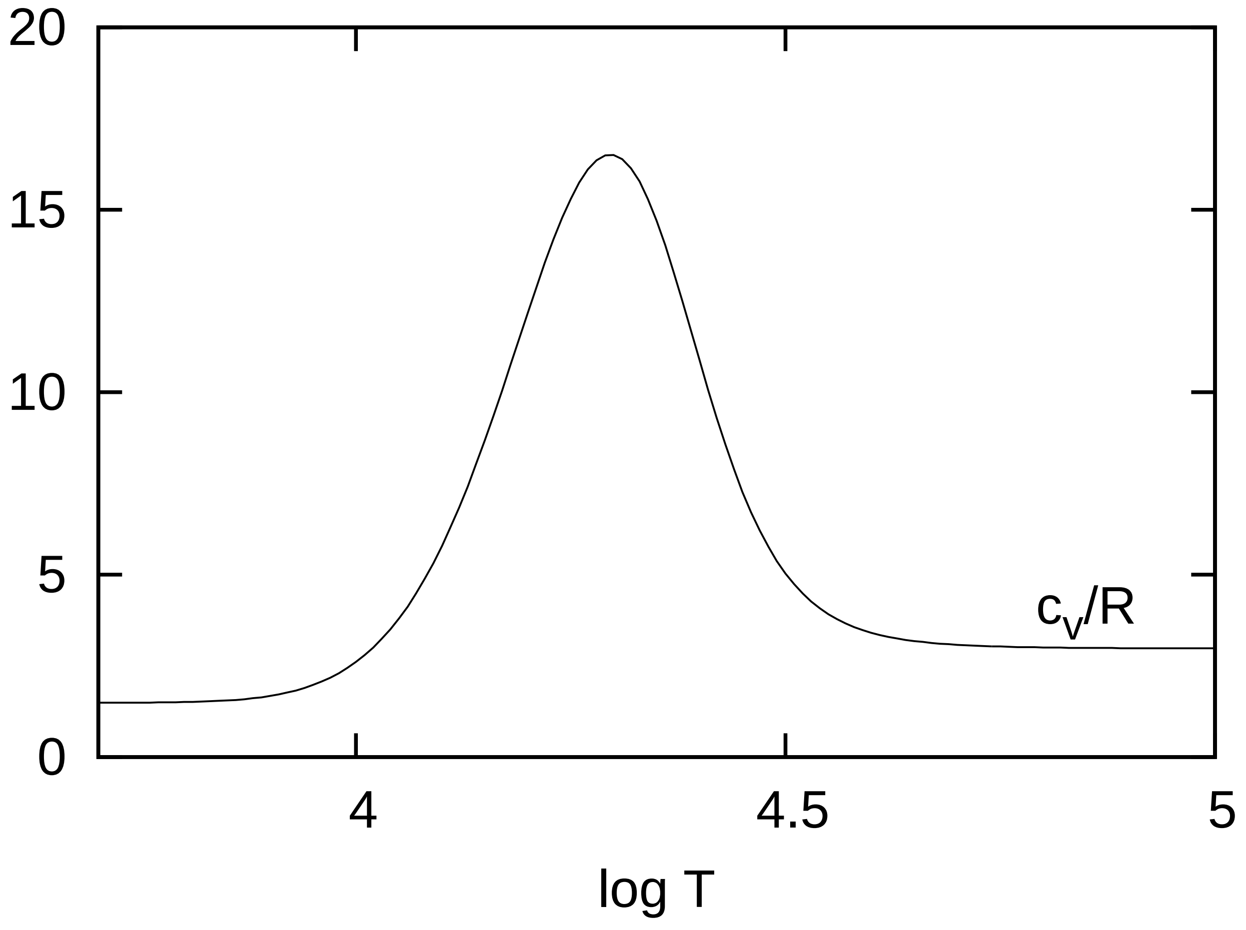
$$
\begin{aligned}
P &= \frac{\mathcal{R}\rho T}{\mu_H}(1+x) \\
U &= \left\{ \frac{3}{2}(1+x) + x\frac{\chi}{kT} \right\} \frac{\mathcal{R}T}{\mu_H} \\
\frac{x^2}{1-x} &= A\frac{T^{3/2}}{\rho} e^{-\chi/kT} = a(\rho, T)
\end{aligned}
$$

$$
\begin{aligned}
x &= \frac{1}{2}\left(-a + \sqrt{a^2 + 4a}\right) \\
x_\rho &= \left(\frac{\partial \ln x}{\partial \ln \rho}\right)_T = -\frac{1-x}{2-x} \\
x_T &= \left(\frac{\partial \ln x}{\partial \ln T}\right)_\rho = \frac{1-x}{2-x}\left(\frac{3}{2} + \frac{\chi}{kT}\right) \\
c_v &= \left(\frac{\partial U}{\partial T}\right)_\rho = \left\{\frac{3}{2}(1+x) + x x_T \left(\frac{3}{2} + \frac{\chi}{kT}\right)\right\} \frac{\mathcal{R}}{\mu_H} \\
P_\rho &= 1 + \frac{x x_\rho}{1+x} \\
P_T &= 1 + \frac{x x_T}{1+x} \\
\Gamma_3 - 1 &= \frac{P_T P}{c_v \rho T} \\
\Gamma 1 &= P_\rho + (\Gamma_3 - 1) P_T
\end{aligned}
$$

Pure hydrogen, $\rho = 10^{-5}$ g cm^{-3}



Pure hydrogen, $\rho = 10^{-5}$ g cm^{-3}



Opacity

$$\frac{\delta\kappa}{\kappa} = \kappa_\rho \frac{\delta\rho}{\rho} + \kappa_T \frac{\delta T}{T}$$

$$\text{with} \quad \kappa_\rho = \left(\frac{\partial \ln \kappa}{\partial \ln \rho}\right)_T \quad \text{and} \quad \kappa_T = \left(\frac{\partial \ln \kappa}{\partial \ln T}\right)_\rho$$

Nuclear energy

During the pulsation, we can no longer assume that the abundances of all nuclei species involved in the chain of nuclear reactions assume their equilibrium values.
If $\tau_i \ll \tau$, species i will remain in equilibrium.
If $\tau_i \gg \tau$, abundance of species i unchanged.
Generally, one must solve the linearized form of the kinetic equations

$$\Longrightarrow \frac{\delta\epsilon}{\epsilon} = \epsilon_\rho(\sigma) \frac{\delta\rho}{\rho} + \epsilon_T(\sigma) \frac{\delta T}{T}$$

In the literature, $\mu_{eff} = \epsilon_\rho(\sigma)$ and $\nu_{eff} = \epsilon_T(\sigma)$.

Example: the pp chains

$$
\begin{aligned}
{}^{1}H + {}^{1}H &\rightarrow {}^{2}H + e^{+} + \nu \\
{}^{2}H + {}^{1}H &\rightarrow {}^{3}He + \gamma
\end{aligned}
$$

$$2\,{}^{3}He \rightarrow {}^{4}He + 2\,{}^{1}H$$

$${}^{3}He + {}^{4}He \rightarrow {}^{7}Be + \gamma$$

$$
\begin{gathered}
{}^{7}Be + e^{-} \rightarrow {}^{7}Li + \nu \\
{}^{7}Li + {}^{1}H \rightarrow 2\,{}^{4}He
\end{gathered}
$$

$$
\begin{gathered}
{}^{7}Be + {}^{1}H \rightarrow {}^{8}B + \gamma \\
{}^{8}B \rightarrow 2\,{}^{4}He + e^{+} + \nu
\end{gathered}
$$

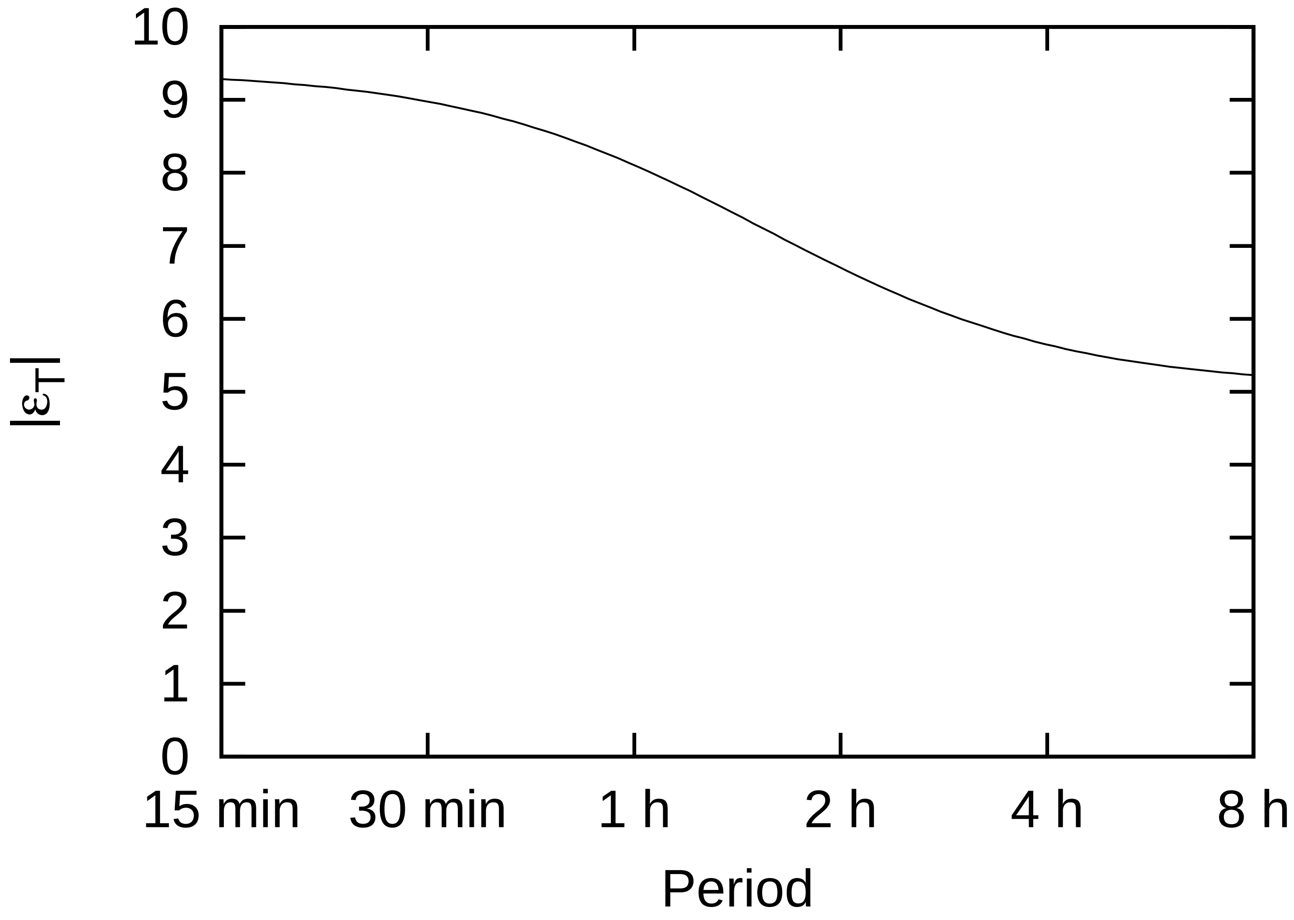
Physical conditions close to those of the central regions of the Sun:

$$\rho = 154\ \mathrm{g\,cm^{-3}} \qquad T = 15.6 \times 10^6\ \mathrm{K} \qquad X = 0.35 \qquad Y = 0.63$$

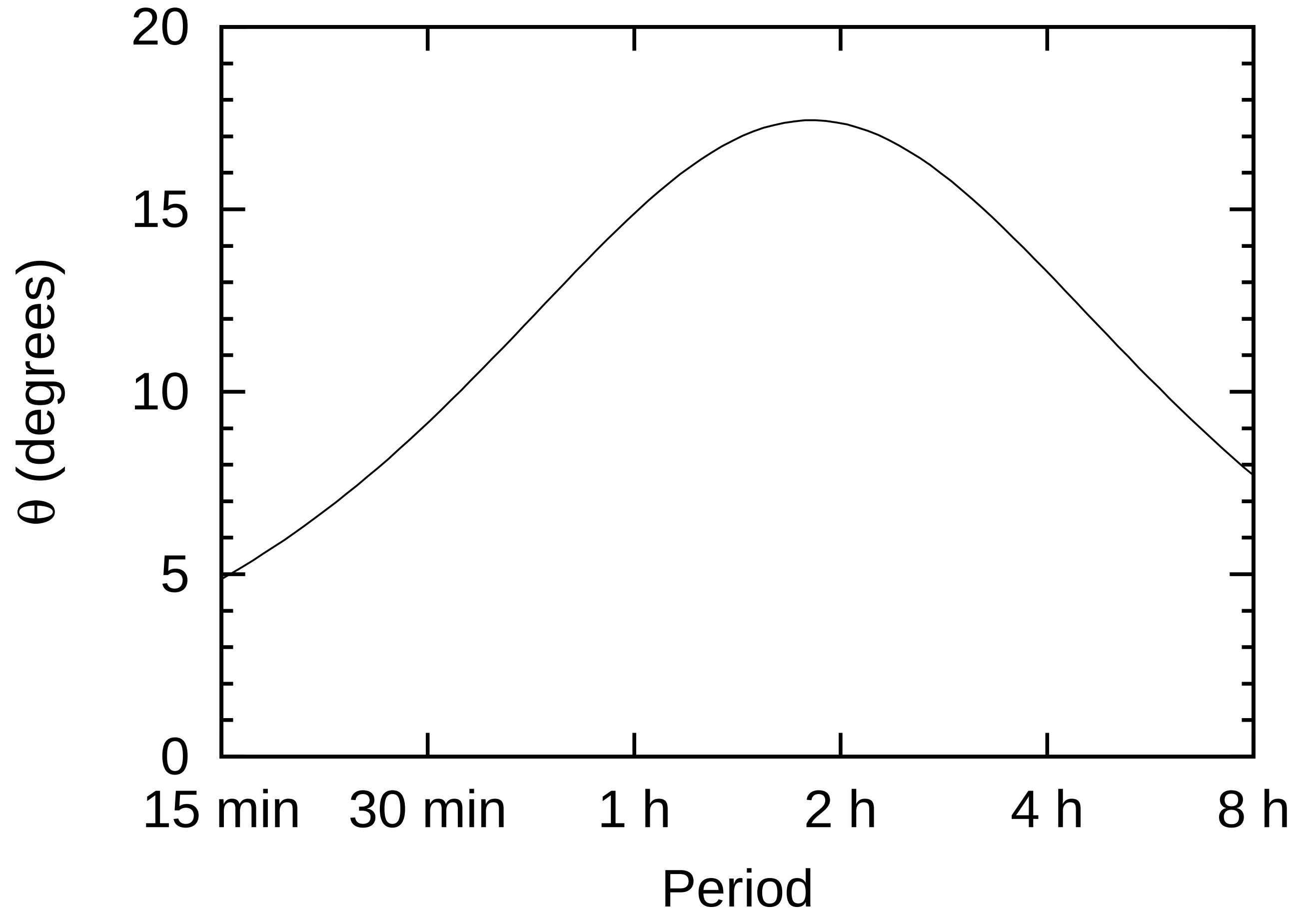
	abund (mol g^{-1})	lifetime
^{2}H	1.084×10^{-18}	1.277 s
^{3}He	2.734×10^{-6}	1.021×10^5 y
^{7}Be	2.604×10^{-12}	0.2409 y
^{7}Li	2.600×10^{-16}	12.7 min
^{8}B	1.590×10^{-21}	1.116 s

If equilibrium, $\epsilon_T = 5.22$, i.e. $\epsilon \propto T^{5.22}$.

But for a periodic oscillation, $\epsilon_T = |\epsilon_T|\, e^{-i\theta}$ with both $|\epsilon_T|$ and θ depending on the period.



10
9
8
7
6
5
4
3
2
1
0
|εT|
15 min
30 min
1 h
2 h
4 h
8 h
Period


20
15
10
5
0
θ (degrees)
15 min
30 min
1 h
2 h
4 h
8 h
Period

pure mechanical problem

	adiabatic	nonadiabatic
radial	adiabatic radial oscillations	nonadiabatic radial oscillations
nonradial	adiabatic nonradial oscillations	nonadiabatic nonradial oscillations

- adiabatic
- radial
- adiabatic radial
- non adiabatic radial
- adiabatic non radial

Adiabatic perturbations

$$\delta S = 0 \Rightarrow \frac{\delta P}{P} = \Gamma_1 \frac{\delta \rho}{\rho} \quad \text{or} \quad \delta P = c^2 \delta \rho$$

No equation of energy, no transfer equation.

We express all variables in terms of $\overrightarrow{\delta r}$

$$\rho' = -\operatorname{div}(\rho \overrightarrow{\delta r})$$

$$\begin{aligned} P' &= \delta P - \overrightarrow{\delta r} \cdot \operatorname{grad} P = \frac{\Gamma_1 P}{\rho} \delta \rho - \overrightarrow{\delta r} \cdot \operatorname{grad} P \\ &= -\Gamma_1 P \operatorname{div} \overrightarrow{\delta r} - \overrightarrow{\delta r} \cdot \operatorname{grad} P \end{aligned}$$

$$\Phi'(P) = -G \int \frac{\rho'_Q \, dV_Q}{|PQ|} = G \int \frac{\operatorname{div}(\rho \overrightarrow{\delta r})_Q \, dV_Q}{|PQ|}$$

The momentum equation

$$\frac{d^2\overrightarrow{\delta r}}{dt^2} = -\,\mathrm{grad}\,\Phi' + \frac{\rho'}{\rho^2}\,\mathrm{grad}\,P - \frac{1}{\rho}\,\mathrm{grad}\,P'$$

can now be written as

$$\frac{d^2\overrightarrow{\delta r}}{dt^2} = \mathcal{L}\overrightarrow{\delta r},$$

where $\mathcal{L}$ is the linear operator defined by

$$\begin{aligned}
\mathcal{L}\overrightarrow{\delta r} \;=\; & -G\,\mathrm{grad}_P \int \frac{\mathrm{div}(\rho\overrightarrow{\delta r})_Q\, dV_Q}{|PQ|} \\
& - \frac{1}{\rho^2}\,\mathrm{div}(\rho\overrightarrow{\delta r})\,\mathrm{grad}\,P \\
& + \frac{1}{\rho}\,\mathrm{grad}(\Gamma_1 P\,\mathrm{div}\,\overrightarrow{\delta r}) \\
& + \frac{1}{\rho}\,\mathrm{grad}(\overrightarrow{\delta r}\cdot\mathrm{grad}\,P)
\end{aligned}$$

The coefficients of $\mathcal{L}$ depend only on the two independent functions $\rho(r)$ and $\Gamma_1(r)$.

The coefficients of $\mathcal{L}$ are independent of t
$\Rightarrow$ There exists *simple* solutions of the form $\vec{\xi}(\vec{r})e^{st}$ (normal modes).
$\Rightarrow$ The equation of motion reduces to an eigenvalue problem

$$\mathcal{L}\xi = s^2\xi$$

Define a scalar product

$$(u, v) = \int_V \rho\vec{u} \cdot \bar{\vec{v}}\, dV = \overline{(v, u)}$$

Then $\mathcal{L}$ is hermitian

$$(\mathcal{L}u, v) = (u, \mathcal{L}v)$$

From the hermiticity of $\mathcal{L}$

1) eigenvalues s^2 real $\Rightarrow$ s real or pure imaginary. The eigenfunctions ξ can be chosen real.

2) The eigenfunctions ξ belonging to different eigenvalues are orthogonal.

3) Define the functional

$$\Lambda(\xi) = \frac{(\mathcal{L}\xi, \xi)}{(\xi, \xi)}$$

The solutions of the eigenvalue problem obey a variational principle:

$$\xi \text{ eigenfunction} \Leftrightarrow \delta\Lambda = 0 \text{ and } s^2 = \Lambda(\xi)$$

Dynamical stability of a mode:
$s^2 > 0 \Rightarrow$ instability
$s^2 < 0 \Rightarrow$ stability, we often write $s = -i\sigma$ plane waves: $\propto e^{i(\vec{k}\cdot\vec{r}-\sigma t)}$

$$\bar{T} = \frac{\sigma^2}{4}(\xi, \xi)$$

Radial oscillations

Differential equations

Simplifications:

- only 1 space coordinate: r
- m is a lagrangian coordinate

$$\frac{\partial}{\partial m} = \frac{1}{4\pi r^2 \rho} \frac{\partial}{\partial r}$$

- Poisson equation integrates once

$$\frac{\partial \Phi}{\partial r} = \frac{Gm}{r^2}$$

The general equations read

$$\frac{1}{\rho} \frac{d\rho}{dt} + 4\pi\rho \frac{\partial}{\partial m}(r^2 v) = 0$$

$$\frac{dv}{dt} = -\frac{Gm}{r^2} - 4\pi r^2 \frac{\partial P}{\partial m}$$

$$T\frac{dS}{dt} = \epsilon - \frac{\partial L}{\partial m}$$

$$L = -\frac{64\pi^2 r^4 acT^3}{3\kappa}\frac{\partial T}{\partial m}$$

Then, the perturbed equations,

$$\delta\frac{\partial X}{\partial m} = \frac{\partial\,\delta X}{\partial m}$$

$$\frac{\delta\rho}{\rho} + 4\pi\rho\frac{\partial}{\partial m}(r^2\delta r) = 0,$$

$$\frac{d^2\delta r}{dt^2} = 2\frac{Gm}{r^2}\frac{\delta r}{r} - 8\pi r^2\frac{\partial P}{\partial m}\frac{\delta r}{r} - 4\pi r^2\frac{\partial\,\delta P}{\partial m},$$

$$T\frac{d\,\delta S}{dt} = \delta\epsilon - \frac{\partial\,\delta L}{\partial m}$$

$$\delta L = -\frac{64\pi^2 r^4 acT^3}{3\kappa}\left\{\frac{\partial T}{\partial m}\left(4\frac{\delta r}{r} + 3\frac{\delta T}{T} - \frac{\delta\kappa}{\kappa}\right) + \frac{\partial\,\delta T}{\partial m}\right\}.$$

Equivalently,

$$\begin{aligned}
\frac{\partial}{\partial r}\left(\frac{\delta r}{r}\right) &= -\frac{1}{r}\left(3\frac{\delta r}{r} + \frac{\delta\rho}{\rho}\right) \\
\frac{\partial}{\partial r}\left(\frac{\delta P}{P}\right) &= -\frac{1}{P}\frac{dP}{dr}\left\{\frac{\delta P}{P} + 4\frac{\delta r}{r} - \frac{r^3}{Gm}\frac{d^2}{dt^2}\left(\frac{\delta r}{r}\right)\right\} \\
\frac{\partial}{\partial r}\left(\frac{\delta L}{L}\right) &= -\frac{1}{L}\frac{dL}{dr}\left(\frac{\delta L}{L} - \frac{\delta\epsilon}{\epsilon}\right) - \frac{4\pi r^2\rho T}{L}\frac{d\,\delta S}{dt} \\
\frac{\partial}{\partial r}\left(\frac{\delta T}{T}\right) &= -\frac{1}{T}\frac{dT}{dr}\left(4\frac{\delta r}{r} + 4\frac{\delta T}{T} - \frac{\delta L}{L} - \frac{\delta\kappa}{\kappa}\right)
\end{aligned}$$

Temporal dependence: $\delta X(r,t) = \delta X(r) e^{st}$

$$
\begin{aligned}
\frac{d}{dr}\left(\frac{\delta r}{r}\right) &= -\frac{1}{r}\left(3\frac{\delta r}{r} + \frac{\delta\rho}{\rho}\right) \\
\frac{d}{dr}\left(\frac{\delta P}{P}\right) &= -\frac{1}{P}\frac{dP}{dr}\left\{\frac{\delta P}{P} + \left(4 - \frac{r^3 s^2}{Gm}\right)\frac{\delta r}{r}\right\} \\
\frac{d}{dr}\left(\frac{\delta L}{L}\right) &= -\frac{1}{L}\frac{dL}{dr}\left(\frac{\delta L}{L} - \frac{\delta\epsilon}{\epsilon}\right) - \frac{4\pi r^2 c_v \rho T}{L}\, s\, \frac{\delta S}{c_v} \\
\frac{d}{dr}\left(\frac{\delta T}{T}\right) &= -\frac{1}{T}\frac{dT}{dr}\left(4\frac{\delta r}{r} + 4\frac{\delta T}{T} - \frac{\delta L}{L} - \frac{\delta\kappa}{\kappa}\right)
\end{aligned}
$$

Boundary conditions

$$
\left.
\begin{aligned}
3\frac{\delta r}{r} + \frac{\delta\rho}{\rho} = 0 \\
\epsilon\left(\frac{\delta L}{L} - \frac{\delta\epsilon}{\epsilon}\right) + sT\,\delta S = 0
\end{aligned}
\right\} \quad \text{at the center}
$$

$$\left.\begin{aligned} &\frac{\delta P}{P} + \left(4 - \frac{R^3 s^2}{GM}\right)\frac{\delta r}{r} = 0 \\ &4\frac{\delta r}{r} + 4\frac{\delta T}{T} - \frac{\delta L}{L} - \frac{\delta \kappa}{\kappa} = 0 \end{aligned}\right\} \quad \text{simple conditions at the surface}$$

Improve the radiative boundary condition at the surface (Eddington approximation of the atmosphere)

$$T^4 = \frac{3}{4}T_e^4(\tau + \frac{2}{3}) \approx \frac{3L}{16\pi r^2\sigma}(\tau + \frac{2}{3}).$$

$$\tau = \int_r^\infty \kappa\rho\, dr \approx \frac{\kappa\, \Delta m}{4\pi r^2}.$$

$$4\frac{\delta T}{T} + 2\frac{\delta r}{r} - \frac{\delta L}{L} - \frac{\tau}{\tau + 2/3}\left(\frac{\delta \kappa}{\kappa} - 2\frac{\delta r}{r}\right) = 0.$$

Still better: join smoothly with a perturbed atmosphere model

It is not possible to perturb directly

$$L = 4\pi R^2 \sigma T_e^4$$

$$\text{or} \quad L = 4\pi r^2 \sigma T^4 \quad \text{with} \quad \tau = 2/3$$

Integral expressions

$$\frac{d}{dr}\left(r^2\delta r\right) = -r^2\frac{\delta\rho}{\rho},$$
$$\frac{d\,\delta P}{dr} + \left(\rho s^2 + \frac{4}{r}\frac{dP}{dr}\right)\delta r = 0,$$
$$sc_vT\frac{\delta S}{c_v} = \delta\epsilon - \frac{d\,\delta L}{dm}.$$

Eq of motion $\times 4\pi r^2\overline{\delta r}$ and $\int\ dr$

$$s^2\int 4\pi r^2\rho|\delta r|^2 dr + \int 4\pi r^2\,\overline{\delta r}\left(\frac{d\,\delta P}{dr} + \frac{4}{r}\frac{dP}{dr}\delta r\right)\,dr = 0.$$

use eq of continuity, integrate by parts

$$s^2\int|\delta r|^2 dm + \int\left\{\frac{\delta P}{\rho}\frac{\overline{\delta\rho}}{\rho} + \frac{4r}{\rho}\frac{dP}{dr}\left|\frac{\delta r}{r}\right|^2\right\}dm = 0$$

Express δP in terms of $\delta\rho$ and δS, then δS from the energy eq

$$s^2 \int |\delta r|^2 dm + \int \left\{ \frac{\Gamma_1 P}{\rho} \left| \frac{\delta\rho}{\rho} \right|^2 + \frac{4r}{\rho} \frac{dP}{dr} \left| \frac{\delta r}{r} \right|^2 \right\} dm$$
$$+ \frac{1}{s} \int (\Gamma_3 - 1) \frac{\overline{\delta\rho}}{\rho} \left(\delta\epsilon - \frac{d\,\delta L}{dm} \right) dm = 0.$$

$$s^3 + As + B = 0$$

$$\text{with } A = \int \left\{ c^2 \left| \frac{\delta\rho}{\rho} \right|^2 - 4 \frac{Gm}{r^3} |\delta r|^2 \right\} dm/I$$
$$= \int \left\{ c^2 r^2 \left| \frac{d}{dr} \left(\frac{\delta r}{r} \right) \right|^2 - \frac{r}{\rho} \frac{d}{dr} [(3\Gamma_1 - 4)P] \left| \frac{\delta r}{r} \right|^2 \right\} dm/I$$

$$
\begin{aligned}
B &= \int (\Gamma_3 - 1)\overline{\frac{\delta\rho}{\rho}}\left(\delta\epsilon - \frac{d\,\delta L}{dm}\right) dm/I \\
I &= \int |\delta r|^2 dm
\end{aligned}
$$

$$
A \approx 1/\tau_{dyn}^2 \quad \text{and} \quad B \approx \frac{1}{\tau_{dyn}^2 \tau_{KH}}
$$

All non adiabatic terms in B

Dynamical and secular modes

• From the differential equations:

$$\sqrt{\frac{r^3}{Gm}} \approx \tau_{dyn} \qquad \frac{4\pi r^3 c_v \rho T}{L} \approx \tau_{KH}$$

• From the cubic equation:
Let $A = A'/\tau_{dyn}^2$, $B = B'/\tau_{dyn}^2 \tau_{KH}$, $s = s'/\tau_{dyn}$ and $\alpha = \tau_{dyn}/\tau_{KH} \ll 1$

$$s'^3 + A's' + \alpha B' = 0$$

$\Rightarrow$ two roots of the order unity

$$s' = \pm\sqrt{-A'}$$

and one of order α

$$s' = -\alpha B'/A'$$

$\Rightarrow$ two roots s of order $1/\tau_{dyn}$ and one of order $1/\tau_{KH}$.

- Other arguments: Baker one-zone model, local analysis, numerical computations.

Dynamical/secular modes: remarks

- weakness of the arguments
- in practice, no problems, except when the model is close to dynamical instability
- Secular stability not related to asteroseismology

Adiabatic radial oscillations

First step in the resolution of the radial pulsation problem: adiabatic approximation.

$$
\begin{aligned}
\frac{d}{dr}\left(\frac{\delta r}{r}\right) &= -\frac{1}{r}\left(3\frac{\delta r}{r} + \frac{1}{\Gamma_1}\frac{\delta P}{P}\right) \\
\frac{d}{dr}\left(\frac{\delta P}{P}\right) &= -\frac{1}{P}\frac{dP}{dr}\left\{\frac{\delta P}{P} + \left(4 - \frac{r^3 s^2}{Gm}\right)\frac{\delta r}{r}\right\}
\end{aligned}
$$

$$
3\frac{\delta r}{r} + \frac{1}{\Gamma_1}\frac{\delta P}{P} = 0 \quad \text{at} \quad r = 0
$$

$$
\frac{\delta P}{P} + \left(4 - \frac{R^3 s^2}{GM}\right)\frac{\delta r}{r} = 0 \quad \text{at} \quad r = R
$$

Dimensionless form

$$x = \frac{r}{R}, \quad q = \frac{m}{M}, \quad \xi = \frac{\delta r}{r}, \quad \eta = \frac{\delta P}{P}, \quad s = -i\sigma, \quad \sigma = \sqrt{\frac{GM}{R^3}}\omega$$

$$\begin{aligned}
\frac{d\xi}{dx} &= -\frac{1}{x}\left(3\xi + \frac{\eta}{\Gamma_1}\right) \\
\frac{d\eta}{dx} &= -\frac{d\ln P}{dx}\left\{\eta + \left(4 + \frac{x^3\omega^2}{q}\right)\xi\right\}
\end{aligned}$$

$$3\xi + \frac{\eta}{\Gamma_1} = 0 \quad \text{at} \quad x = 0$$

$$\eta + (4 + \omega^2)\xi = 0 \quad \text{at} \quad x = 1$$

Homologous stellar models

$$r' = \alpha r\,, \quad m' = \beta m\,, \quad \rho' = \alpha^{-3}\beta\rho\,,$$

$$P' = \alpha^{-4}\beta^2 P\,, \quad \tau'_{dyn} = \alpha^{3/2}\beta^{-1/2}\tau_{dyn}\,, \ldots$$

These relations define a Lie group with 2 parameters. It is a symmetry group of the problem.

$$\Rightarrow \omega' = \omega \Rightarrow \sigma' = \alpha^{-3/2}\beta^{1/2}\sigma$$

or, for the periods: $\tau'/\tau = \tau'_{dyn}/\tau_{dyn}$

Sturm-Liouville problem

The system can be written as an equivalent 2nd order equation

$$\frac{d}{dr}\left(\Gamma_1 P r^4 \frac{d\xi}{dr}\right) + \left\{r^3 \frac{d}{dr}[(3\Gamma_1 - 4)P] + \sigma^2 \rho r^4\right\}\xi = 0$$

with the boundary conditions

$$\frac{d\xi}{dr} = 0 \quad \text{for} \quad r = 0$$

$$\Gamma_1 R \frac{d\xi}{dr} + \left(3\Gamma_1 - 4 - \frac{R^3\sigma^2}{GM}\right)\xi = 0 \quad \text{for} \quad r = R$$

It is a Sturm-Liouville problem. It has a countable infinity of solutions with

$$\sigma_0^2 < \sigma_1^2 < \ldots < \sigma_k^2 < \ldots \quad \text{and} \quad \lim_{k\to\infty} \sigma_k^2 = +\infty$$

ξ_k has exactly k nodes in the interval $]0, R[$ and the set of all ξ_k is a basis in the functional space of all the allowable displacements.

Variational principle

$$\mathcal{L}\xi = \sigma^2 \xi$$

with

$$\mathcal{L}\xi = -\frac{1}{\rho r^4}\frac{d}{dr}\left(\Gamma_1 P r^4 \frac{d\xi}{dr}\right) - \frac{1}{\rho r}\frac{d}{dr}[(3\Gamma_1 - 4)P]\xi$$

Define a scalar product

$$(u, v) = \int \rho r^4 u \bar{v} \, dr$$

The σ_k^2 are the stationary values of the functional

$$\Lambda(u) = \frac{(u, \mathcal{L}u)}{(u, u)}$$

In particular,

$$\sigma_0^2 = \min_u \Lambda(u)$$

We can write

$$(u, \mathcal{L}u) = \int \{\Gamma_1 P r^4 \left|\frac{du}{dr}\right|^2 - r^3|u|^2 \frac{d}{dr}\left[(3\Gamma_1 - 4)P\right]\}dr\,.$$

From this variational principle we can deduce:

1) If Γ_1 is constant through the star, then the star is dynamically stable if $\Gamma_1 > 4/3$ and unstable if $\Gamma_1 < 4/3$.

2) If Γ_1 is constant through the star and $> 4/3$, we have

$$(3\Gamma_1 - 4)\frac{GM}{R^3} < \sigma_0^2 < (3\Gamma_1 - 4)\frac{GM}{R^3}\,\frac{\displaystyle\int \frac{q\,dq}{x}}{\displaystyle\int x^2\,dq}$$

Energy of a radial mode

momentum equation

$$\rho\frac{d^2\delta r}{dt^2} = \frac{4Gm\rho}{r^3}\delta r - \frac{\partial\,\delta P}{\partial r}\,,$$

multiply by $d\,\delta r/dt$

$$\rho\frac{d}{dt}(\frac{1}{2}v^2 - \frac{2Gm}{r^3}\delta r^2) = -\frac{\partial\,\delta P}{\partial r}\frac{d\,\delta r}{dt}\,.$$

transform the right-hand side

$$\begin{aligned} \ldots &= -\vec{v}\cdot\mathrm{grad}\,\delta P = -\,\mathrm{div}(\vec{v}\,\delta P) + \delta P\,\mathrm{div}\,\vec{v} \\ &= -\,\mathrm{div}(\vec{v}\,\delta P) - \frac{1}{2}\rho c^2\frac{d}{dt}\left(\frac{\delta\rho}{\rho}\right)^2 \end{aligned}$$

and finally

$$\frac{d}{dt}\{\rho[\frac{1}{2}v^2 + \frac{1}{2}c^2(\frac{\delta\rho}{\rho})^2 - 2\frac{Gm}{r^3}(\delta r)^2]\} = -\,\mathrm{div}(\delta P\,\vec{v})$$

This can be written

$$\frac{d}{dt}(\rho\mathcal{E}) = -\,\mathsf{div}\,\overrightarrow{\mathcal{F}}$$

with

$$\mathcal{E} = \underbrace{\frac{1}{2}v^2}_{\mathcal{E}_K} + \underbrace{\underbrace{\frac{1}{2}c^2\left(\frac{\delta\rho}{\rho}\right)^2}_{\mathcal{E}_A} - \underbrace{2\frac{Gm}{r}\left(\frac{\delta r}{r}\right)^2}_{\mathcal{E}_G}}_{\mathcal{E}_P}$$

and

$$\overrightarrow{\mathcal{F}} = \delta P\,\vec{v}$$

Let $\delta r(r,t) = \delta r(r)\cos\sigma t$, then

$$\begin{aligned} \mathcal{E}_K(r,t) &= \mathcal{E}_K(r)\sin^2\sigma t \\ \mathcal{E}_P(r,t) &= \mathcal{E}_P(r)\cos^2\sigma t \end{aligned}$$

Integrate over the whole star

$$
\begin{aligned}
E_K(t) &= \int 4\pi r^2 \rho \mathcal{E}_K(r,t)\,dr = E_K \sin^2 \sigma t \\
E_P(t) &= \int 4\pi r^2 \rho \mathcal{E}_P(r,t)\,dr = E_P \cos^2 \sigma t \\
E(t) &= E_K(t) + E_P(t) = const
\end{aligned}
$$

then

$$
E_K = E_P
$$

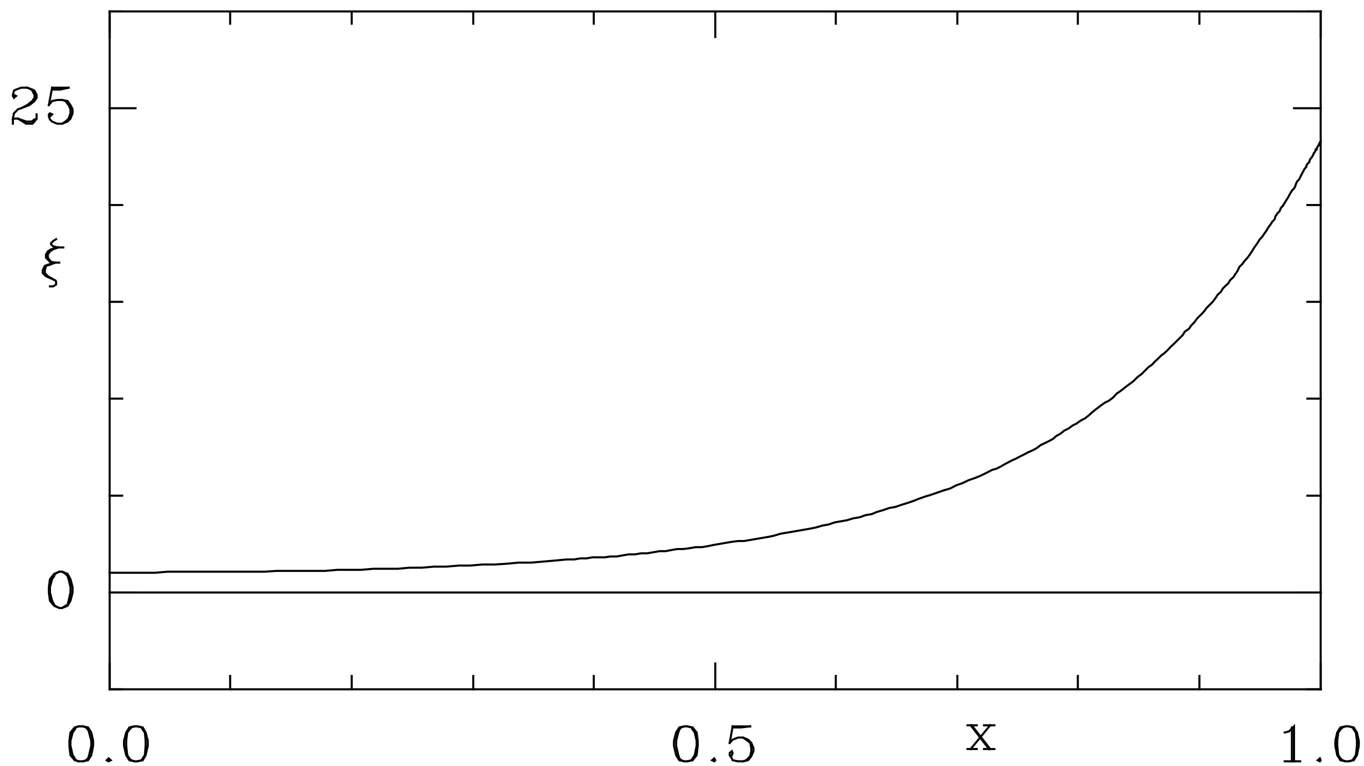
$$
\overline{E_K(t)} = \overline{E_P(t)} = \frac{1}{2}E
$$

$$
E = \frac{\sigma^2}{2}\int \delta r^2 dm
$$

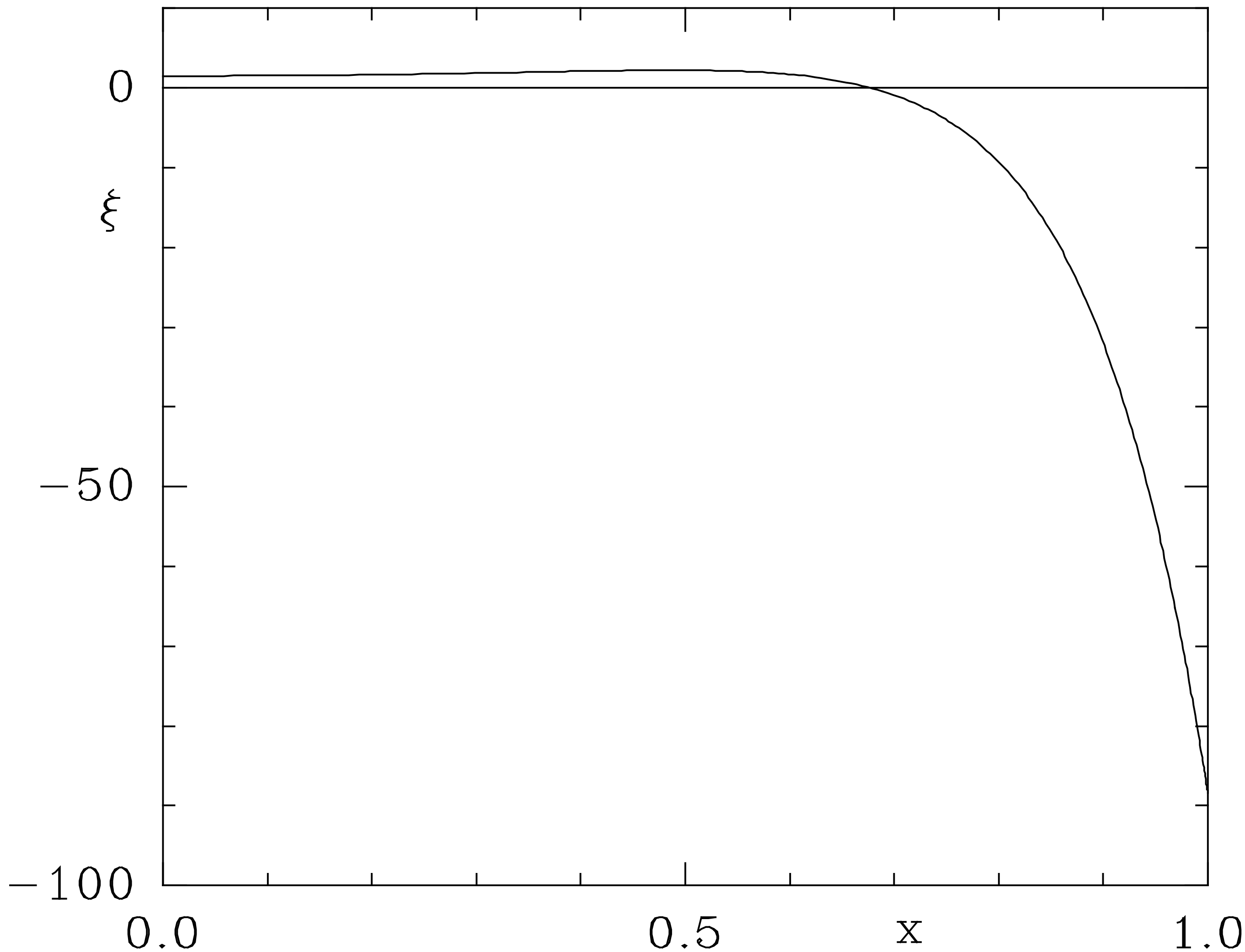
Behaviour of the eigenfunctions

Polytropic model, $n = 3$

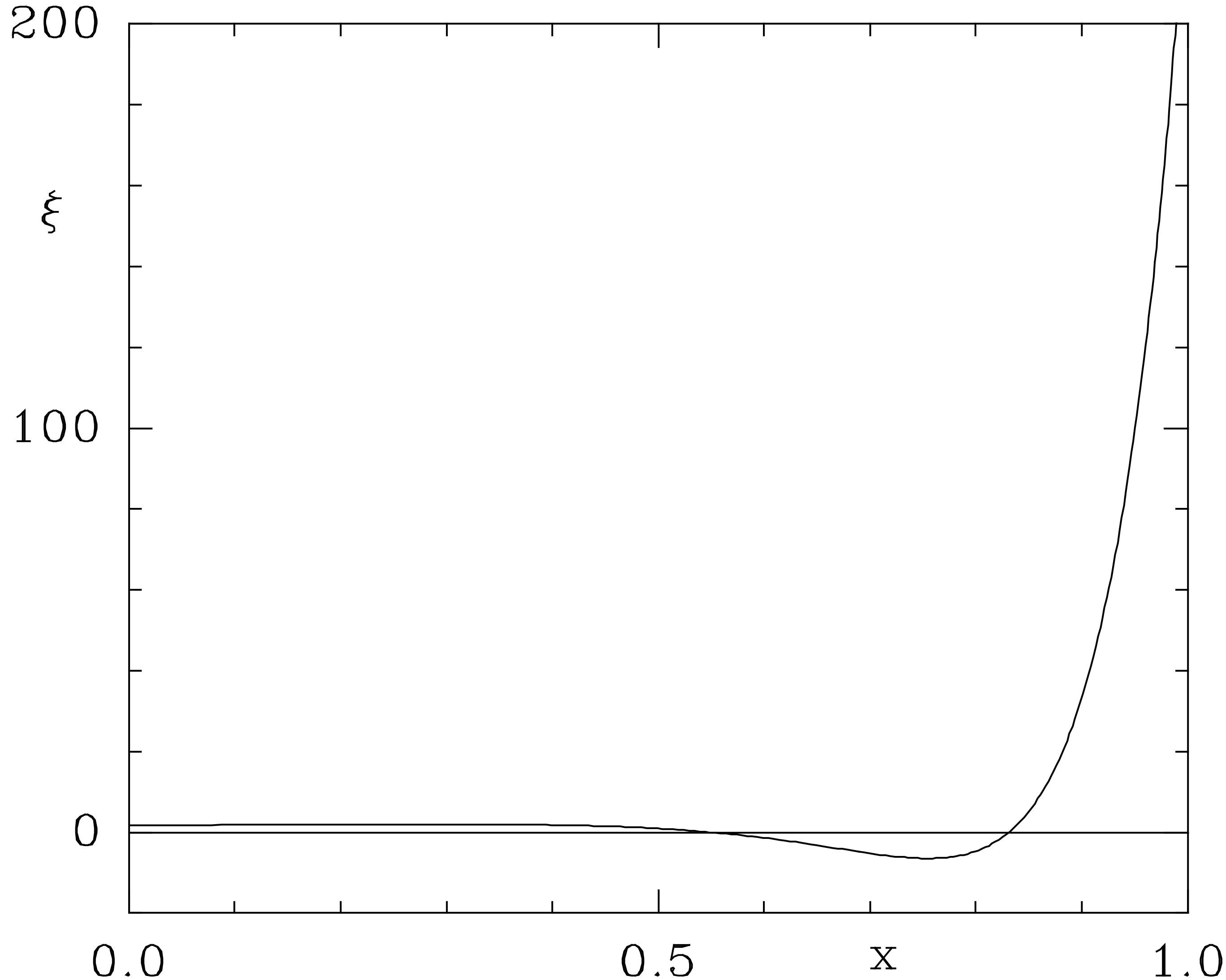
$$(\xi_0, \xi_0) = (\xi_1, \xi_1) = \dots$$



Radial p_1 mode (fundamental)



Radial p_2 mode (1st harmonic)



Radial p_3 mode (2nd harmonic)

A few cases of dynamical instability

Cause: $\Gamma_1 < \Gamma_{1\,cr}$

1) Initial phases of the contraction of a proto-star: dissociation of H_2, ionization of H and He in a large fraction of the mass

2) Final phases of the evolution of massive stars, collapse of the core, initial phase of the supernova: photodesintegration of heavy nuclei, nuclear equilibrium

$$^{56}_{26}Fe \rightleftarrows 13\alpha + 4\mathrm{n}$$

3) Very high density white dwarfs:

- relativistic degeneracy: $\Gamma_1 \approx 4/3$
- inverse β decay

$$(Z, A) + e^- \rightleftarrows (Z-1, A) + \nu$$

- general relativity

$$\Gamma_{1\,cr} = \frac{4}{3} + \Lambda \frac{GM}{Rc^2}$$

$\Rightarrow$ no stable stellar configuration with 3×10^9 g cm$^{-3} < \rho_c < 10^{14}$ g cm^{-3}

4) Maybe a situation close to instability in the envelope of S Dor variables (LBV): $\beta \approx 0 \Rightarrow \Gamma_1 \approx 4/3$ stengthens the effect of ionization.

Asymptotic expression of radial frequencies

- Useful to understand which factors influence the frequencies
- Not precise enough for the computations

The principle: change of variables to transform the differential equation to an approximation of a well-known equation.

$$\frac{d}{dr}\left(\Gamma_1 P r^4 \frac{d\xi}{dr}\right) + \left\{r^3 \frac{d}{dr}[(3\Gamma_1 - 4)P] + \sigma^2 \rho r^4\right\}\xi = 0$$

Define

$$\tau = \int_0^r \frac{dr}{c} \qquad w = r^2(\Gamma_1 P \rho)^{1/4}\xi$$

Then

$$\frac{d^2 w}{d\tau} + \{\sigma^2 + f(\tau)\}w = 0$$

First approximation

$$\frac{d^2w}{d\tau} + \sigma^2 w = 0 \Rightarrow w_k \propto \sin\sigma_k\tau$$

with $\sigma_k = k\pi/\tau_R$ for $k = 1, 2, 3, \ldots$
w_k has $k-1$ nodes in the interval $]0, \tau_R[$.

Next approximation

- Study of the singularity at the centre

$$\frac{d^2w}{d\tau^2} + \{\sigma^2 - \frac{2}{\tau^2} + g(\tau)\}w = 0$$

If far enough from the surface neglect $g(\tau)$, define $z = \sigma\tau$ and $w = \sqrt{z}u(z)$

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{9}{4z^2}\right)u = 0$$

$\Rightarrow u(z) = J_{3/2}(z)$ and $w(\tau) \approx \sin(\sigma\tau - \frac{\pi}{2})$ far from the center

• Study of the singularity at the surface

Effective polytropic index n_e such that
$\rho \propto (R-r)^{n_e}$ and $P \propto (R-r)^{n_e+1}$.

$\tau_R - \tau \propto (R-r)^{1/2}$, $\rho \propto (\tau_R - \tau)^{2n_e}$,
$P \propto (\tau_R - \tau)^{2n_e+1}$ and $c \propto (\tau_R - \tau)$.

$$\frac{d^2w}{d\tau^2} + \left[\sigma^2 - \frac{n_e^2 - \frac{1}{4}}{(\tau_R - \tau)^2} + h(\tau)\right] w = 0,$$

If far enough from the center neglect $h(\tau)$,
define $z = \sigma(\tau_R - \tau)$ and $w = \sqrt{z}u(z)$

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{n_e^2}{z^2}\right) u = 0$$

$\Rightarrow u(z) = J_{n_e}(z)$ and $w(\tau) \propto \sin(\sigma\tau - \sigma\tau_R - \frac{\pi}{4} + \frac{n_e\pi}{2})$ far from the surface

Join both pieces of solution in τ^*

You obtain the condition

$$\sigma\tau^* - \frac{\pi}{2} = \sigma\tau^* - \sigma\tau_R - \frac{\pi}{4} + \frac{n_e\pi}{2} + k\pi$$

or

$$\sigma_k = \left(k + \frac{n_e}{2} + \frac{1}{4}\right)\frac{\pi}{\tau_R} \quad \text{for } k = 1, 2, \ldots$$

Vibrational stability

Take the imaginary part of

$$s^2 \int |\delta r|^2 dm + \int \left\{ \frac{\delta P}{\rho} \frac{\overline{\delta\rho}}{\rho} + \frac{4r}{\rho} \frac{dP}{dr} \left| \frac{\delta r}{r} \right| \right\} dm = 0$$

$$2\Re s \, \Im s = - \frac{\Im \int \frac{\delta P}{\rho} \frac{\overline{\delta\rho}}{\rho} \, dm}{\int |\delta r|^2 dm}$$

Tranform the numerator

- eq of state + conservation of the energy
- or cubic equation in the form $s^2 + A + B/s = 0$

$$2\Re s \, \Im s = \frac{\Im \frac{1}{s} \int (\Gamma_3 - 1) \frac{\overline{\delta\rho}}{\rho} \left(\delta\epsilon - \frac{d\,\delta L}{dm} \right) dm}{\int |\delta r|^2 dm}$$

Quasi-adiabatic approximation

$s = \sigma' - i\sigma$, nonadiabatic solution considered as a small perturbation of the adiabatic solution $\Rightarrow$ a simple expression for σ'

$$\sigma' = \frac{1}{2\sigma^2}\frac{\int \frac{\delta T}{T}\left(\delta\epsilon - \frac{d\,\delta L}{dm}\right) dm}{\int |\delta r|^2 dm}$$

Physical interpretation

Denominator $\sim$ energy of the pulsation

$$E = \frac{\sigma^2}{2}\int |\delta r|^2\, dm$$

Numerator $\sim$ power of a thermodynamic cycle

$$\begin{cases} P(t) &= P_0 + a\cos(\phi - \sigma t) \\ \rho(t) &= \rho_0 + b\cos(\psi - \sigma t) \end{cases}$$

or

$$\begin{cases} \delta P(t) &= \delta P\, e^{-i\sigma t} \quad (\delta P = a e^{i\phi}) \\ \delta\rho(t) &= \delta\rho\, e^{-i\sigma t} \quad (\delta\rho = b e^{i\psi}) \end{cases}$$

$$\mathcal{T} = \oint P\, dV = \frac{\pi ab}{\rho^2}\sin(\phi - \psi) = \pi \Im\left(\frac{\delta P}{\rho}\frac{\overline{\delta\rho}}{\rho}\right)$$

$$\mathcal{W} = \frac{\mathcal{T}}{\tau} = \frac{\sigma}{2}\Im\left(\frac{\delta P}{P}\frac{\overline{\delta\rho}}{\rho}\right)$$

$$W = \frac{\sigma}{2}\Im\int\frac{\delta P}{\rho}\frac{\overline{\delta\rho}}{\rho}\,dm = \frac{1}{2}\int\frac{\delta T}{T}\left(\delta\epsilon - \frac{d\,\delta L}{dm}\right)dm$$

The expression for σ' reduces to expected result

$$\sigma' = \frac{W}{2E}$$

If amplitude $\propto e^{\sigma' t}$ then $E \propto e^{2\sigma' t}$ and

$$2\sigma' = \frac{1}{E}\frac{dE}{dt} = \frac{W}{E}$$

Interest of the integral expression for σ':

- the mechanism of the excitation
- the seat of the instability

Value of the quasi-adiabatic approximation:

- very good in the interior
- very poor in the external layers

center	adiabatic zone	transition zone	non adiabatic zone	surface

The transition zone may be defined such that

$$c_v T \, \Delta m \approx L\tau$$

In the nonadiabatic zone $c_v T \, \Delta m \ll L\tau$ and from the equation of energy conservation

$$\Delta \frac{\delta L}{L} \approx \frac{c_v T \, \Delta m}{L\tau} \frac{\delta S}{c_v}$$

shows that $\delta L \approx const.$

Nuclear excitation

$$\int \frac{\delta T}{T} \delta\epsilon \, dm$$
$$= \int (\Gamma_3 - 1)[\epsilon_\rho + (\Gamma_3 - 1)\epsilon_T] \left(\frac{\delta\rho}{\rho}\right)^2 \epsilon \, dm > 0$$

Contribution from the internal layers. For main sequence stars, this term is reponsible for instability for masses above $M_{cr} \sim 90 - 120\ M_\odot$.

The transfer term

$$-\int \frac{\delta T}{T} \frac{d\,\delta L}{dm} \, dm = -\int \frac{\delta T}{T} \frac{d\,\delta L}{dr} \, dr$$

Physical interpretation: if δL decreases outwards when $\delta T > 0$, then the matter absorbs heat at high temperature and releases it at low temperature, like an engine, and produces positive work.

The main contribution comes from the external layers.

In a radiative zone, the main terms of the equation of transfer give

$$\frac{\delta L}{L} \approx [(4 - \kappa_T)(\Gamma_3 - 1) - \kappa_\rho]\frac{\delta \rho}{\rho}$$

and

$$\frac{\delta T}{T} = (\Gamma_3 - 1)\frac{\delta \rho}{\rho}$$

The effect of this term is mainly determined by the sign of the coefficient

$$-[(4 - \kappa_T)(\Gamma_3 - 1) - \kappa_\rho]$$

Generally $\Gamma_3 \approx 5/3$, $\kappa_\rho = 1$, $\kappa_T = -3.5$ and the term of transfer has a stabilizing effect.

Conditions for excitation (κ-mechanism):
1) $\Gamma_3 - 1$ small (partial ionization)
2) $\kappa_T > 0$ (opacity due to H^-)

The variables of the instability strip (δ Sct, RR Lyr, δ Cep, W Vir, RV Tau) owe their instability to the κ-mechanism taking place in the zone where $He^+ \rightleftarrows He^{++}$

For the Mira variables the partial ionization of hydrogen $H \rightleftarrows H^+$ is responsible for the instability.

In β Cep variables, an increase of opacity due to Fe at $T \approx 2 \times 10^5$ K is the cause of the instability.

Ratio between the amplification time $\tau' = 1/\sigma'$ and the period τ

Variable type	τ'/τ
δ Sct	10^4 – 10^6
δ Cep and RR Lyr	10^2 – 10^3
W Vir	10 – 20
Long period var (Mira)	1 – 10

The instability strip

Computations

- κ-mechanism: 2nd He ionization zone $+$
  small contribution 1st He and H ioniz. zones
- blue edge: OK
- red edge: problems with convection

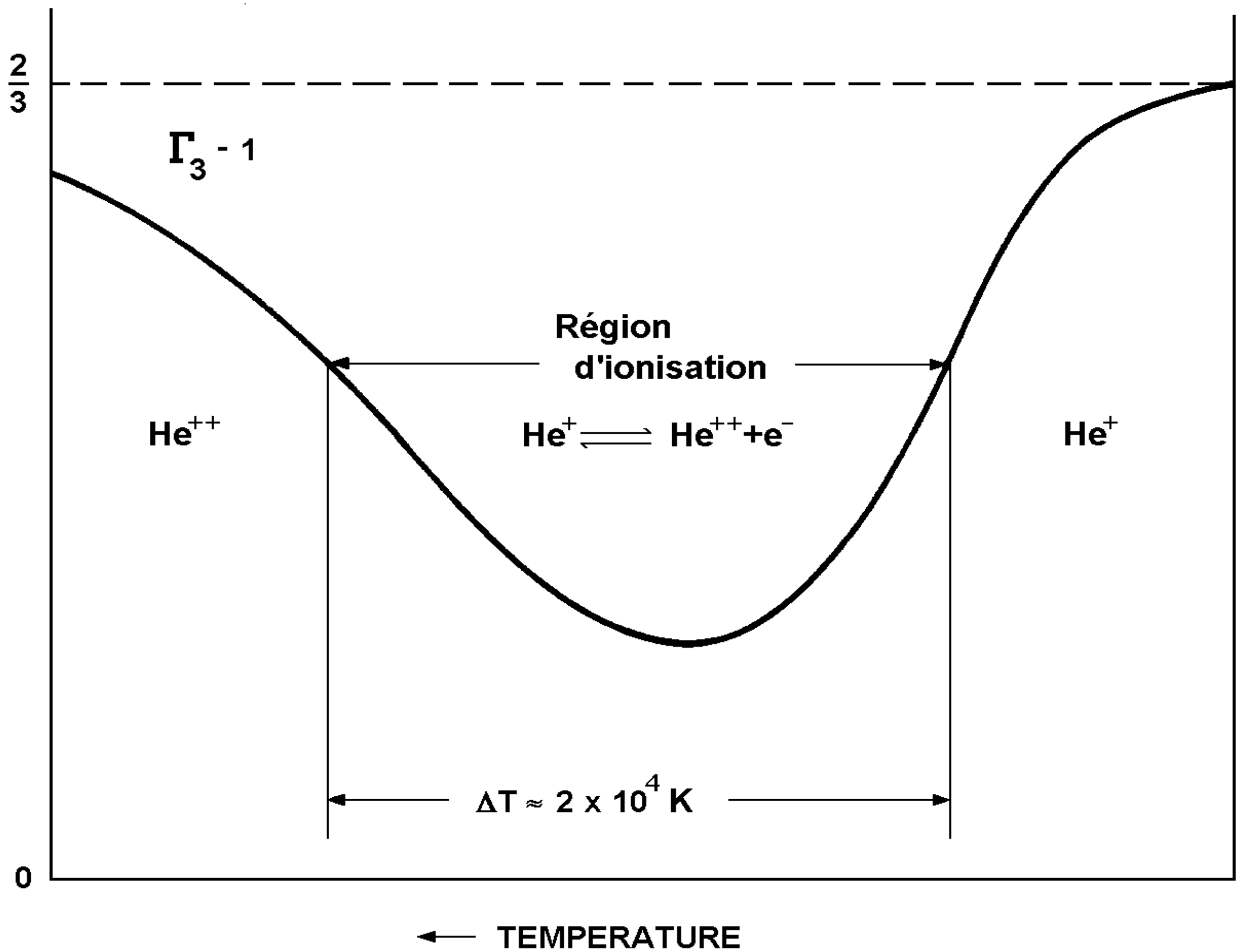
Simple explanations for the existence of the instability strip and for the phase lag of the light.

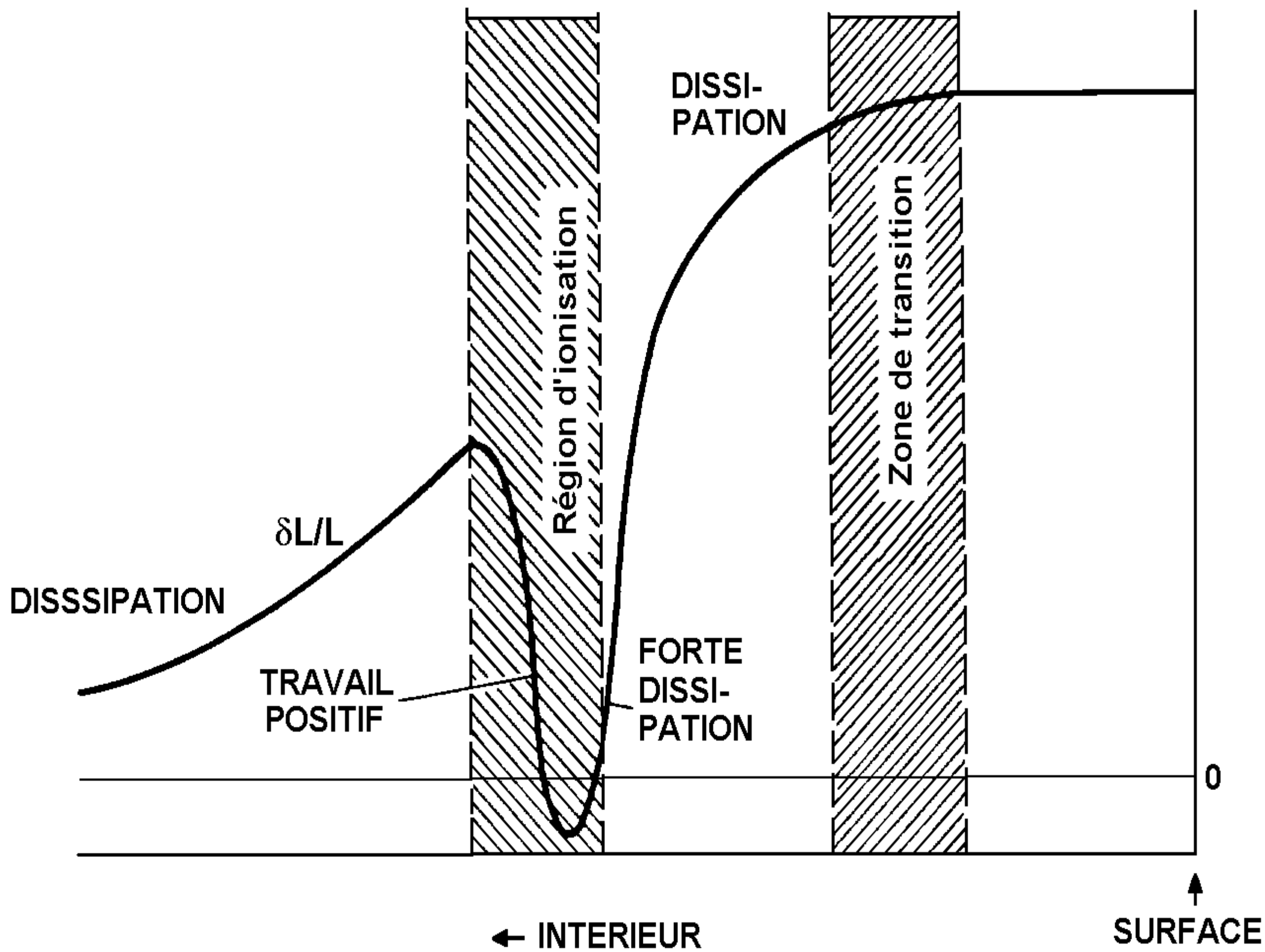
Interpretation of J.P. Cox

1) $\dfrac{\delta L}{L} \approx [(\Gamma_3 - 1)(4 - \kappa_T) - \kappa_\rho]\dfrac{\delta\rho}{\rho}$

2) In the external layers, $\delta L/L$ increases $\sim$ exponentially with r in adiabatic region, but remains constant in non adiabatic region

We suppose $\delta T/T > 0$

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PATION
Région d'ionisation
Zone de transition
δL/L
DISSSIPATION
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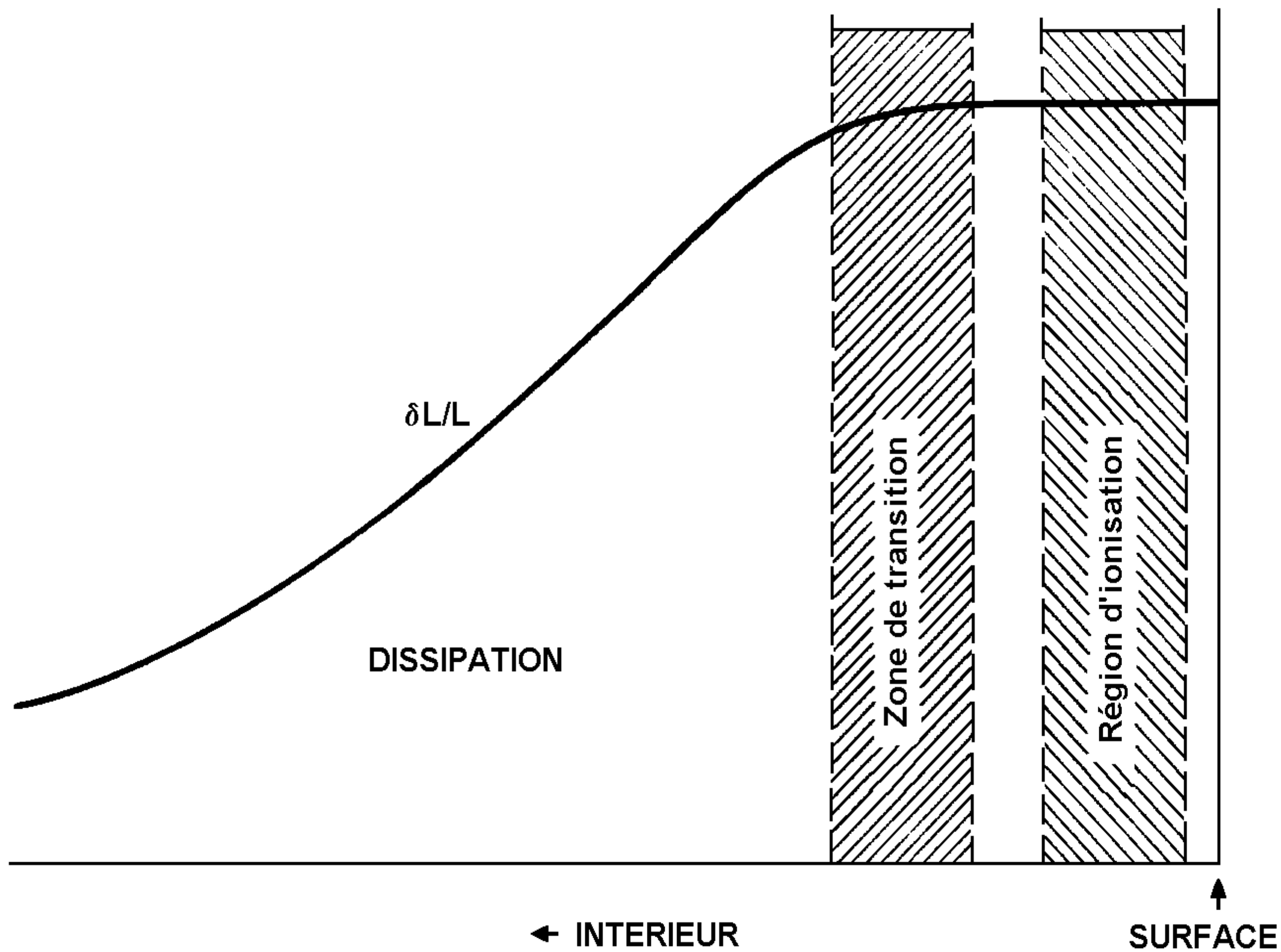
SURFACE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Région d'ionisation
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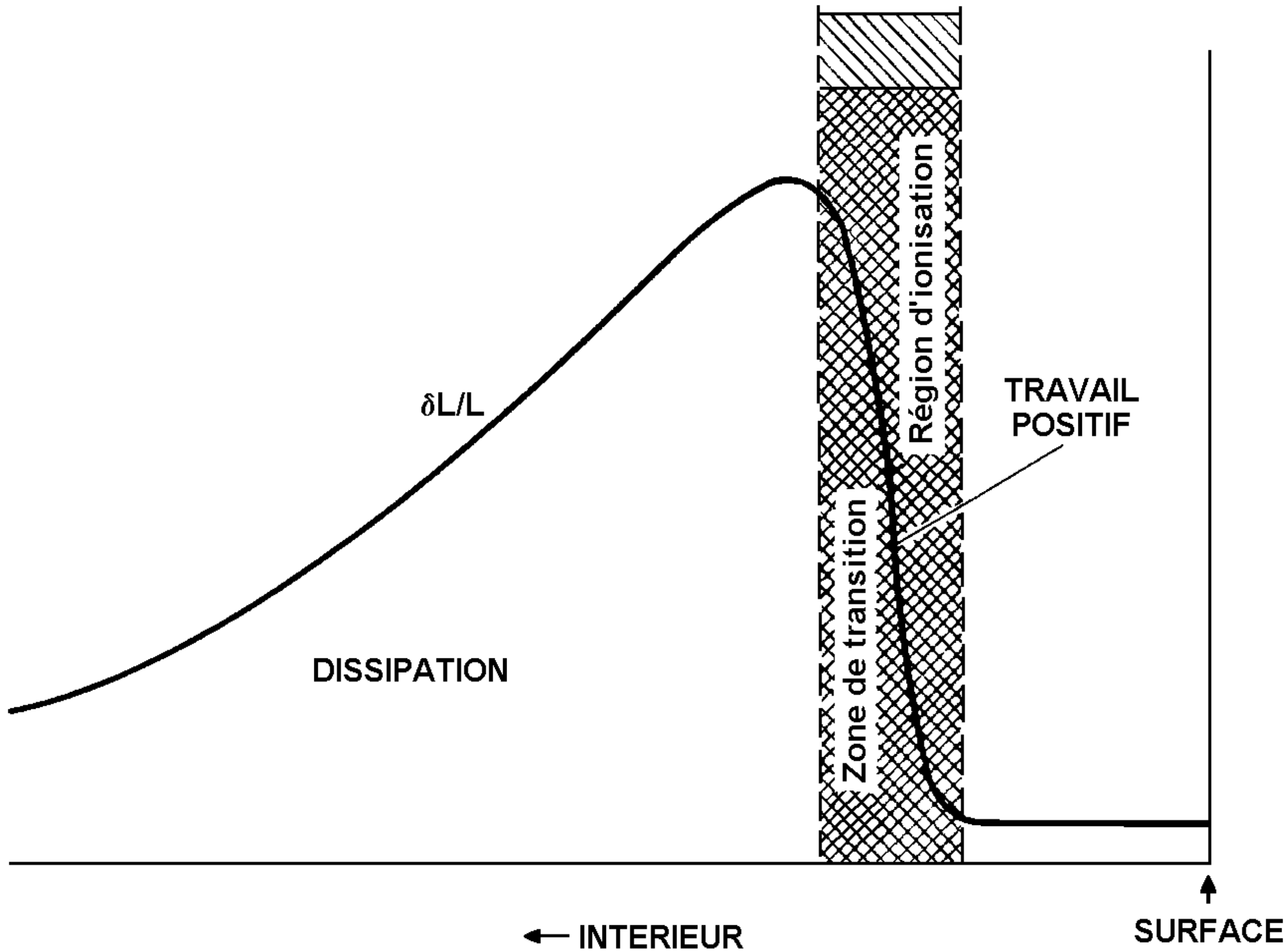
SURFACE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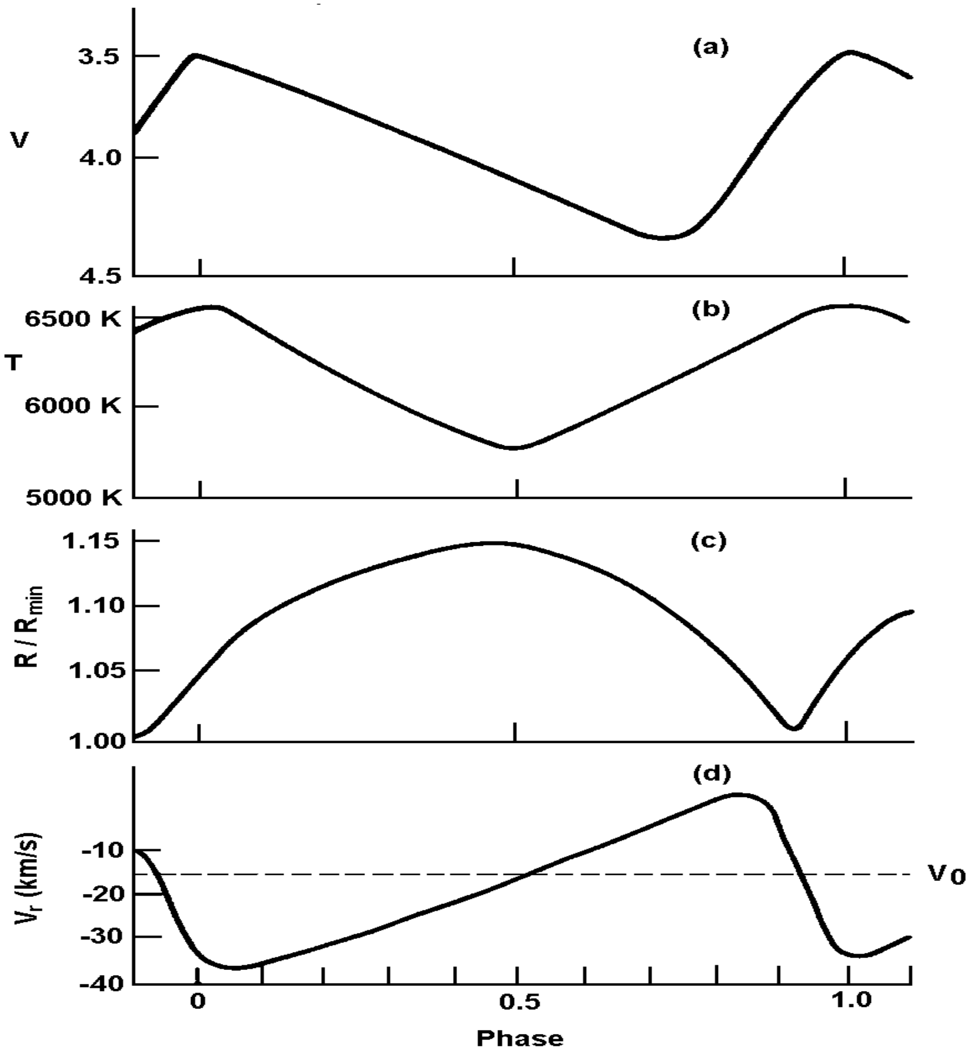
SURFACE

The phase lag



The pulsation of δ Sct

(a) light curve,

(b) temperature,

(c) radius,

(d) radial velocity

Adiabatic theory: max of $\delta\rho$, δT, δL and min of δr simultaneous.

True below the H ionization zone.

The luminosity acquires its phase lag in the H ionization zone (in the non adiabatic region !). Thanks to its high c_v, the ionization front can absorb energy and moves through the stellar material. Its position (and the outgoing luminosity) lags behind the inner luminosity in the same way as the charge of a condenser lags behind the difference of potential at its terminals.

This mechanism cannot exist in stars with
$T_e > 10^4$ K. This is in agreement with observations: no phase lag in β Cep variables.

Secular stability

secular modes

associated with s in energy equation

s^2 term in motion equation now negligible

$$s^3 + As + B = 0 \qquad \text{with}$$

$$A \approx 1/\tau_{dyn}^2 \qquad B \approx 1/\tau_{dyn}^2 \tau_{HK}$$

s^3 term negligible

$$s = -\frac{B}{A} = -\frac{\int (\Gamma_3 - 1)\frac{\overline{\delta\rho}}{\rho}\left(\delta\epsilon - \frac{d\,\delta L}{dm}\right) dm}{\int \{c^2 r^2 \left|\frac{d}{dr}\left(\frac{\delta r}{r}\right)\right|^2 - \frac{r}{\rho}\frac{d}{dr}[(3\Gamma_1 - 4)P]\left|\frac{\delta r}{r}\right|^2\} dm}$$

Approximation: eigenfunctions replaced by a perturbation describing a homologous transformation.

$$\frac{\delta r}{r} = -1\,, \quad \frac{\delta \rho}{\rho} = 3\,, \quad \frac{\delta P}{P} = 4\,, \quad \frac{\delta T}{T} = \frac{4 - 3P_\rho}{P_T}$$

Then

$$\frac{\delta L}{L} = 4\frac{\delta r}{r} + 4\frac{\delta T}{T} - \frac{\delta \kappa}{\kappa} = -4 - 3\kappa_\rho + \frac{4 - 3P_\rho}{P_T}(4 - \kappa_T)$$

and

$$s \approx -\frac{(\Gamma_3 - 1)L}{(\Gamma_1 - \frac{4}{3})|\Omega|}\{3\kappa_\rho + 3\epsilon_\rho + 4 + \frac{4 - 3P_\rho}{P_T}(\kappa_T + \epsilon_T - 4)\}$$

where $\Omega = -\int \frac{Gm\,dm}{r}$ is the gravitational potential energy of the star.

Consequences:

1) $s \approx 1/\tau_{KH}$

2) We suppose dynamical stability, $\Gamma_1 > 4/3$, then the secular stability criterion reads

$$3\kappa_\rho + 3\epsilon_\rho + 4 + \frac{4 - 3P_\rho}{P_T}(\kappa_T + \epsilon_T - 4) > 0$$

Perfect gas

Secular stability criterion

$$3\kappa_\rho + \kappa_T + 3\epsilon_\rho + \epsilon_T > 0$$

For a main sequence star

$$\kappa_\rho \approx 1\,, \quad \kappa_T \approx -3,5\,, \quad \epsilon_\rho \approx 1$$

For pp chains $\epsilon_T \approx 6$ for $T \approx 5 \times 10^6$ K and for the carbon cycle $\epsilon_T \approx 13$ for $T \approx 5 \times 10^7$K. Main sequence stars are secularly stable.

Consider an infinitesimal homologous transformation

$$\frac{\delta r}{r} = -1\,, \quad \frac{\delta\rho}{\rho} = 3 \quad \frac{\delta P}{P} = 4\,, \quad \frac{\delta T}{T} = 1$$

Then

$$\frac{\delta\epsilon}{\epsilon} - \frac{\delta L}{L} = 3\kappa_\rho + \kappa_T + 3\epsilon_\rho + \epsilon_T > 0$$

The increase in nuclear energy production is not entirely compensated by the variation of the luminosity. It results in an increase in temperature and pressure able to oppose a further contraction.

Degenerate matter

$$P_\rho \approx 5/3\,, \quad P_T \approx 0$$

and the secular stability criterion reads

$$\epsilon_T + \kappa_T - 4 < 0$$

In degenerate matter, energy transport is provided by conduction and

$$\kappa \propto \rho^{-2} T^2 \quad \text{et} \quad \kappa_T \approx 2$$

The presence of nuclear fuel in degenerate matter leads to instability.

Consider a perturbation described by

$$\frac{\delta r}{r} = \frac{\delta \rho}{\rho} = \frac{\delta P}{P} = 0\,, \quad \frac{\delta T}{T} = 1$$

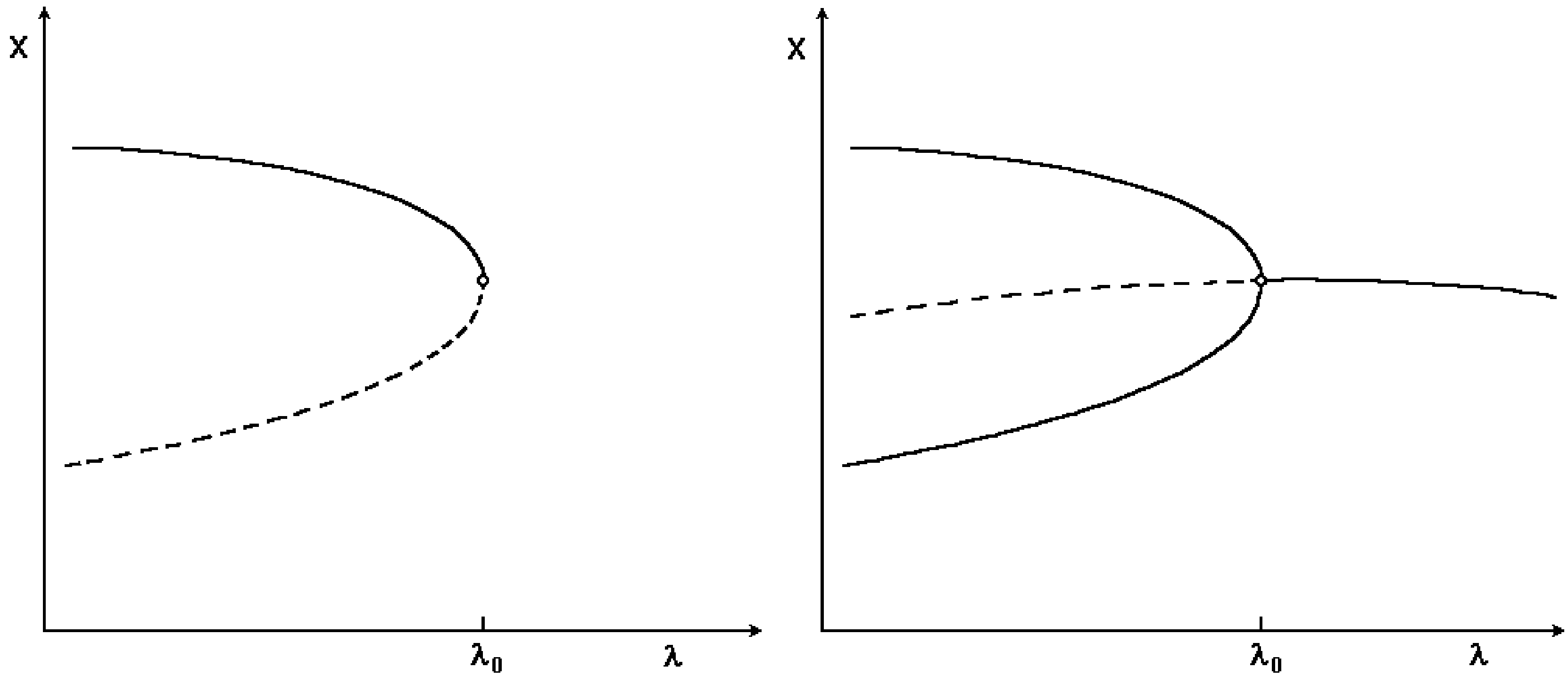
Then

$$\frac{\delta \epsilon}{\epsilon} - \frac{\delta L}{L} = \epsilon_T + \kappa_T - 4 > 0$$

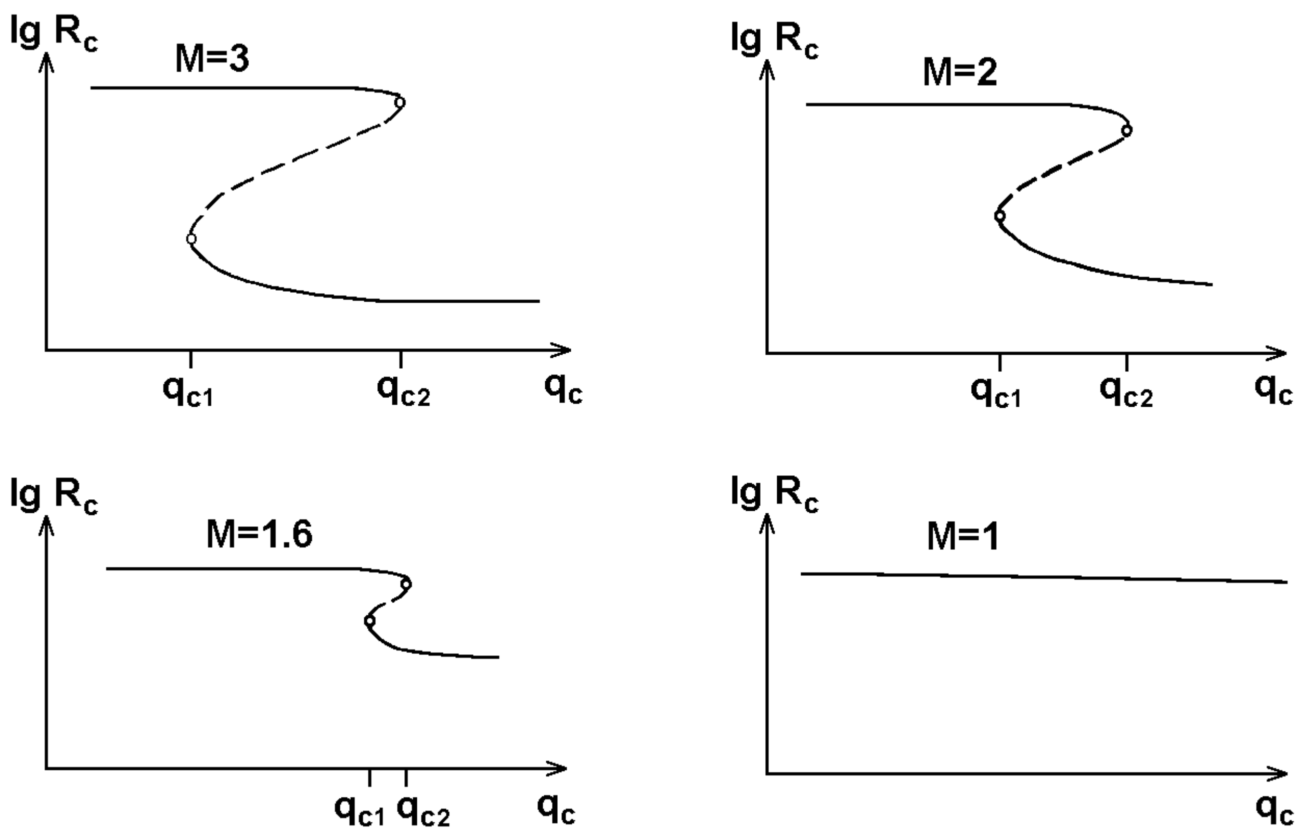
The increase in nuclear energy production is not entirely compensated by the variation of luminosity. It results in a further increase in temperature. The pressure is almost independent of temperature and is unable to oppose the resulting thermal runaway. This runaway is only stopped when the temperature is high enough so that the matter is no longer degenerate.

Application to the stellar evolution

1) Linear series: local unicity of stellar models, bifurcation diagram, critical values, and $s = 0$.



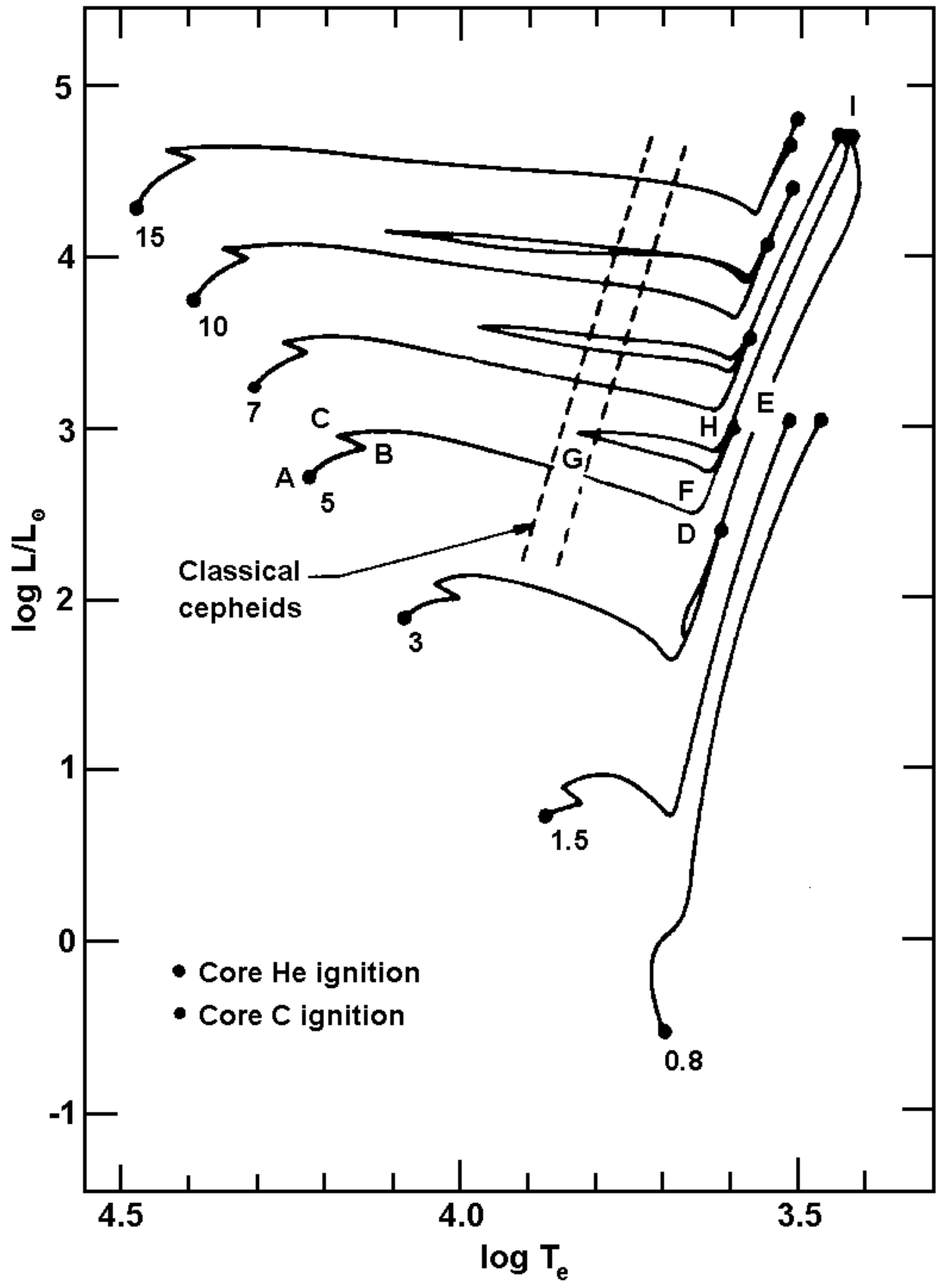
Stars with a hydrogen shell source: the Schönberg-Chandrasekhar limit



$$q_{SC} = q_{c2} = 0.37 \left(\frac{\mu_e}{\mu_c}\right)$$

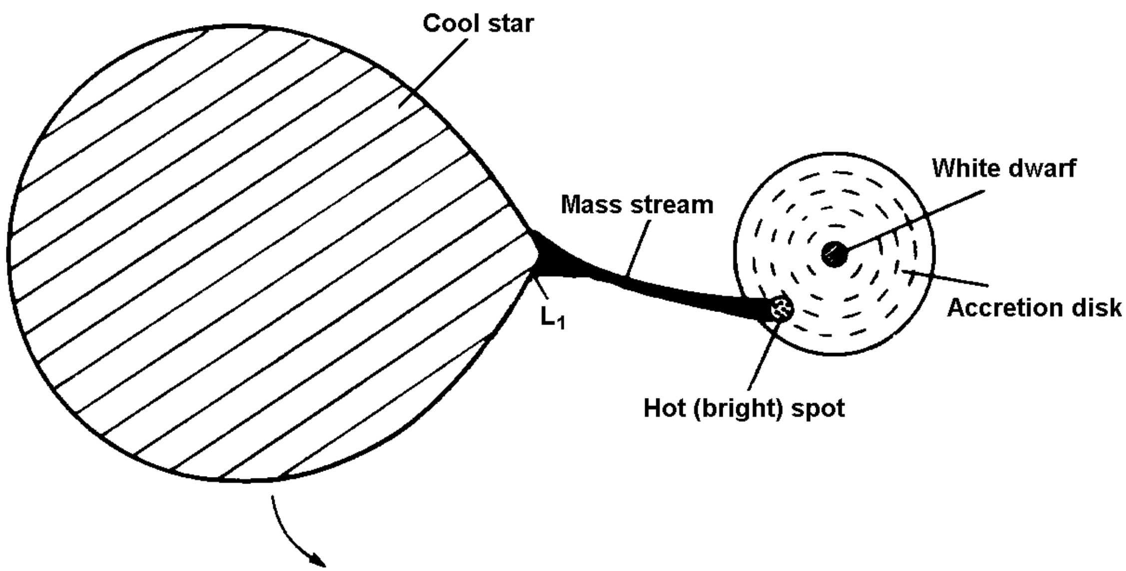
2) Cepheids

$5M_{\odot} < M < 10M_{\odot}$



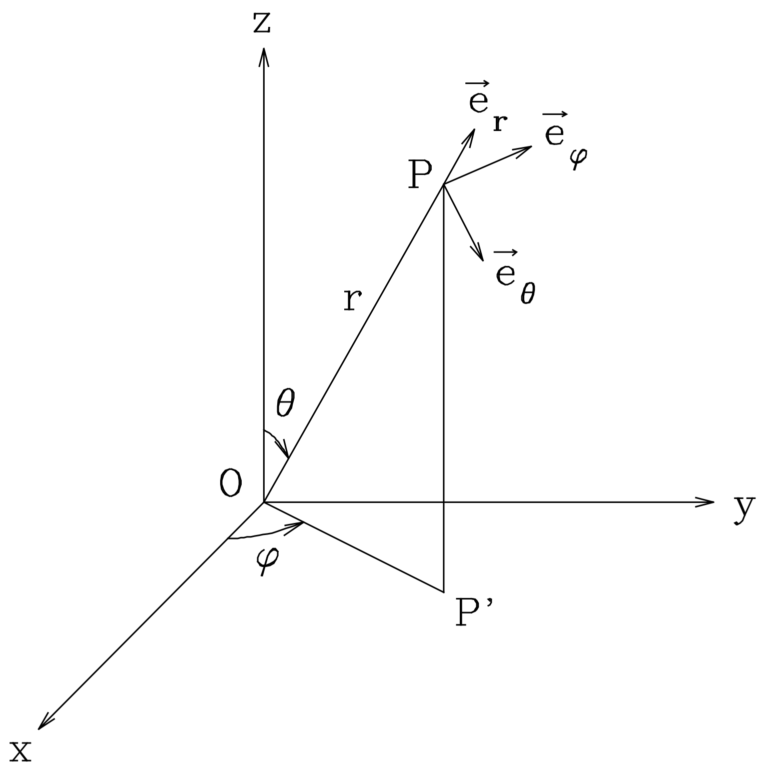
3) Helium flash

4) Nova phenomenon



Non radial oscillations

Spherical coordinates



Some differential operators in spherical coordinates

$$\begin{aligned}\Delta\alpha &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\alpha}{\partial r}\right) \\ &\quad+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\alpha}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\alpha}{\partial\phi^2} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\alpha}{\partial r}\right)-\frac{1}{r^2}L^2\alpha\end{aligned}$$

where

$$L^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Perturbation equations

$$\overrightarrow{\delta r} = \delta r\,\vec{e}_r + r\,\delta\theta\,\vec{e}_\theta + r\,\delta\phi\,\sin\theta\,\vec{e}_\phi$$

We suppose we have already separated the factor e^{st}

Equation of continuity

$$\rho' + \delta r\,\frac{d\rho}{dr} + \rho\left\{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\delta r) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\delta\theta) + \frac{\partial\,\delta\phi}{\partial\phi}\right\} = 0$$

Equations of motion

$$\begin{aligned}
s^2\delta r &= -\frac{\partial\Phi'}{\partial r} + \frac{\rho'}{\rho^2}\frac{dP}{dr} - \frac{1}{\rho}\frac{\partial P'}{\partial r} \\
s^2 r\,\delta\theta &= -\frac{1}{r}\frac{\partial\Phi'}{\partial\theta} - \frac{1}{\rho r}\frac{\partial P'}{\partial\theta} \\
s^2 r\sin\theta\,\delta\phi &= -\frac{1}{r\sin\theta}\frac{\partial\Phi'}{\partial\phi} - \frac{1}{\rho r\sin\theta}\frac{\partial P'}{\partial\phi}
\end{aligned}$$

Poisson equation

$$\frac{1}{r^2}\left(r^2\frac{\partial\Phi'}{\partial r}\right) - \frac{1}{r^2}L^2\Phi' = 4\pi G\rho'$$

Energy equation

$$\begin{aligned} sT\left(S' + \delta r\,\frac{dS}{dr}\right) &= \epsilon' + \frac{\rho'}{\rho^2}\frac{1}{r^2}\frac{d}{dr}(r^2F) \\ &\quad -\frac{1}{\rho}\Big\{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F'_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,F'_\theta) + \frac{1}{r\sin\theta}\frac{\partial F'_\phi}{\partial\phi}\Big\}\,. \end{aligned}$$

Transport equations

$$\begin{aligned} F'_r &= -\lambda'\frac{dT}{dr} - \lambda\frac{\partial T'}{\partial r}\,, \\ F'_\theta &= -\frac{\lambda}{r}\frac{\partial T'}{\partial\theta}\,, \\ F'_\phi &= -\frac{\lambda}{r\sin\theta}\frac{\partial T'}{\partial\phi}\,. \end{aligned}$$

A difficult problem. We turn directly to adiabatic approximation.

Separation of the coordinates

$$Y_{\ell m}(\theta,\phi) = a_{\ell m} P_\ell^{|m|}(\cos\theta) e^{im\phi} \quad \text{and} \quad L^2 Y_{\ell m}(\theta,\phi) = \ell(\ell+1) Y_{\ell m}(\theta,\phi)$$

Eq. of motion $\Rightarrow \delta\theta, \delta\phi \Rightarrow$ eq. of continuity

$$\rho' + \delta r \frac{d\rho}{dr} + \frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 \delta r) + \frac{\rho}{s^2 r^2} L^2 \chi = 0 \quad \text{with } \chi = \Phi' + P'/\rho$$

$$\delta r(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \delta r_{\ell m}(r) Y_{\ell m}(\theta,\phi)\,, \qquad \rho'(r,\theta,\phi) = \ldots$$

$$\begin{cases} \rho'_{\ell m} + \delta r_{\ell m}\dfrac{d\rho}{dr} + \dfrac{\rho}{r^2}\dfrac{d}{dr}(r^2 \delta r_{\ell m}) + \dfrac{\rho\ell(\ell+1)}{s^2 r^2}\chi_{\ell m} = 0 \\ s^2 \delta r_{\ell m} = -\dfrac{d\Phi'_{\ell m}}{dr} + \dfrac{\rho'_{\ell m}}{\rho^2}\dfrac{dP}{dr} - \dfrac{1}{\rho}\dfrac{dP'_{\ell m}}{dr} \\ \dfrac{1}{r^2}\dfrac{d}{dr}\left(r^2 \dfrac{d\Phi'_{\ell m}}{dr}\right) - \dfrac{\ell(\ell+1)}{r^2}\Phi'_{\ell m} = 4\pi G \rho'_{\ell m} \end{cases}$$

• Boundary conditions at the center

Regularity of the solution

$\Rightarrow$ 2 boundary conditions

$\Rightarrow \delta r \propto r^{\ell-1}$, P' and $\Phi' \propto r^{\ell}$

• Boundary conditions at the surface

1) $\delta P = 0 \Rightarrow \delta P_{\ell m} = 0$

2) Continuity of Φ' and $\mathrm{grad}\,\Phi'$

For simplicity, we omit the ℓ, m indices. Let Φ'_e be the exterior solution.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi'}{dr}\right) - \frac{\ell(\ell+1)\Phi'}{r^2} = 0\,,$$

Its regular solution is

$$\Phi'_e = \frac{A}{r^{\ell+1}}$$

We impose

$$\delta\Phi = \delta\Phi_e$$
$$\delta\frac{d\Phi}{dr} = \delta\frac{d\Phi_e}{dr}$$

or

$$\Phi' + \delta r\frac{d\Phi}{dr} = \Phi'_e + \delta r\frac{d\Phi_e}{dr}$$
$$\frac{d\Phi'}{dr} + \delta r\frac{d^2\Phi}{dr^2} = \frac{d\Phi'_e}{dr} + \delta r\frac{d^2\Phi_e}{dr^2}$$

For the equilibrium configuration we have

$$\begin{aligned}
\Phi &= \Phi_e \\
\frac{d\Phi}{dr} &= \frac{d\Phi_e}{dr} \\
\frac{d^2\Phi}{dr^2} &= \frac{d^2\Phi_e}{dr^2} + 4\pi G\rho
\end{aligned}$$

The continuity conditions then give

$$\begin{aligned} \Phi' &= \frac{A}{r^{\ell+1}} \\ \frac{d\Phi'}{dr} &= -\frac{(\ell+1)A}{r^{\ell+2}} - 4\pi G\rho\,\delta r \end{aligned}$$

And the elimination of A gives the required condition

$$\frac{d\Phi'}{dr} + \frac{\ell+1}{r}\Phi' + 4\pi G\rho\,\delta r = 0\,.$$

Degeneracy

$$s_{k\ell m} = s_{k\ell m'}$$

Cowling approximation

$\Phi' = 0 \Rightarrow$ 2nd order system.
Define $v = r^2 \delta r\, P^{1/\Gamma_1}$, $w = P'/P^{1/\Gamma_1}$ and
$s = -i\sigma$, then

$$\begin{aligned}\frac{dv}{dr} &= \left(\frac{L_\ell^2}{\sigma^2} - 1\right)\frac{r^2 P^{2/\Gamma_1}}{\rho c^2} w \\ \frac{dw}{dr} &= (\sigma^2 - n^2)\frac{\rho}{r^2 P^{2/\Gamma_1}} v\end{aligned}$$

with

$$\begin{aligned}L_\ell^2 &= \frac{\ell(\ell+1)c^2}{r^2} \\ n^2 &= -Ag \qquad \text{where} \quad A = \frac{d\ln\rho}{dr} - \frac{1}{\Gamma_1}\frac{d\ln P}{dr}\end{aligned}$$

- Don't use for computations

- Useful for analytical discussion:

- link with plane waves (Lamb frequency, Brunt-Väisälä frequency)
- classification of the modes

Properties of non radial modes

Components of the displacement

$$
\begin{aligned}
\overrightarrow{\delta r} &= \delta r\,\vec{e}_r + r\,\delta\theta\,\vec{e}_\theta + r\,\sin\theta\,\delta\phi\,\vec{e}_\phi \\
&= \delta r\,\vec{e}_r + \frac{1}{r\sigma^2}\left(\frac{\partial\chi}{\partial\theta}\vec{e}_\theta + \frac{1}{\sin\theta}\frac{\partial\chi}{\partial\phi}\,\vec{e}_\phi\right) \\
&= \left[a(r)\vec{\epsilon} + b(r)\vec{\eta}\right]e^{-i\sigma t}
\end{aligned}
$$

with

$$
\begin{aligned}
\chi &= \Phi' + \frac{P'}{\rho} \\
\vec{\epsilon} &= Y_{\ell m}(\theta,\phi)\vec{e}_r \\
\vec{\eta} &= \frac{\partial Y_{\ell m}}{\partial\theta}\,\vec{e}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_{\ell m}}{\partial\phi}\,\vec{e}_\phi
\end{aligned}
$$

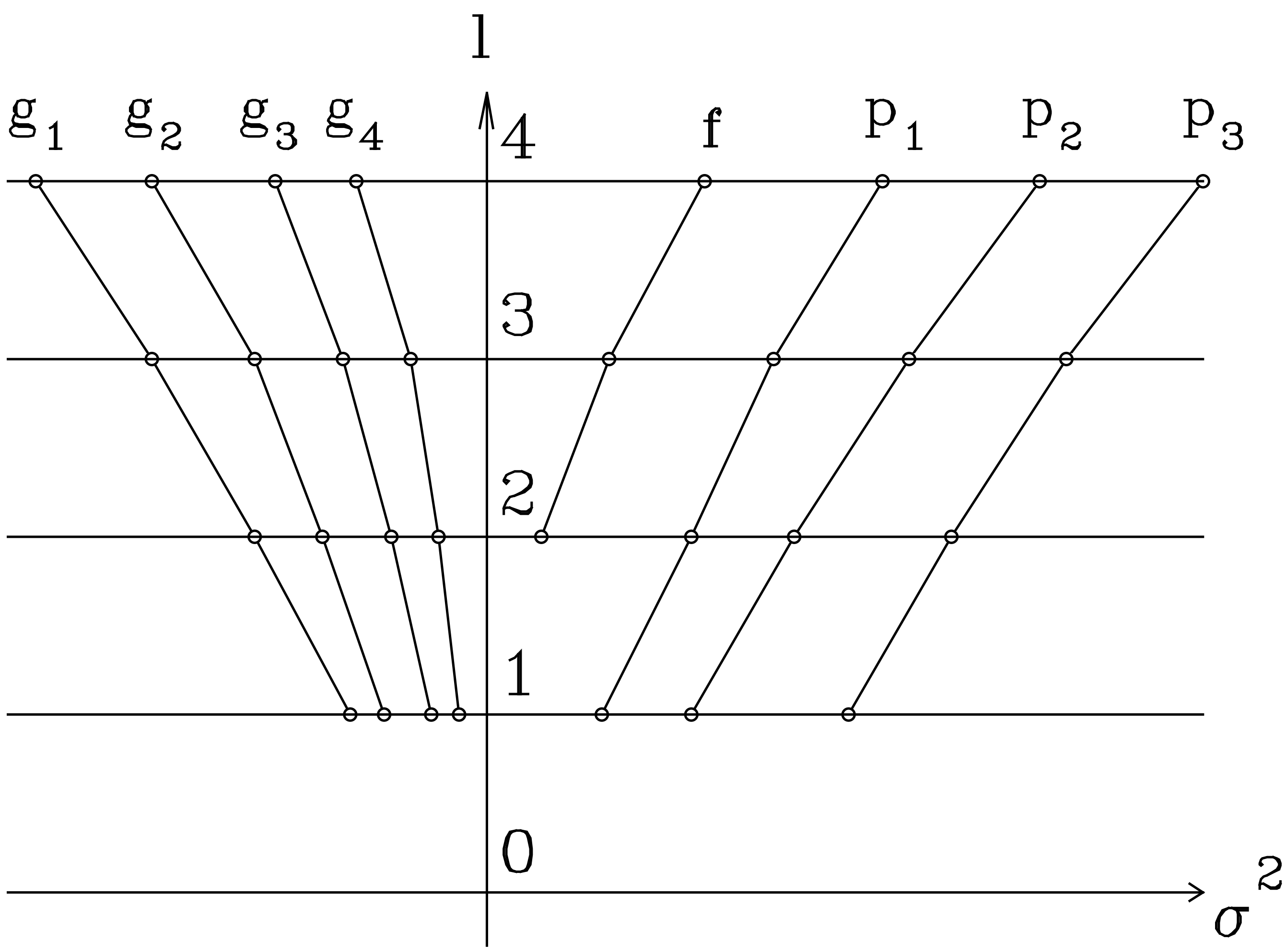
It is easy to show that

$$\begin{aligned} \int |\vec{\epsilon}|^2 d\Omega &= 1 \\ \int |\vec{\eta}|^2 d\Omega &= \ell(\ell+1) \end{aligned}$$

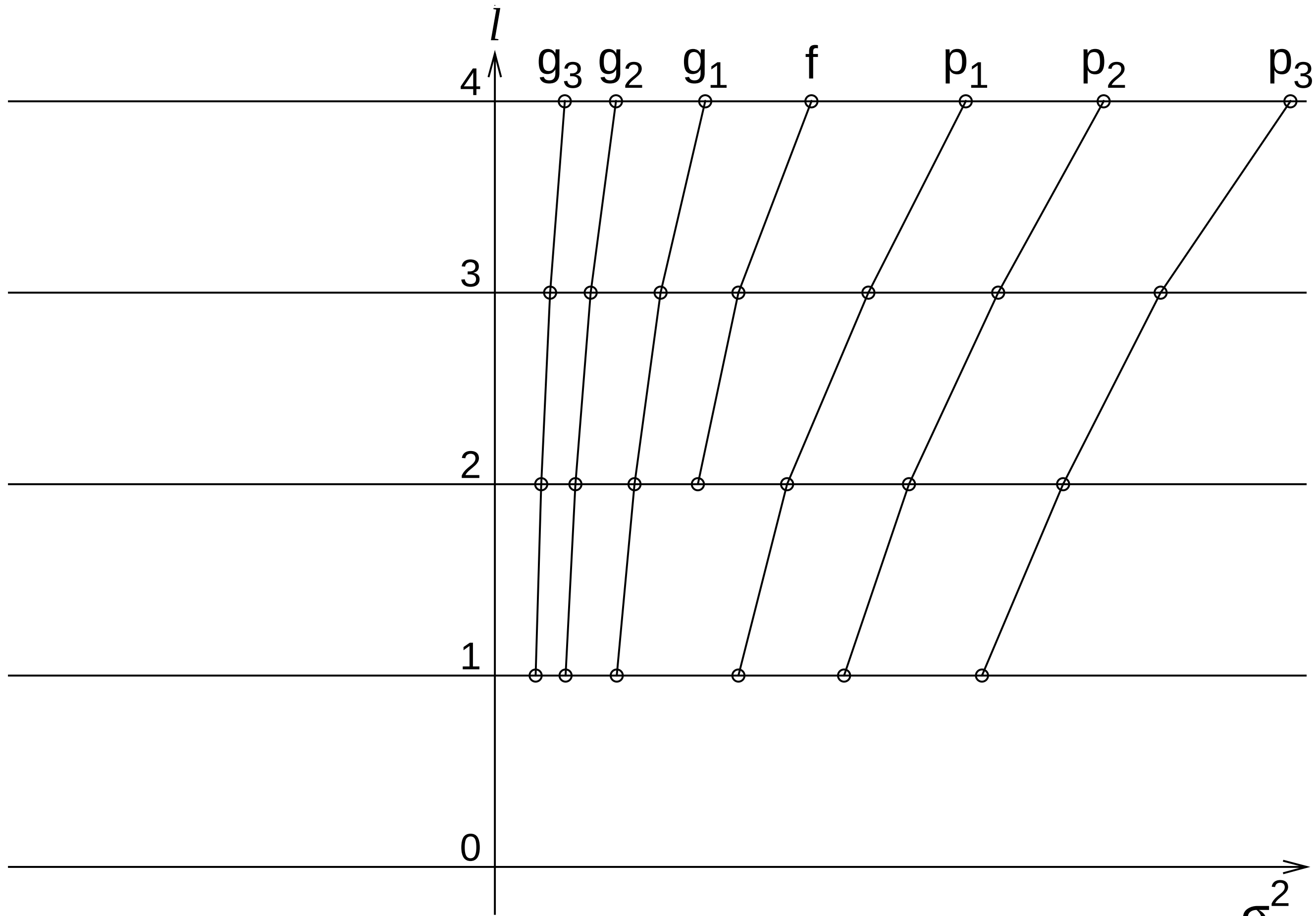
Then

$$\int |\overrightarrow{\delta r}|^2 dm = \int \rho r^2 \left[a^2 + \ell(\ell+1) b^2 \right] dr$$

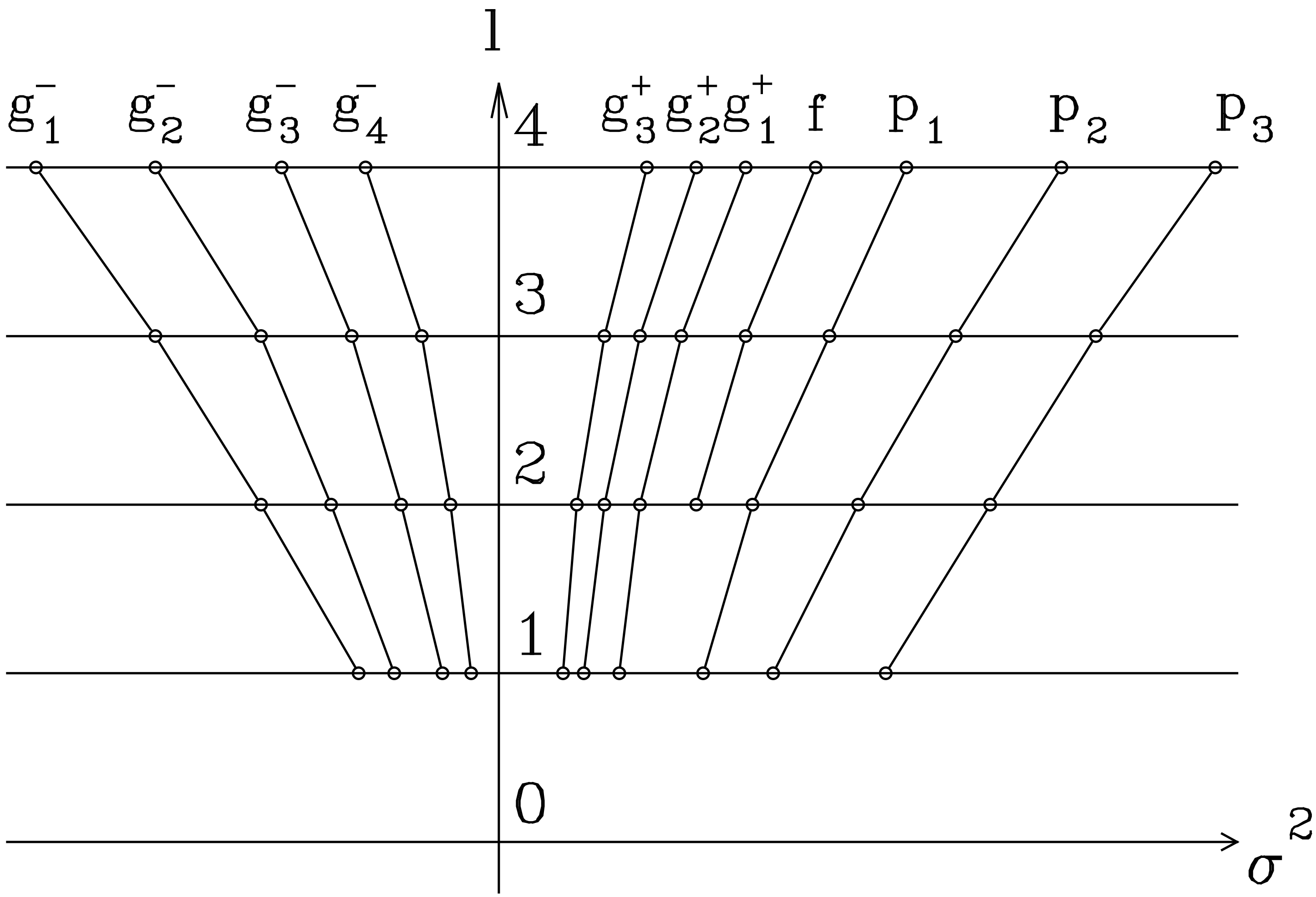
f, p and g modes



Fully convective model



Fully radiative model



Model with radiative and convective zones

Nature of the modes

p-modes: compressibility, disappear if $\Gamma_1 = \infty$

sound waves

g-modes: buoyancy, disappear if $A = 0$

g^+: internal gravity waves

evanescent in convective zones

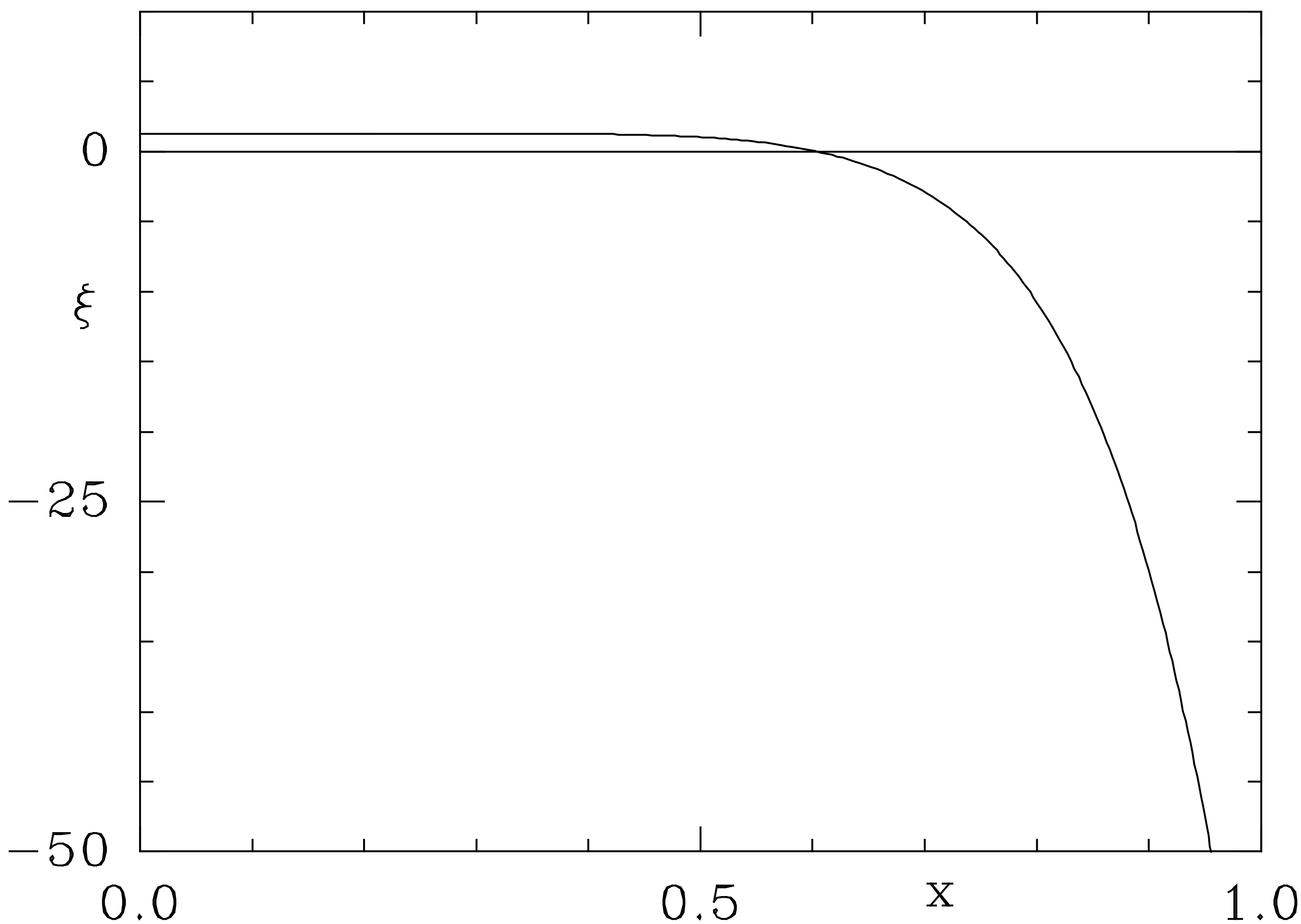
g^-: convection

evanescent in radiative zones

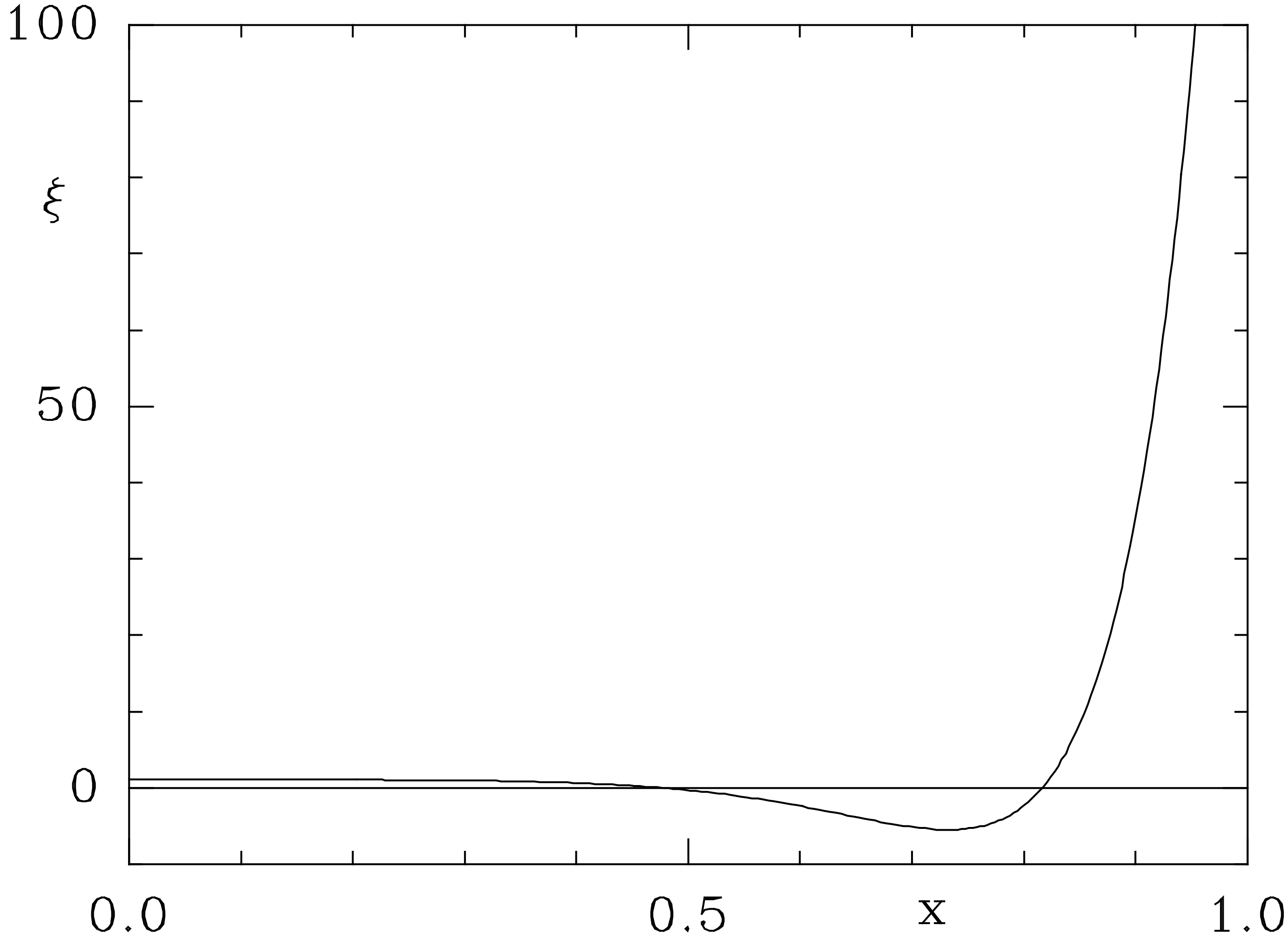
f-mode: ?

surface gravity wave

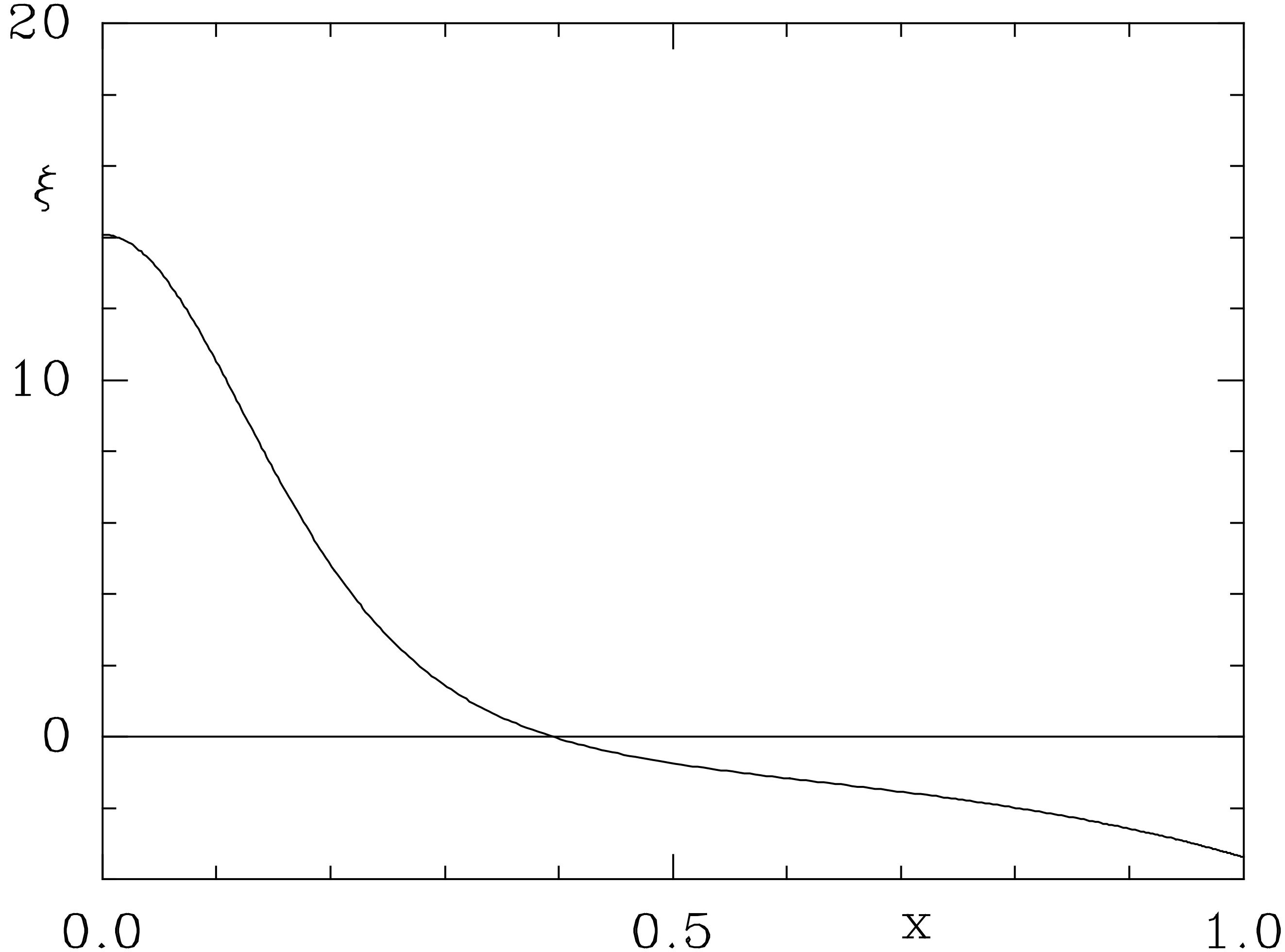
Eigenfunctions of the standard model ($\delta r/R = x^{\ell-1}\xi$)



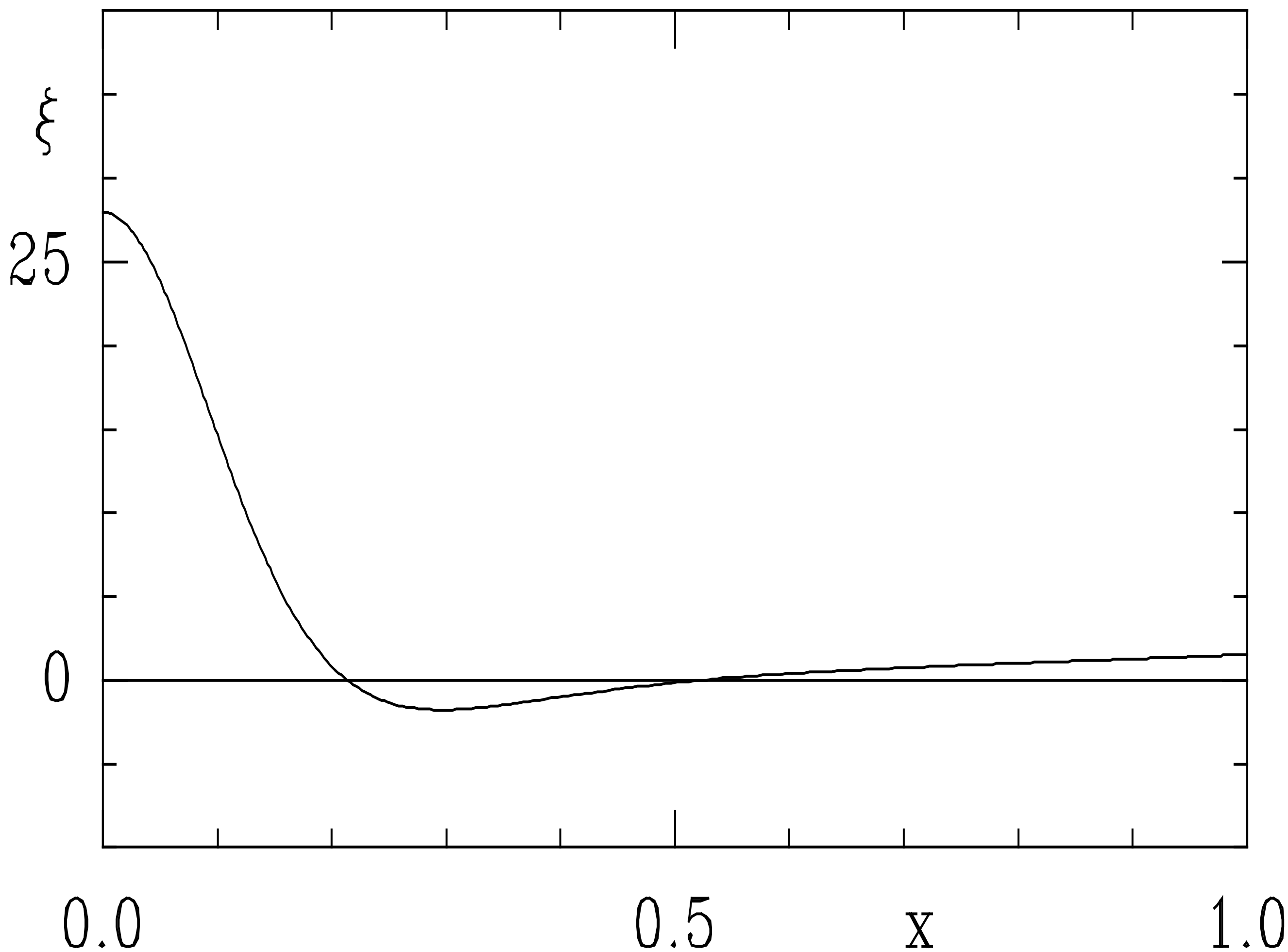
$\ell = 2$, p_1



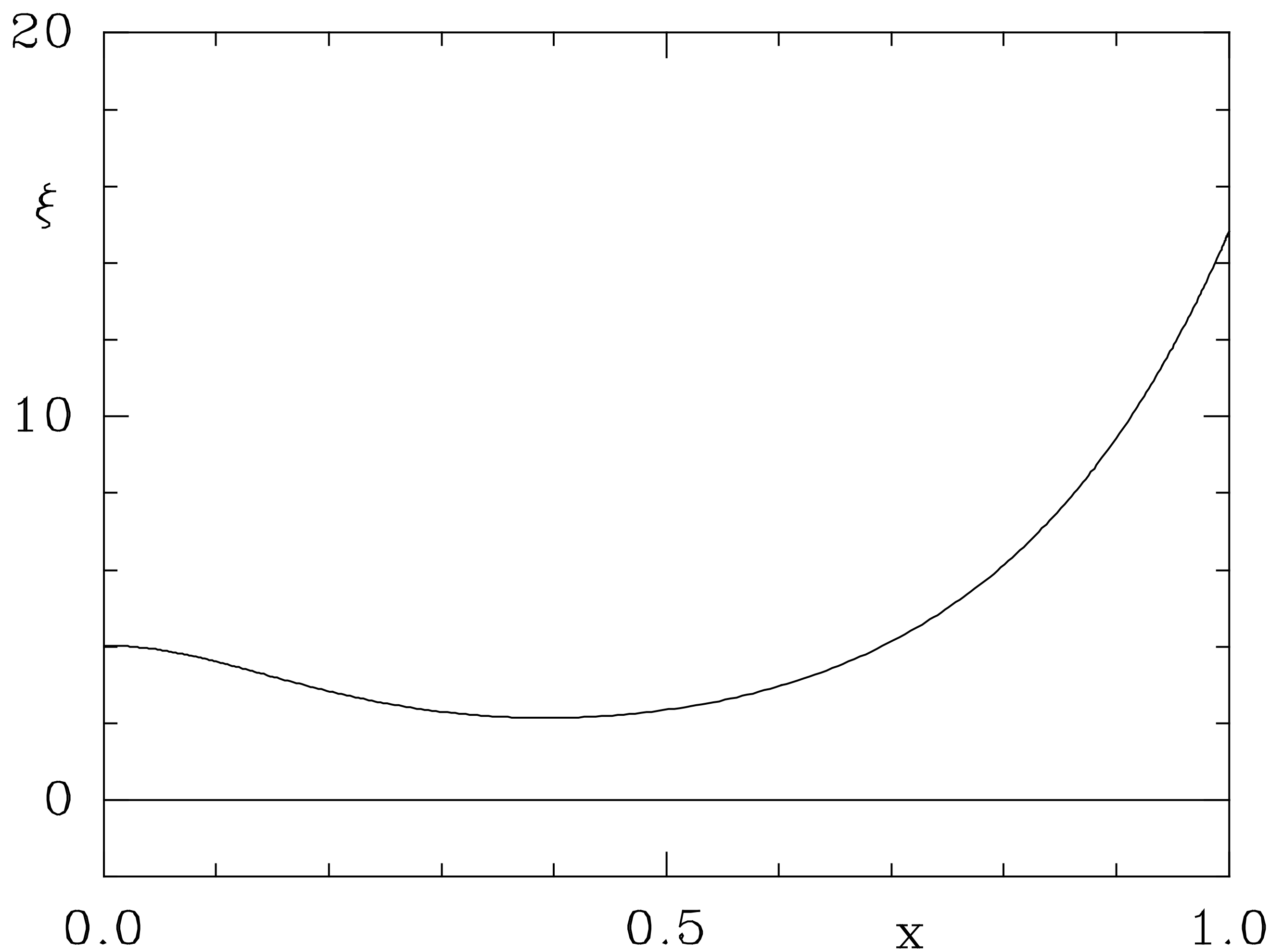
$\ell = 2,\, p_2$



$\ell = 2,\ g_1$



$\ell = 2,\ g_2$



$\ell = 2,\ f$

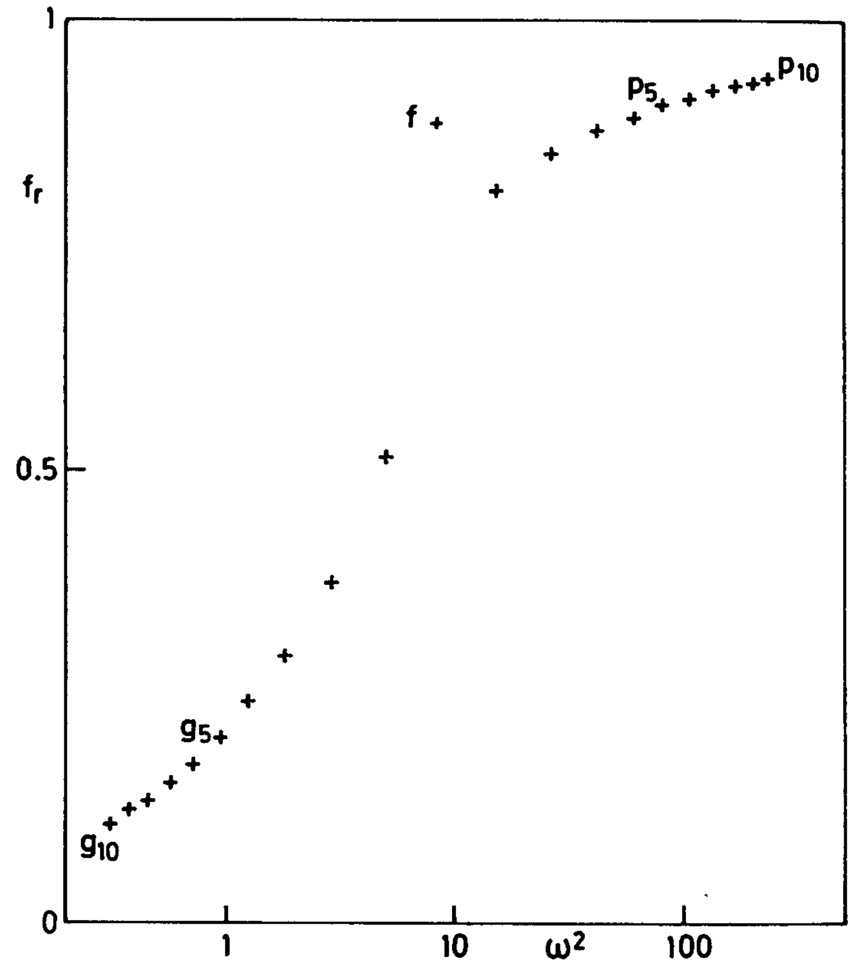
Energy of a non radial mode

$$E = E_K(t) + E_P(t) = E_K \sin^2 \sigma t + E_P \cos^2 \sigma t$$

$$\begin{aligned} E_K(t) &= E_{Kr}(t) + E_{Kh}(t) \\ E_P(t) &= E_A(t) + E_G(t) + E_B(t) \end{aligned}$$

$$\begin{aligned} E_{Kr}(t) &= \int \frac{1}{2}\rho v_r^2 dV \\ E_{Kh}(t) &= \int \frac{1}{2}\rho v_h^2 dV \\ E_A(t) &= \int \frac{P'^2}{2\rho c^2} dV \\ E_G(t) &= \int \frac{1}{2}\rho' \Phi' dV \\ E_B(t) &= \int \frac{1}{2}\rho n^2 \delta r^2 dV \end{aligned}$$

Polytrope $n = 3$
$\Gamma_1 = 5/3$
$\rho_c/\bar{\rho} = 54.18$



Fraction of the kinetic energy in the radial component

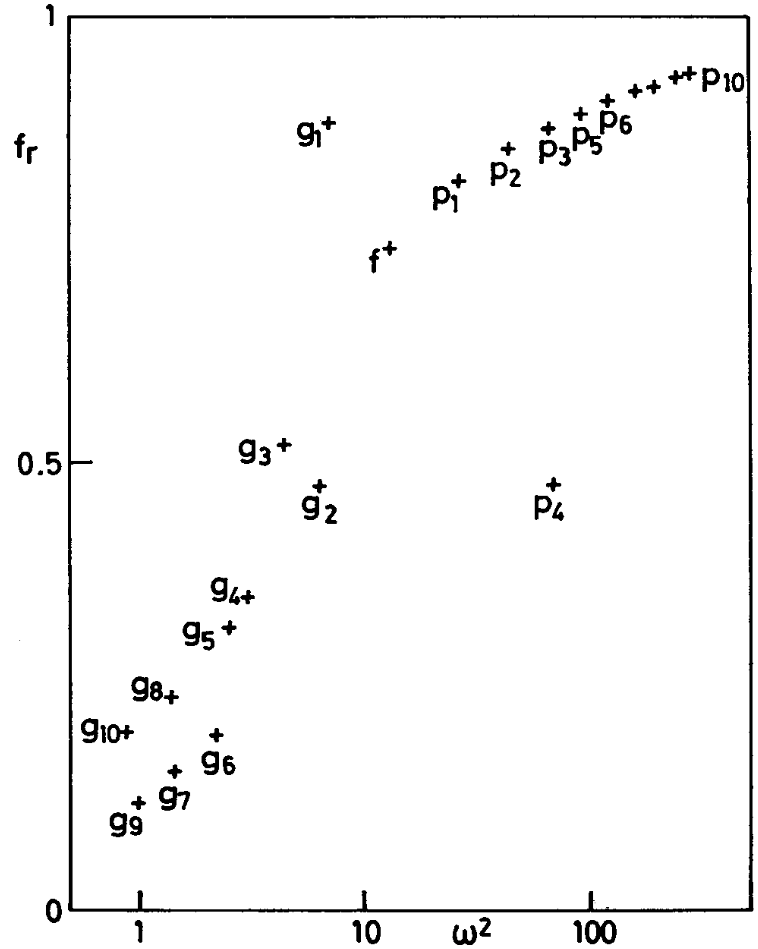
Physical model, 1 $M_\odot$

$\rho_c/\bar{\rho} = 168.3$

density discontinuity at

$x = 0.0615$, $q = 0.03$

$\frac{\rho_1-\rho_2}{\rho_1+\rho_2} = 0.32$



Fraction of the kinetic energy in the radial component

Physical model, 1 $M_\odot$

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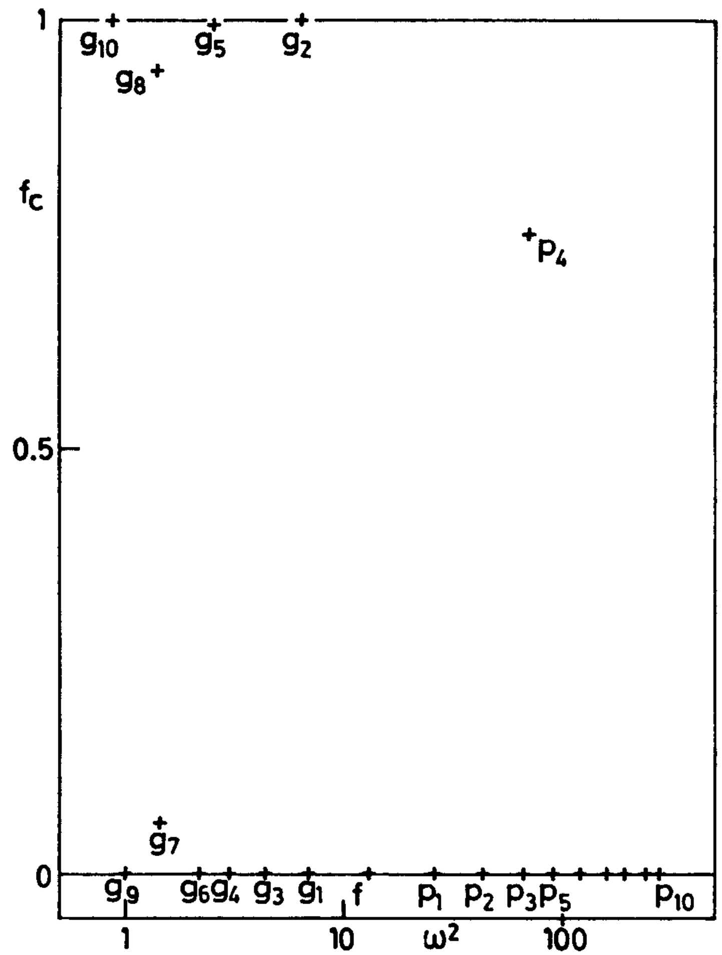
$\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} = 0.32$



Fraction of the kinetic energy in the core

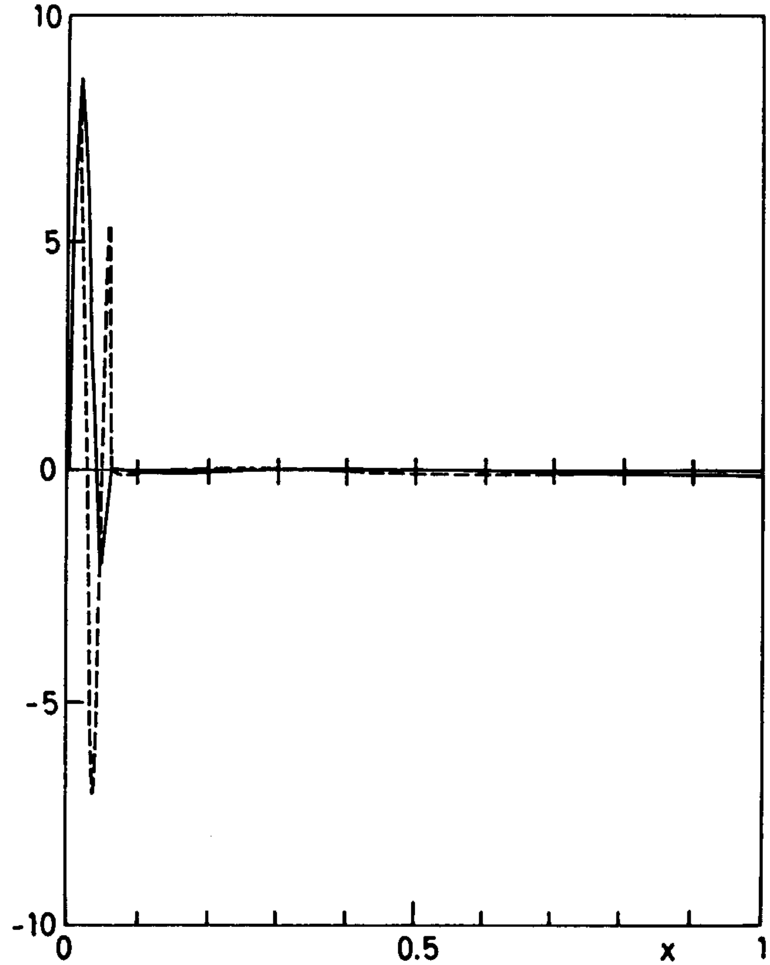
Physical model, 1 $M_\odot$

$\rho_c/\bar{\rho} = 168.3$

density discontinuity at

$x = 0.0615$, $q = 0.03$

$\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} = 0.32$



Radial and horizontal components of the displacement for $l = 2$ g_5 mode

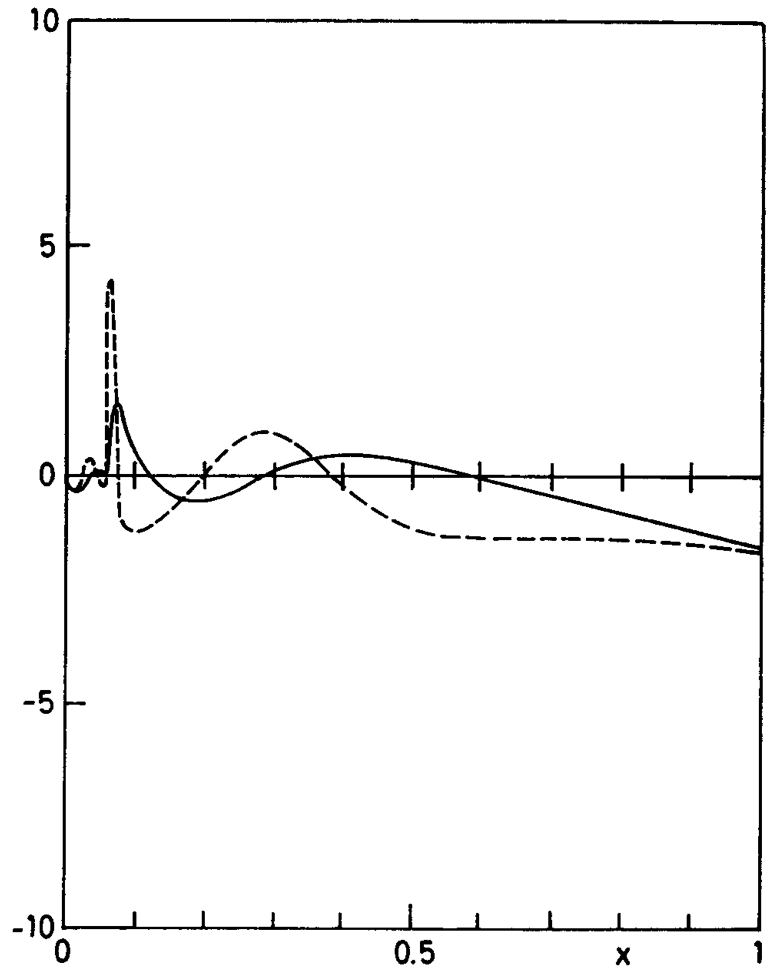
Physical model, 1 $M_\odot$

$\rho_c/\bar{\rho} = 168.3$

density discontinuity at

$x = 0.0615$, $q = 0.03$

$\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} = 0.32$



Radial and horizontal components of the displacement for $l = 2$ g_6 mode

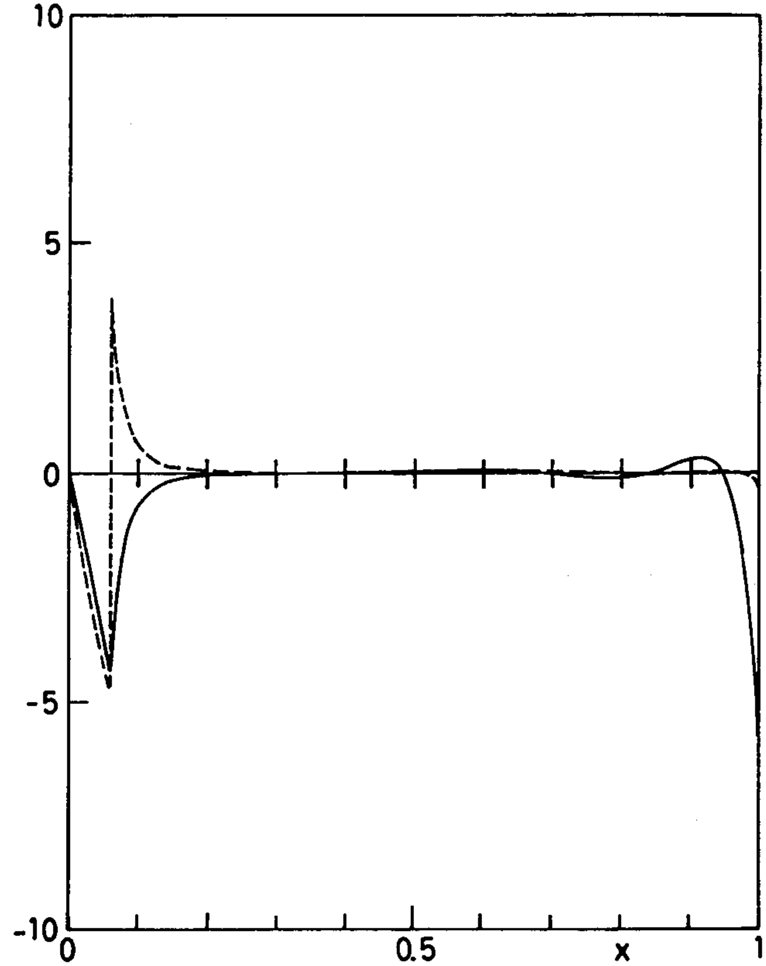
Physical model, 1 $M_\odot$

$\rho_c/\bar{\rho} = 168.3$

density discontinuity at

$x = 0.0615$, $q = 0.03$

$\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} = 0.32$



Radial and horizontal components of the displacement for $l = 2$ p_4 mode

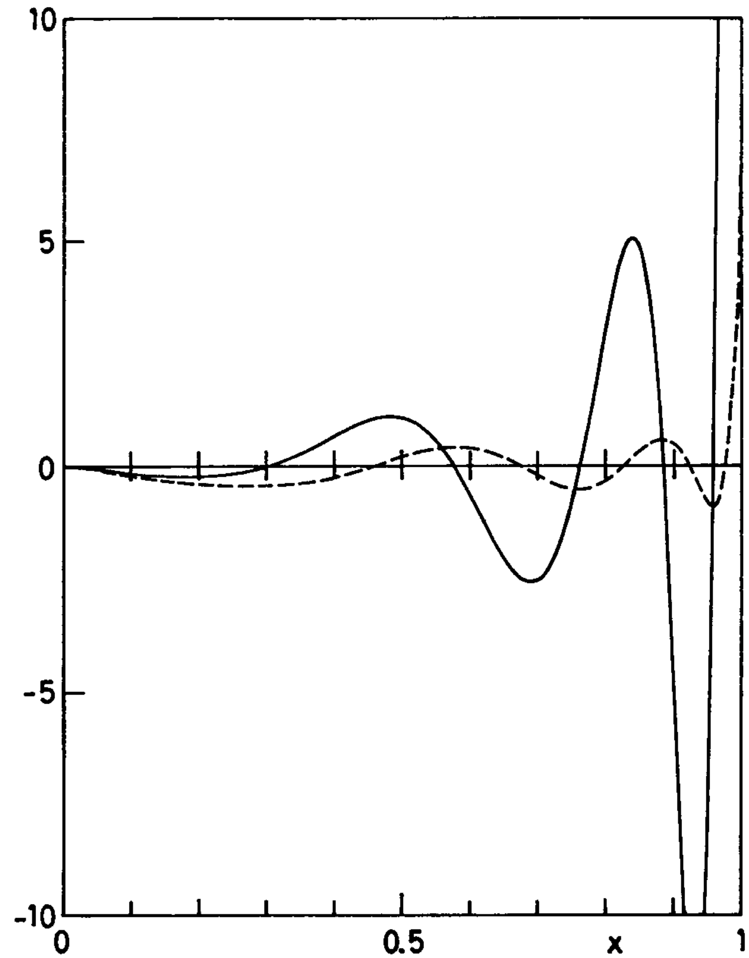
Physical model, 1 $M_\odot$

$\rho_c/\bar{\rho} = 168.3$

density discontinuity at

$x = 0.0615$, $q = 0.03$

$\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} = 0.32$



Radial and horizontal components of the displacement for $l = 2$ p_5 mode

Spheroidal and toroidal modes

The modes studied up to now **do not** form a basis in the space of all possible perturbations of a star.

A general vector field may be written in terms of three independant scalar fields

$$\overrightarrow{\delta r} = \underbrace{\alpha(\vec{r})\vec{e}_r + \text{grad}\,\beta(\vec{r})}_{\text{spheroidal}} + \underbrace{\text{rot}[\gamma(\vec{r})\vec{e}_r]}_{\text{toroidal}}$$

The momentum equation can be written as

$$s^2\overrightarrow{\delta r} = -\,\text{grad}\,\Phi' - \frac{1}{\rho}\text{grad}\,P' + \frac{\rho'}{\rho^2}\text{grad}\,P\,.$$

Using the adiabatic relation and the continuity equation we get

$$s^2\overrightarrow{\delta r} = -\,\text{grad}\,\chi + c^2\vec{A}\,\text{div}\,\overrightarrow{\delta r}\,,$$

So that $\vec{\delta r}$ is of the form

$$\overrightarrow{\delta r} = \alpha(\vec{r})\vec{e}_r + \mathrm{grad}\,\beta(\vec{r})$$

In our analysis, we have lost all the zero-frequency modes.

1) Three spheroidal modes with $\ell = 1$,
$m = -1, 0, 1$ describing solid translations of the star.

$$\overrightarrow{\delta r} = a\{Y_{\ell m}\vec{e}_r + \frac{\partial Y_{\ell m}}{\partial \theta}\vec{e}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_{\ell m}}{\partial \phi}\vec{e}_\phi\}$$

$\ell = 1$ f-modes ?

2) Toroidal modes: horizontal and divergenceless

$$\overrightarrow{\delta r} = a(r)\{\frac{1}{\sin\theta}\frac{\partial Y_{\ell m}}{\partial \phi}\vec{e}_\theta - \frac{\partial Y_{\ell m}}{\partial \theta}\vec{e}_\phi\}\,.$$

They acquire non-zero frequencies in presence of rotation and are of the same nature as Rossby waves.

Asymptotic expression for the frequencies

Difficulty: even with Cowling approximation, moving singularities.

p-modes

$$\sigma_{k\ell} \approx \frac{\left(k + \frac{\ell}{2} + \frac{n_e}{2} + \frac{1}{4}\right)\pi}{\int_0^R \frac{dr}{c}}$$

$$\Longrightarrow$$

$$\begin{aligned} \sigma_{k+1,\ell} - \sigma_{k,\ell} &\approx const \\ \sigma_{k,\ell} &\approx \sigma_{k-1,\ell+2} \\ \sigma_{k,\ell+1} &\approx (\sigma_{k,\ell} + \sigma_{k+1,\ell})/2 \end{aligned}$$

g-modes

$$\frac{\sqrt{\ell(\ell+1)}}{|\sigma_{k\ell}|} \approx \frac{\left(k + \frac{\ell}{2} + const\right)\pi}{\int \frac{|n|}{r}\, dr}$$

Effect of a slow rotation on the frequencies

small rotation around the z-axis: $\Omega(r,\theta)$
- Ω taken into account in the Coriolis force
- Ω^2 neglected in the centrifugal force

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot \text{grad}\right)^2 \overrightarrow{\delta r} = \mathcal{L}\,\overrightarrow{\delta r}$$

with $\vec{v} = \Omega r \sin\theta\,\vec{e}_\phi$.

Looking for solutions $\overrightarrow{\delta r} = \vec{\xi} e^{-i\sigma t}$ and neglecting terms in Ω^2,

$$\sigma^2\,\xi + 2\sigma\mathcal{M}\,\xi + \mathcal{L}\,\xi = 0$$

with $\mathcal{M}\xi = i(\vec{v}\cdot\text{grad})\xi$.

$\mathcal{M}$ is hermitian and linear in Ω. The problem can be solved by a perturbation method presented in elementary textbooks of quantum mechanics (degenerate case !)

We write $\sigma = \sigma_0 + \sigma_1$ and $\xi = \xi_0 + \xi_1$ and we obtain easily

$$\sigma_1 = -\frac{(\mathcal{M}\xi_0, \xi_0)}{(\xi_0, \xi_0)}$$

The explicit expression of σ_1 is rather tedious to calculate. It is given by an integral expression involving Ω and the eigenfunctions of the problem without rotation.

rotational splitting $\Leftrightarrow$ degeneracy entirely lifted

If $\Omega = \Omega(r)$, the expression simplifies to

$$\sigma_1 = m \int K_{k\ell}(r)\Omega(r)\, dr$$

with

$$K_{k\ell}(r) = \frac{\rho r^2[a^2 + \ell(\ell+1)b^2 - 2ab - b^2]}{\int \rho r^2[a^2 + \ell(\ell+1)b^2]\, dr}$$

For a uniform rotation, we have the usual expression

$$\sigma_{k\ell m} = \sigma^0_{kl} + m\beta_{kl}\Omega$$

with

$$\beta = \int K_{k\ell}(r)dr$$

Toroidal modes

In presence of rotation, *toroidal* modes acquire non zero frequencies. Their dynamics is governed by the Coriolis force as Rossby waves or planetary waves. They have low frequencies. For uniform rotation

$$\sigma = m\Omega - \frac{2m\Omega}{\ell(\ell+1)}$$

Non-linear radial oscillations

Why non-linear oscillations ?

- In δ Cep and RR Lyr variables, $\delta r/r \approx$ 5–10%
  and $\delta P/P = (4 + \omega^2)\delta r/r$
- shock wave in atmosphere of W Vir variables
- non sinusoidal light curves
- finite amplitudes

Lagrangian formalism

$$\rho r^2 \frac{\partial r}{\partial r_0} = \rho_0 r_0^2$$

$$\frac{\partial \Phi}{\partial r} = \frac{Gm}{r^2}$$

$$\frac{d^2r}{dt^2} = -\frac{Gm}{r^2} - \frac{r^2}{\rho_0 r_0^2}\frac{\partial P}{\partial r_0}$$

$$T\frac{dS}{dt} = \epsilon - \frac{1}{4\pi\rho_0 r_0^2}\frac{\partial L}{\partial r_0}$$

$$L = -\frac{16\pi r^4 acT^3}{3\kappa\rho_0 r_0^2}\frac{\partial T}{\partial r_0}$$

Adiabatic approximation

$$\frac{dP}{dt} = c^2\frac{d\rho}{dt}$$

If Γ_1 is constant,

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\Gamma_1} = \left(\frac{r_0^2}{r^2\partial r/\partial r_0}\right)^{\Gamma_1}$$

$$\frac{d^2r}{dt^2} = -\frac{Gm}{r^2} - \frac{r^2}{\rho_0 r_0^2}\frac{\partial}{\partial r_0}\left[P_0\left(\frac{r_0^2}{r^2\partial r/\partial r_0}\right)^{\Gamma_1}\right]$$

• Separation of variables

1) if $\Gamma_1 = 4/3$
2) homogeneous model (ρ_0 independent of r_0)

• Series developments

$$r = r_0(1+\xi) \qquad \text{et} \qquad \xi = \sum_{i=0}^{\infty} f_i(r_0)q_i(t),$$

1) $q_i(t) =$ harmonic functions (Fourier development, Eddington)
2) $f_i(r_0) =$ eigenfunctions

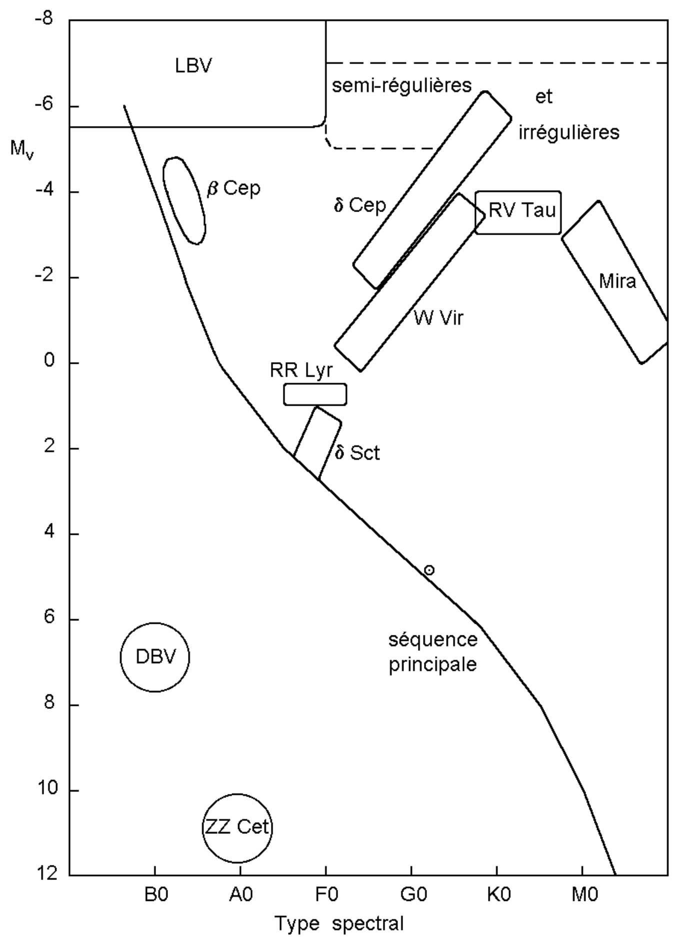
Numerical integrations

In the sixties, equations for radial pulsations with initial conditions were solved by hydrodynamic lagrangian codes with ~ 50 shells and ~ 200 time steps
- success for Cepeids and RR Lyr
- limit cycle
- little change in periods
- rather good light curves

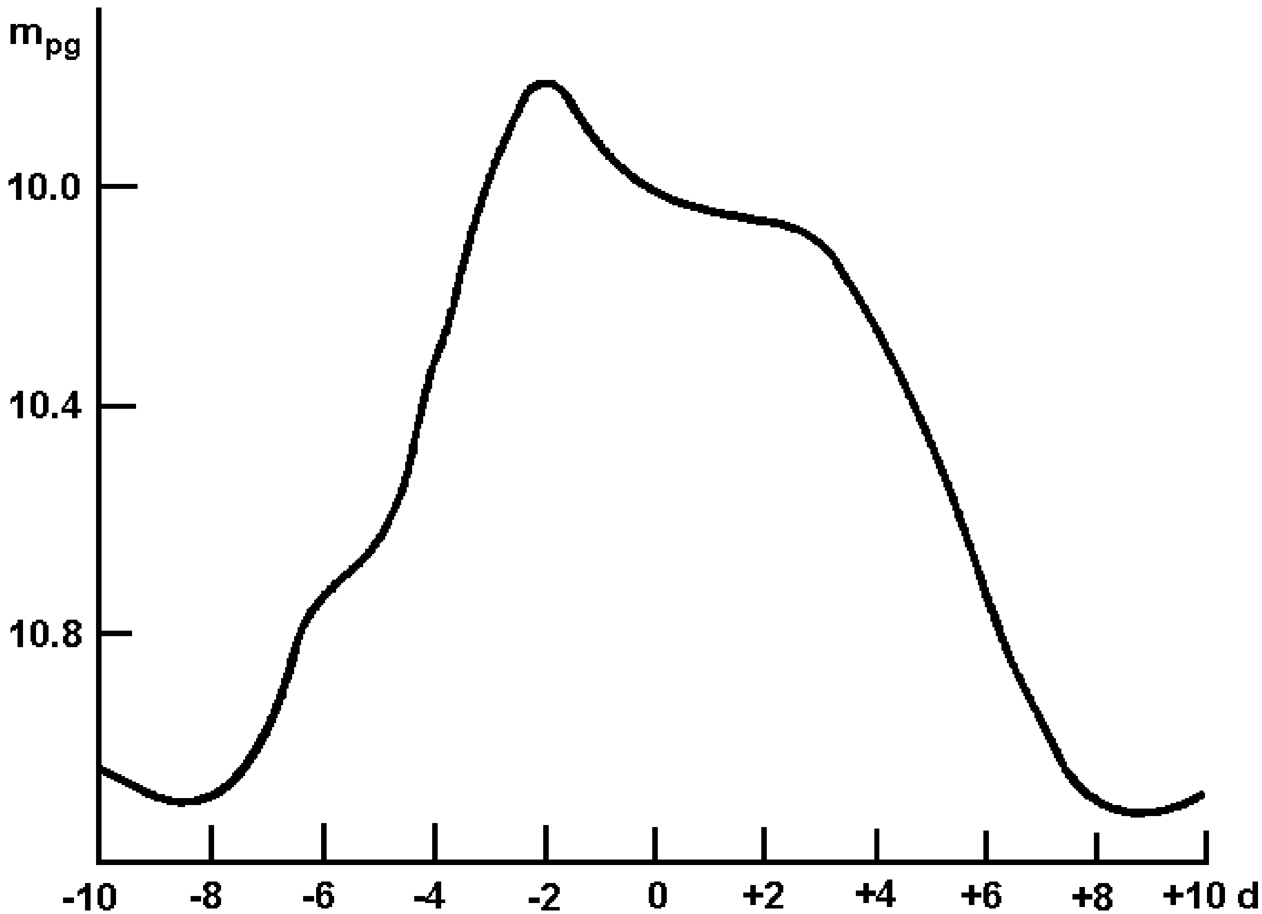
But
- numerical tricks to ensure the stability of the code or to follow shock waves
- depending on the type of variable, huge computation time may be necessary for the damping of stable modes and to reach full amplitude for the unstable ones

Alternative procedure
- direct search for a limit cycle, but delicate to work out

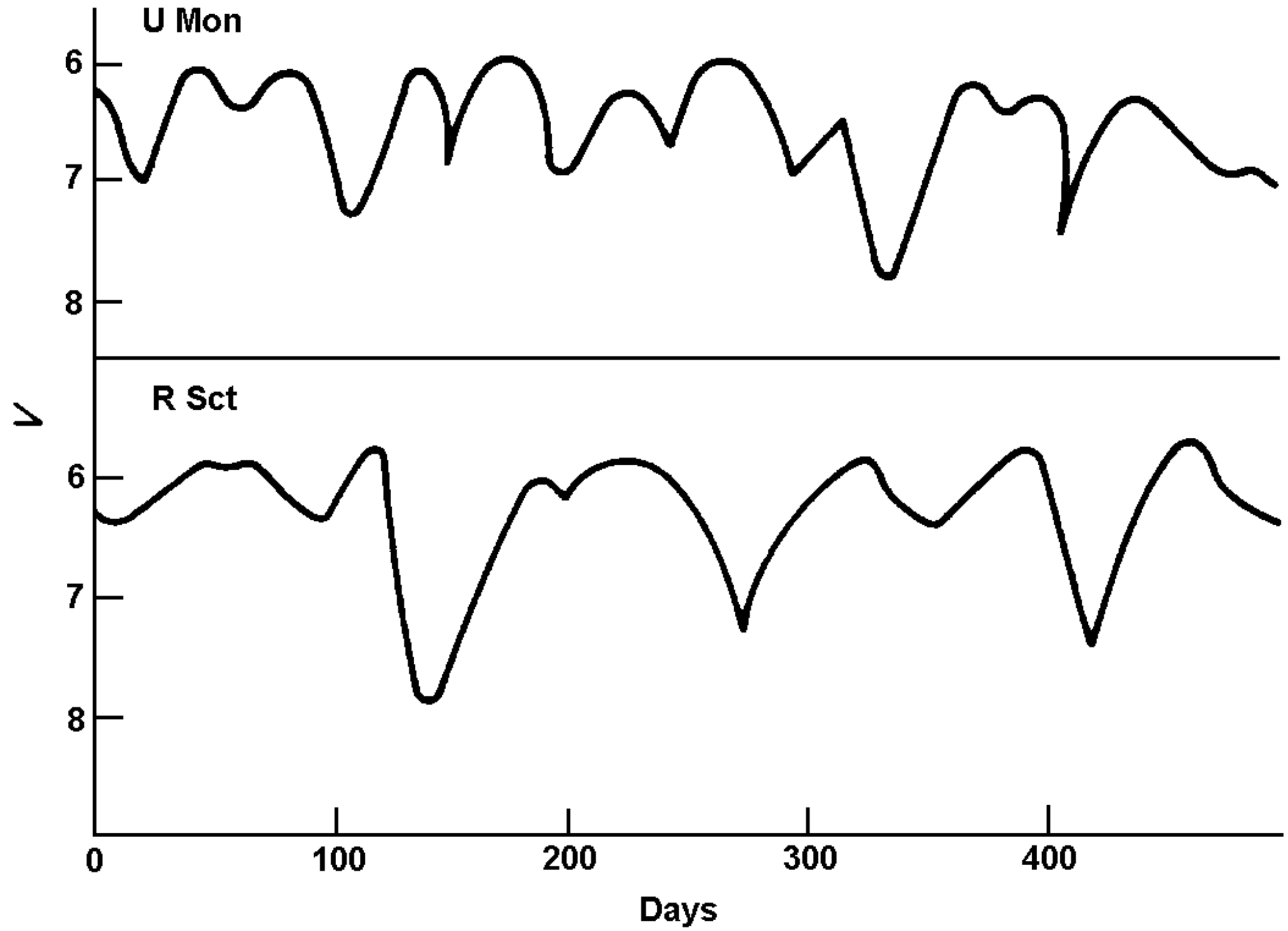
Regular and chaotic pulsations



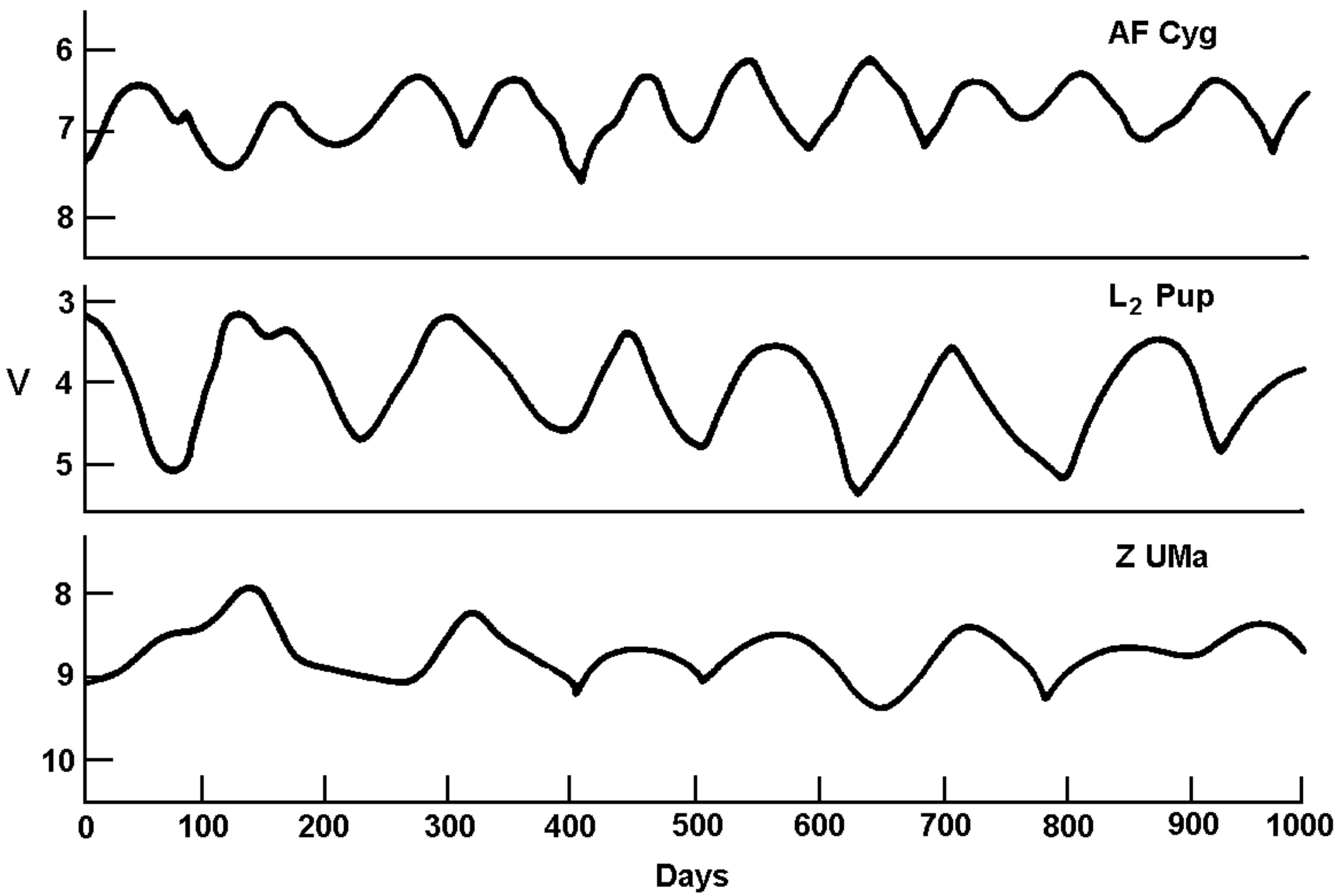
A few classes of variable stars in the HR diagram



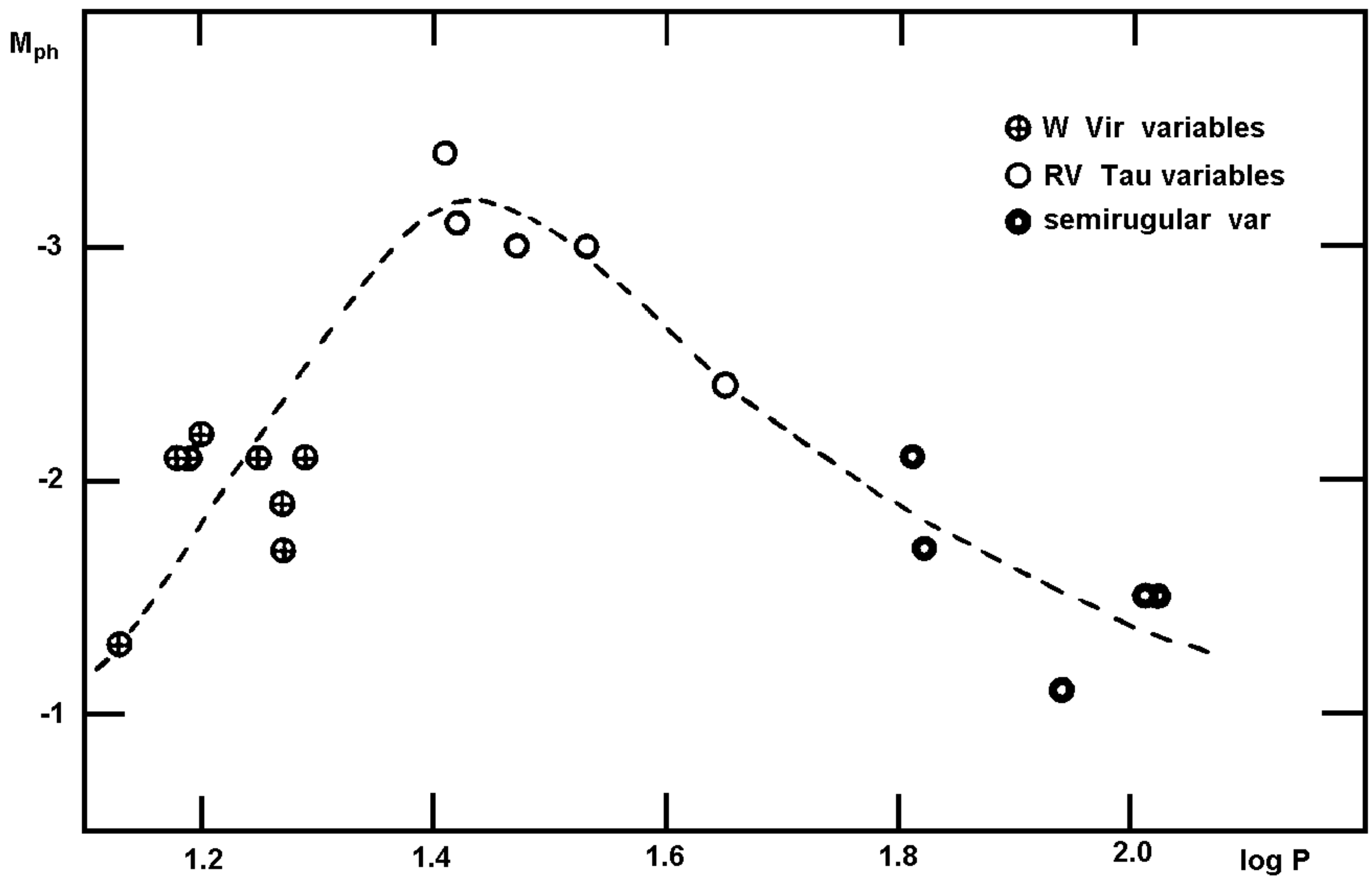
Light curve of W Vir



Light curves of two RV Tauri variables: U Mon (92.3 d) and R Sct (140.2 d)

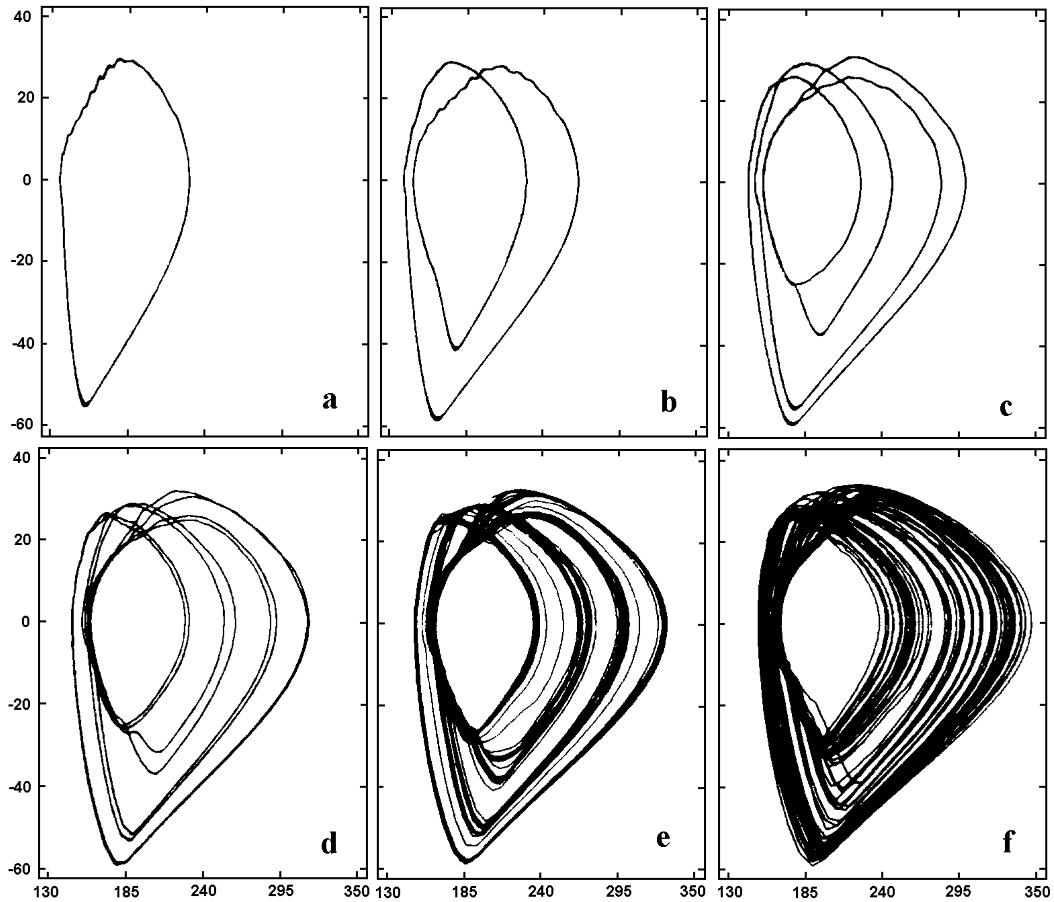


Light curves of three SRb variables: AF Cyg (94.1 d), L_2 Pup (140.8 d) et Z UMa (196 d)



Period-luminosity relation of W Vir, RV Tau and semi-regular variables of globular clusters (periods in days) For RV Tau variables, the half-periods have been used

Stability of pop II envelope models of decreasing T_e from a to f.



$r/10^{10}$cm (abscissa) versus $v/\mathrm{km\,s^{-1}}$ (ordinate) for a given shell