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Chapter 1

A brief history of celestial mechanics

One of the most important problems in ancient astronomy was to understand (and predict) the apparent motion of the so-called wandering stars. In fact, whilst the vast majority of the stars in the sky do not change their relative positions over a human lifetime, the Sun, the Moon and five other objects (the planets Mercury, Venus, Mars, Jupiter and Saturn) apparently move among the other stars. Until the 16th century, the most popular theory to explain these properties was the geocentric model summarized by Claudius Ptolemy (85 - 165) in the *Almagest*. This model assumed that all the planets (including the Sun and the Moon) move on epicycles that are small circular trajectories. The centers of these epicycles in turn move around the Earth on so-called deferents, i.e. circles that are centred on a point halfway between the Earth and a point called the equant. The center of the epicycle rotates on the deferent with a uniform motion not with respect to the center, but with respect to the equant.

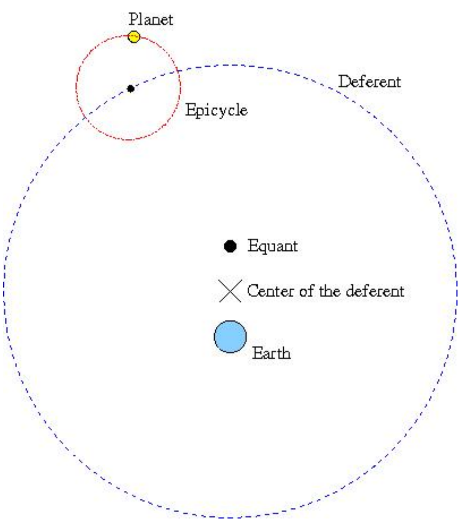


Figure 1.1: Illustration of the basic ingredients of the Ptolemaic model. The deferent is centred on a point midway between the Earth and the equant and the planet moves around a circle (the epicycle) centred on a point of the deferent. The epicycle rotates on the deferent with a uniform motion not with respect to the center, but with respect to the equant.

In 1543, on the very day he died, the Polish astronomer Nicolas Copernicus (1473 - 1543) published his book *De revolutionibus orbium coelestium*, where he actually proposed the idea that the model could be simplified if the planets, including the Earth itself, were to move around the Sun rather than around the Earth. Copernicus was actually not the first to propose a heliocentric architecture of the Universe[1], but it was his book that stimulated the work of his successors. It has to be stressed that Copernicus also had to assume a complex system (although less complicated than for the geocentric model) of epicycles and deferents.

[1]Several Greek, Indian and Arab scientists had envisaged heliocentric hypotheses many centuries before Copernicus.

Motivated by the fact that even Copernicus' tables failed to correctly predict the dates of some particular planetary configurations, the Danish astronomer Tycho Brahe (1546 - 1601) started to work on an alternative model. He concluded that the difficulties to distinguish between the geocentric and heliocentric model was mainly due to the lack of precise measurements and so he started, assisted by his sister Sophia, to systematically collect extremely accurate observations with positional errors[2] of 2 arcsec. In this way, Brahe compiled a huge set of high-quality data, and he was looking for a skilled mathematician able to explore these data to confirm his own, geoheliocentric model of the Universe, where the Moon and the Sun orbit the Earth and all other planets orbit the Sun.
This mathematician was the German Johannes Kepler (1571 - 1630). Kepler did his studies at the University of Tübingen where he became a student of Michael Mästlin (1550 - 1631) who was a supporter of Copernicus' ideas. In 1600, Johannes Kepler moved to Prague to become an assistant of Tycho Brahe. However, the relation between Kepler and Brahe turned out to be rather difficult and Kepler had to wait until Brahe's death[3] in 1601 before he could eventually use the data gathered by his Danish colleague[4]. Kepler started to study Brahe's measurements of the position of planet Mars. Thanks to the amazing accuracy of these measurements he was able to show that the planet was moving around the Sun on a slightly elliptical orbit. In 1609, he published his book *Astronomia Nova*, where he formulated the first two of his famous laws: (1) the planets move on elliptical orbits with the Sun at one focus of the ellipse and (2) an imaginary line joining the planet to the Sun sweeps out equal areas in equal times. In 1619, he published another book (*Harmonices mundi*) where he presented his third law, relating the square of the period of the orbital revolution to the cube of the semi-major axis of the orbit. It has to be stressed that it took several decades before Kepler's results received the attention they deserved. This was mainly because the demonstrations were not easily understood. As it was customary at the epoch, his results were reported in latin without the use of equations and were based on geometrical arguments since the differential and integral calculation had not yet been invented.

Figure 1.2: From left to right: Nicolas Copernicus, Johannes Kepler, Isaac Newton and Pierre-Simon Laplace.

Roughly at the same time, the Italian scientist Galileo Galilei (1564 - 1642) performed several experiments that led him to the discovery of the theory of a free falling body (the distance is proportional to the square of the time, implying that the acceleration is constant) as well as to the principle of inertia. The latter was reformulated (and actually corrected) by the French mathematician René Descartes (1596 - 1650).

[2]Previouly, typical astronomical observations had positional errors of 10 arcsec or larger.

[3]The official version is that Brahe died as a result of a strained bladder. Modern forensic investigations revealed highly toxic levels of mercury in his hair. Some authors therefore speculate that Kepler could have poisoned Brahe to steal his data, although this concentration of mercury could also result from Brahe's numerous alchemy experiments.

[4]Actually Kepler illegally appropriated Brahe's data and had to subsequently negotiate the permission to use the data with Brahe's heirs. This led to a four years delay in the publication of Kepler's results.

The English genius Isaac Newton (1643 - 1727) combined all these results into a single, coherent, theory. Newton was a very eccentric personality and had an obsessive interest in alchemy[5]. Newton made most of his discoveries and calculations while he was staying at his home in Lincolnshire after he had to leave Trinity College in Cambridge because of the plague. Edmond Halley (1665 - 1742) encouraged Newton to publish his results and this happened in 1687 with the publication of the *Philosophiae Naturalis Principia Mathematica*. In this masterpiece, Newton formulated the three fundamental laws of Newtonian mechanics: (1) an object that is not submitted to the action of any external force remains either at rest or in a straight line uniform motion; (2) the time changes of an object's momentum are proportional to the force that is applied and have the same sense and direction; (3) for every action there is a reaction of same intensity and direction, but opposite sense.
These fundamental laws apply to all kinds of forces, but Newton in addition specified the behaviour of one of these forces, the **gravity** which acts between any pair of massive bodies and decreases with the inverse of the square of the distance between these bodies[6]. Newton developed many mathematical tools such as differential and integral calculations. However, his demonstration of Kepler's laws being a consequence of his theory was formulated in classical (geometrical) terms.

The success of Newton's theory was overwhelming. It was now possible to understand the motion of the planets around the Sun. The most impressive early confirmation of his theory was the prediction by Edmond Halley that the great comet of 1682 (nowadays known as Halley's comet) would return in 1758. However, whilst the comet actually returned, it did not so exactly at the date predicted by Halley. The delay was due to the gravitational influence of the planets of the outer Solar System, first of all Jupiter and Saturn, the most massive ones. To account for this effect, Newton's theory had to be applied to more than two massive bodies. This so-called three body problem was first formulated as a set of differential equations in the 1740's thanks to the work of the Swiss mathematician Leonhard Euler (1707 - 1783). Euler was one of the most productive mathematicians ever. He invented many precious tools for calculus, including the concept of trigonometric series, and was the first to introduce analytical rather than geometrical methods to handle the problems of mechanics. For the first time, it was possible to account for the mutual gravitational influence of the various planets.

Based on Euler's work, the Italian mathematician Joseph-Louis Lagrange (1736 - 1813) proposed an efficient way to compute the coefficients in these trigonometric series. Lagrange further developed a new approach to solve the equations of mechanics, summarized in his book *Mécanique Analytique* and introduced the theory of perturbations of the orbital parameters to treat effects that are not included in the classical formulation of the two-body problem. In 1764 Lagrange studied the libration of the Moon and in 1772 he discovered the Lagrangian points, a set of analytical solutions to the restricted three-body problem.
The theory of perturbations led to the introduction of so-called secular variations that are monotonic changes altering the orbital elements unidirectionally. Such variations were (observationally) found for the Earth-Moon system as well as for Jupiter and Saturn. Therefore, the question came up whether the Solar System would remain stable on long time-scales. This question was addressed by Lagrange and his French colleague Pierre Simon Laplace (1749 - 1827) who showed that the mean motion of the planets is apparently immune to these perturbations (see however below). In 1785, Laplace described the effects of the resonance between Jupiter's and Saturn's orbital period that leads to a periodic perturbation with a period of about 900 years. What had been interpreted as a genuine secular variation was in fact a long-term periodic one.

[5] The British economist John Maynard Keynes (1883 - 1946) even considered him 'the last of the magicians, rather than the first of the age of reason'.

[6] Well before the publication of Newton's results, Robert Hooke (1635 - 1703) came near to an experimental proof that gravity follows an inverse square of the distance law. Hooke hypothesised that such a relation could govern the motion of the planets and mentioned these results in his correspondence with Newton. Hooke later claimed priority for proposing the idea that gravity and hence the planetary motion are caused by an inverse square central force. This led to a violent dispute between Hooke and Newton.

In 1783, Laplace proposed the use of spherical harmonics to develop the gravitational potential of the Earth, thereby setting the stage for the modern theory of satellite orbits. Laplace summarized all his findings in his four-volume opus **Mécanique Céleste** that appeared between 1798 and 1805[7]. Meanwhile, William Herschel (1738 - 1822) had discovered Uranus, the seventh planet of the Solar System, in 1781. Around 1820, it was realised that the motion of this planet could not be explained theoretically with the influence of the known planets of the Solar System. In fact, by 1830 Uranus had departed by 15 arcsec from the calculated orbit. Therefore, it was speculated that Uranus was perturbed by another planet and Urbain Le Verrier (1811 - 1877) started to compute what had to be the properties of the perturbing planet that could explain Uranus' motion. Le Verrier communicated his predictions of the position of the new planet to colleagues at the Berlin Observatory and in September 1846 Johann Gottfried Galle (1812 - 1910) and Heinrich Louis d'Arrest (1822 - 1875) discovered Neptune near the position predicted by Le Verrier. The context of this discovery triggered a lively debate between the French and Anglo-Saxon world. Indeed, the British astronomer John Couch Adams (1819 - 1892) had also made predictions about the parameters of the unknown planet that perturbed the motion of Uranus. However, the accuracy of Adam's calculations would probably not have been sufficient to actually discover Neptune. Le Verrier was appointed director of the Paris Observatory in 1852, where he became extremely unpopular among his subordinates because of his authoritarian personality. He continued his work on the orbits of planets and discovered the variations of the perihelion of Mercury. He suggested that this effect was due to another unknown planet that he named Vulcan. This triggered an intensive search (with false detections) that ended only in 1915, when Albert Einstein (1879 - 1955) explained this anomaly with the theory of General Relativity.

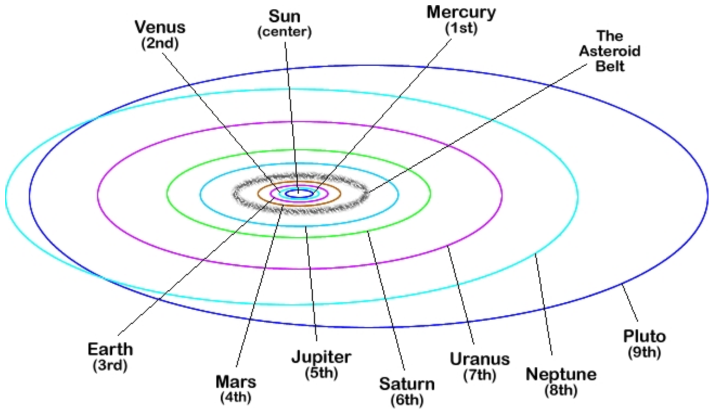


Figure 1.3: Schematic illustration of the (almost) current view of our Solar System. (Find the intruder!)

In the nineteenth century, many improvements to the handling of the equations of celestial mechanics were proposed. These were mainly the achievements of scientists like William Rowan Hamilton (1805 - 1865), Charles Delaunay (1816 - 1872) or George William Hill (1838 - 1914). Henri Poincaré (1854 - 1912) showed that there exists no uniformly convergent solution for N-body problems, thus rendering the global stability proofs of the Solar System by Laplace and Lagrange obsolete[8].

Whilst General Relativity has nowadays replaced Newtonian mechanics as the main theory of gravity, it has to be stressed that as far as the motion of the planets in the Solar System and the question of spaceflight are concerned, relativistic effects are usually very small and celestial mechanics remains the important theory. With the advent of the current generation of computers, it has become possible to resolve the numerical equations much faster than what was possible in the past. Still, we should not forget the enormous achievements that past generations of scientists have accomplished. Our current knowledge is their heritage.

[7]Laplace was actually the first to use the terminology *Celestial Mechanics*.

[8]We note here that recent calculations suggest that Jupiter's action on Mercury could lead to an increase in the eccentricity of the orbit of the latter that might eventually lead to a collision with another planet and could potentially trigger a chaotic situation.

Chapter 2

The two-body problem

The so-called two-body problem deals with the motion of two point-like masses m_1 and m_2 that form an isolated system and interact through the sole forces of gravity according to Newton's law. In this chapter, we recall the main equations of this problem.

2.1 The equations of the motion

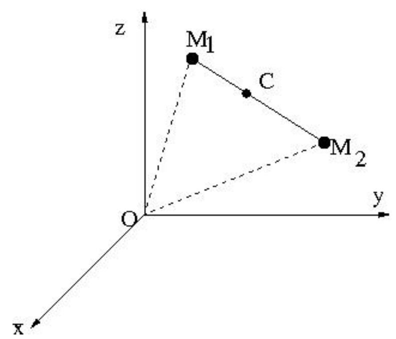
Consider a cartesian frame of reference $Oxyz$. According to Newton's law of gravity, the differential equations of the motion of two point-like masses can be written

$$m_1 \, \ddot{\vec{OM_1}} = \frac{G\, m_1\, m_2}{d^3} \, \vec{M_1 M_2}$$

and

$$m_2 \, \ddot{\vec{OM_2}} = -\frac{G\, m_1\, m_2}{d^3} \, \vec{M_1 M_2}$$

where $d = |M_1 M_2|$ and G is the gravitational constant ($G = 6.674 \times 10^{-8}\,\mathrm{cm}^2\,\mathrm{g}^{-1}\,\mathrm{s}^{-2}$). A priori, the resolution of these two equations requires twelve constants of integration (initial positions and velocities of two points). The resolution of the problem can however be simplified by noting that the center of mass C follows a straight line. Indeed, $m_1 \, \ddot{\vec{OM_1}} + m_2 \, \ddot{\vec{OM_2}} = \vec{0}$, hence



$$\frac{m_1}{m_1 + m_2} \, \vec{OM_1} + \frac{m_2}{m_1 + m_2} \, \vec{OM_2} = \vec{OC} = \vec{a}\, t + \vec{b}$$

where $\vec{a}$ and $\vec{b}$ are respectively the initial (at time $t = 0$) velocity and the initial position of C. The center of mass thus follows a straight line motion at constant velocity. The motion of m_1 and m_2 can therefore be described in an inertial frame of reference centred on C.

$$m_1\,C\vec{M}_1 + m_2\,C\vec{M}_2 = \vec{0}$$

Defining $\vec{r_1} = C\vec{M}_1$ and $\vec{r_2} = C\vec{M}_2$, we obtain

$$\ddot{\vec{r_1}} = -\frac{G\,m_2^3}{(m_1+m_2)^2}\,\frac{\vec{r_1}}{r_1^3}$$

and

$$\ddot{\vec{r_2}} = -\frac{G\,m_1^3}{(m_1+m_2)^2}\,\frac{\vec{r_2}}{r_2^3}$$

Only one of these differential equations needs to be solved, the coordinates of the other point can be obtained directly from the fact that

$$m_1\,\vec{r_1} + m_2\,\vec{r_2} = \vec{0}$$

2.1.1 The relative motion

The motion of the mass m_2 is frequently expressed relative to m_1 in an inertial frame of reference parallel to $Oxyz$:

$$M_1\ddot{\vec{M}}_2 = -\frac{G\,(m_1+m_2)}{d^3}\,M_1\vec{M}_2$$

Whether the motion of m_2 is expressed relative to the center of mass or to the other mass m_1, the equations can be expressed as

$$\ddot{\vec{r}} = -\mu\,\frac{\vec{r}}{r^3} = -\vec{\nabla}\left(-\frac{\mu}{r}\right) \tag{2.1}$$

where $\vec{r} = \vec{r_2}$, $r = |\vec{r_2}|$ and $\mu = \frac{G\,m_1^3}{(m_1+m_2)^2}$ if we express the motion relative to the center of mass, whilst $\vec{r} = M_1\vec{M}_2$, $r = |M_1M_2|$ and $\mu = G\,(m_1+m_2)$ if the motion of m_2 is expressed relative to m_1.

Therefore, whatever choice is adopted, it is sufficient to solve equation 2.1 and to subsequently substitute the correct expressions of $\vec{r}$ and μ. This equation can be seen as describing the motion of a **single unit mass under the action of a force that derives from the potential** $-\mu/r$. Thus the number of free parameters needed to characterize this motion is reduced to six: three for the initial position and three for the initial velocity. A seventh parameter is μ, that specifies the force acting on the unit mass. Since this is a central force, we know that the motion occurs in a constant plane. Indeed, from Eq. 2.1 it is obvious to write that

$$\vec{r} \wedge \ddot{\vec{r}} = \vec{0}$$

hence,

$$\vec{r} \wedge \dot{\vec{r}} = \vec{h} = Co\vec{n}st$$

Therefore, the velocity and the position are always perpendicular to a constant vector $\vec{h}$ and the motion thus occurs in a fixed **plane**.
The orientation of this plane in three dimensional space is shown in Fig. 2.1. The intersection between the plane of the motion and the inertial xy plane is called the line of the nodes and N is the ascending node. In the plane of the motion, we then adopt the polar coordinate system. Hence, the velocity of the mass can be expressed as $\dot{r}\,\vec{e_r} + r\,\dot{\psi}\,\vec{e_\psi}$. The conservation of the angular momentum

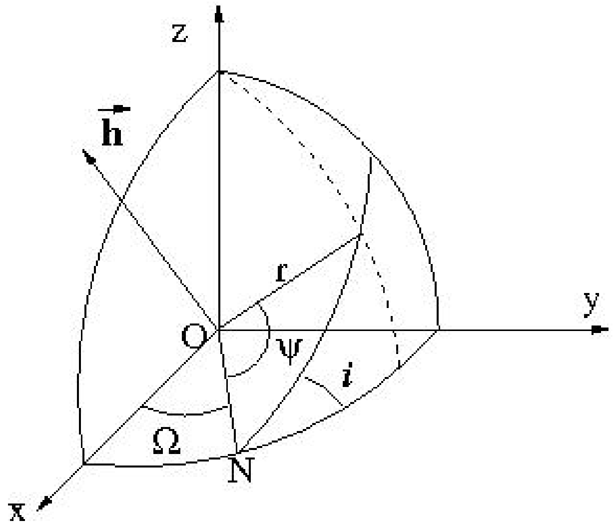


Figure 2.1: Definition of the angles Ω and i that specify the orientation of the plane of the motion with respect to the inertial frame of reference $Oxyz$ (see also below). Here N is the ascending node of the orbit.

$$\vec{r} \wedge \dot{\vec{r}} = \vec{h}$$

can now be expressed as

$$r^2\,\dot{\psi} = h = Const \tag{2.2}$$

This is actually **Kepler's second law** of planetary motion as we will show below.
Equation 2.1 also allows to describe the evolution of the kinetic energy per unit mass $T = \frac{1}{2}(\dot{r}^2 + r^2\,\dot{\psi}^2)$:

$$\frac{dT}{dt} = -\frac{\mu}{r^3}\,\vec{r}\cdot\dot{\vec{r}}$$

We thus find that the total energy per unit mass $\mathcal{E}$ is preserved:

$$\frac{1}{2}(\dot{r}^2 + r^2\,\dot{\psi}^2) - \frac{\mu}{r} = \mathcal{E} \tag{2.3}$$

Yet another vector quantity can be shown to be constant:

$$\frac{\dot{\vec{r}} \wedge \vec{h}}{\mu} - \vec{e_r} = \vec{l} = \vec{Const} \tag{2.4}$$

This result is called the **Laplace vector integral** or the **Laplace-Runge-Lenz vector**. The vector $\vec{l}$ lies in the plane of the orbit (since $\vec{l}\cdot\vec{h} = 0$). From this relation, we infer that

$$\vec{l}\cdot\vec{r} = \frac{h^2}{\mu} - r \tag{2.5}$$

This result tells us that r reaches a minimum whenever $\vec{l}$ and $\vec{r}$ are aligned. Hence, $\vec{l}$ points towards the pericenter, i.e. the point of minimum separation between the mass and the center of the force. If we call ϕ the angle between $\vec{l}$ and $\vec{r}$, equation 2.5 further leads to

$$\begin{aligned} |\vec{l}|\,r\,\cos\phi &= \frac{h^2}{\mu} - r \\ \Rightarrow r &= \frac{\frac{h^2}{\mu}}{1 + |\vec{l}|\,\cos\phi} \end{aligned} \tag{2.6}$$

From the Laplace integral we obtain that

$$\frac{h^2}{\mu}\,\dot{\vec{r}} = \vec{h} \wedge (\vec{l} + \vec{e_r}) \tag{2.7}$$

In the right hand side member of the latter relation, the only time-dependent vector is $\vec{e_r}$. Therefore, we conclude that the extremity Q of the velocity vector $\dot{\vec{r}} = \vec{FQ}$ lies on a circle of radius $\frac{\mu}{h}$ and centred on the extremity of $\frac{\mu}{h^2}\,(\vec{h} \wedge \vec{l})$. This property will be used below to build the so-called **hodograph** or velocity diagram of the motion[1] which provides a graphical representation of the locus of the point Q. For this purpose, we introduce the unit vectors $\vec{u_0} = \frac{\vec{l}}{|\vec{l}|}$ and $\vec{v_0} = \frac{\vec{h}}{h} \wedge \vec{u_0}$ as well as the vectors $\vec{V_0} = \frac{\mu}{h}\,\vec{v_0}$ and $\vec{V} = \mu\,\frac{\vec{h} \wedge \vec{e_r}}{h^2}$. As a result we can reformulate equation 2.7:

$$\dot{\vec{r}} = \vec{V} + |\vec{l}|\,\vec{V_0} \tag{2.8}$$

Finally, the Lagrangian can be written as $\mathcal{L} = \frac{1}{2}(\dot{r}^2 + r^2\,\dot{\psi}^2) + \frac{\mu}{r}$

If $h = 0$, then m_2 moves along a straight line away or towards the center of force. If $h \neq 0$, we can write

$$\left(\frac{d\,r}{d\psi}\right)^2 \frac{h^2}{r^4} + \frac{h^2}{r^2} - \frac{2\,\mu}{r} = 2\,\mathcal{E}$$

If we now define an ancillary variable $u = 1/r$, the latter equation becomes

$$\left(\frac{d\,u}{d\psi}\right)^2 + u^2 - \frac{2\,\mu\,u}{h^2} = \frac{2\,\mathcal{E}}{h^2}$$

Let us introduce yet another ancillary variable $v = u - \frac{\mu}{h^2}$. This then leads to

$$\left(\frac{d\,v}{d\psi}\right)^2 + v^2 = \frac{\mu^2}{h^4} + \frac{2\,\mathcal{E}}{h^2} = H^2 \geq 0$$

Hence

$$\frac{d\,v}{d\psi} = \pm\sqrt{H^2 - v^2}$$

$$\Rightarrow \frac{d\,v}{\sqrt{H^2 - v^2}} = \pm d\psi$$

The $\pm$ sign corresponds respectively to the derivative of the $\arcsin$ and $\arccos$ functions which differ by an additive constant $\pi/2$. Thus, we can drop the $\pm$ sign and we finally obtain: $v = H\,\cos(\psi - \omega)$ which leads to

$$r = \frac{1}{\frac{\mu}{h^2} + H\,\cos(\psi - \omega)}$$

This is equivalent to the equation of a conic in polar coordinates

$$r = \frac{C}{1 + e\,\cos(\psi - \omega)} \tag{2.9}$$

where $e = \frac{H\,h^2}{\mu}$ is the eccentricity, $C = \frac{h^2}{\mu}$, and the center of force F is one focus of the conic. Comparing equations 2.6 and 2.9, we immediately find that $e = |\vec{l}|$.
The trajectory of the mass is

[1] A hodograph is a diagram that provides a vectorial representation of the motion of a body or a fluid. It is the locus of one end of a variable vector, with the other end fixed.

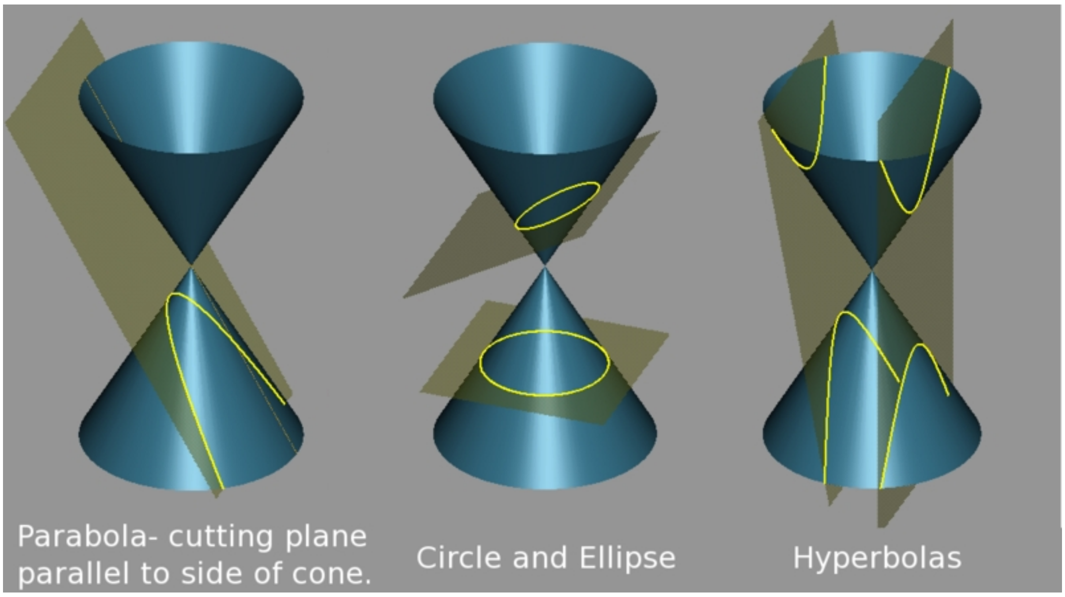


Figure 2.2: Illustration of the different types of conical sections.

- a circle if $e = 0$ (hence $\mathcal{E} = -\frac{\mu^2}{2\,h^2}$),
- an ellipse if $0 < e < 1$ (hence $-\frac{\mu^2}{2\,h^2} < \mathcal{E} < 0$)
- a parabola if $e = 1$ (hence $\mathcal{E} = 0$), and
- a hyperbola if $e > 1$ (hence $\mathcal{E} > 0$).

This is the mathematical formulation of **Kepler's first law**.

The closest approach of the mass and the center of the force is reached for $\psi = \omega$. This position is called the pericenter and ω is the argument of the pericenter. The angle $\phi = \psi - \omega$ is the true anomaly. Hence

$$r = \frac{h^2}{\mu} \frac{1}{1 + e\,\cos\phi}$$

At pericenter passage, $r = r_P$ and $\phi = 0$ yielding $r_P = \frac{h^2}{\mu\,(1+e)}$. Therefore, we can finally express the equation of the trajectory:

$$r = r_P \frac{1+e}{1 + e\,\cos\phi}$$

Finally, we can also express the fact that $\dot{r} = 0$ at pericenter passage. Hence,

$$\mathcal{E} = \frac{1}{2}\frac{h^2}{r_P^2} - \frac{\mu}{r_P} = \frac{\mu}{2\,r_P}\,(e-1)$$

These relations are valid whatever the nature of the trajectory (hence whatever the value of e).

2.2 Elliptical orbits

For an ellipse, one has the following relations between the semi-major axis (a) and the semi-minor axis (b) on the one hand and the distance between the center of the ellipse and its focus on the other hand: $b^2 = a^2(1-e^2)$ and $|CF| = a\,e$. Therefore $r_P = a\,(1-e)$ and thus $h = \sqrt{\mu\,a\,(1-e^2)}$ and

$$r = \frac{a\,(1-e^2)}{1 + e\,\cos\phi} \tag{2.10}$$

We still need to establish the relation between the position of the mass on its orbit and the time. To this aim, we use the equation of conservation of energy (equation 2.3).

$$\frac{1}{2}(\dot{r}^2 + \frac{\mu\,a\,(1-e^2)}{r^2}) - \frac{\mu}{r} = -\frac{\mu}{2\,a}$$

therefore

$$\left(r\,\frac{d\,r}{dt}\right)^2 = \frac{\mu}{a}\,[a^2\,e^2 - (r-a)^2]$$

If we introduce the ancillary variable E such that $\frac{a-r}{a\,e} = \cos E$, we finally obtain that

$$E - e\,\sin E = \sqrt{\frac{\mu}{a^3}}\,(t - t_0) \tag{2.11}$$

This is **Kepler's equation** where t_0 is the time of pericenter passage ($E = 0$). This is a non-linear relation between E and the time t. E is called the eccentric anomaly. Note that $\dot{E} = \frac{a}{r}\,\dot{M}$ where $M = \sqrt{\frac{\mu}{a^3}}\,(t - t_0)$ is the mean anomaly.

Consider an auxiliary circle of radius a, centred on the point C, center of the ellipse. If we raise a straight line across the current position of the orbiting mass and perpendicular to the major axis, it intersects the auxiliary circle at the point T. E is the angle between CT and the major axis.
In fact, from Fig. 2.3 we see that $r\,\cos\phi = a\,\cos E - a\,e$. Moreover $\left(\frac{x'}{a}\right)^2 + \left(\frac{y'}{b}\right)^2 = 1$ and $\left(\frac{x'}{a}\right)^2 + \left(\frac{a\,\sin E}{a}\right)^2 = 1$. Hence $b\,\sin E = y' = r\,\sin\phi$. Finally, combining the expressions of $r\,\cos\phi$ and $r\,\sin\phi$, we obtain $r = (a - a\,e\,\cos E)$ which is equivalent to the definition of E introduced above.
We can go one step further and express E as a function of ϕ. It is quite straightforward to show that

$$\cos\phi = \frac{\cos E - e}{1 - e\,\cos E}$$

$$\sin\phi = \frac{\sqrt{1-e^2}\,\sin E}{1 - e\,\cos E}$$

$$\tan\frac{\phi}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{E}{2}$$

Let P be the orbital period (i.e. the time between two consecutive pericenter passages). Integrating the equation of conservation of angular momentum over a full period one gets

$$\int r^2\,d\,\phi = h\,P = \sqrt{\mu\,a\,(1-e^2)}\,P$$

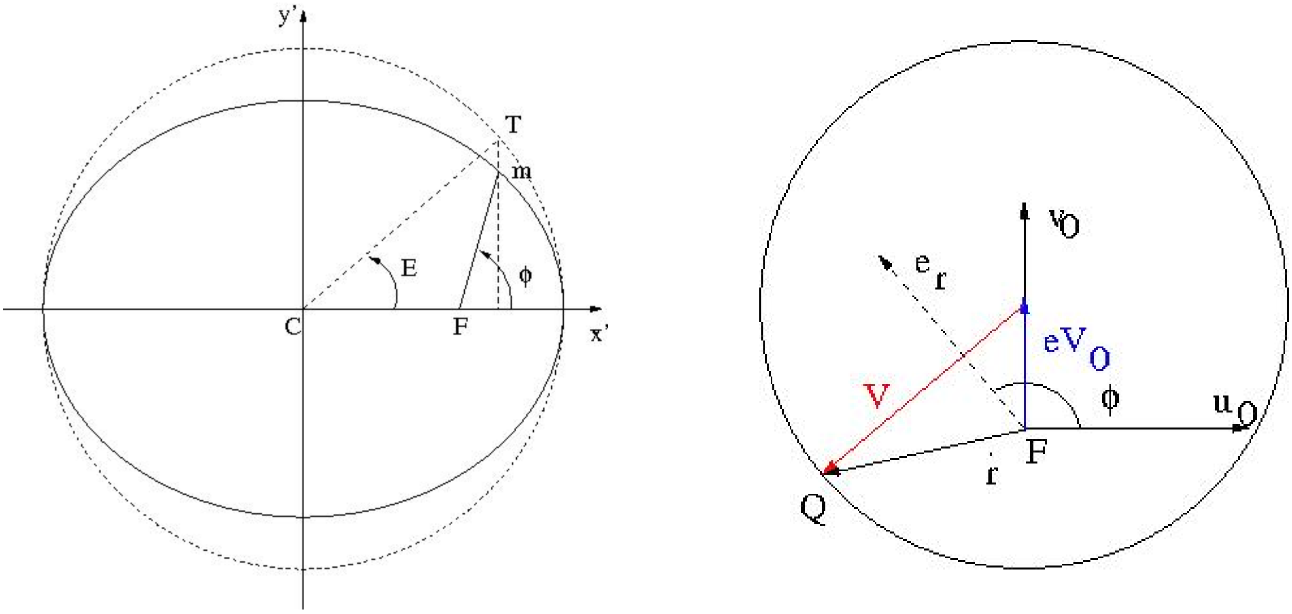


Figure 2.3: Left: definition of the eccentric anomaly E as function of ϕ. Right: hodograph of the Keplerian motion in the case of an ellipse ($e < 1$).

On the other hand, $\int r^2\, d\,\phi = 2\,\pi\,a\,b = 2\,\pi\,a^2\sqrt{1-e^2}$ and therefore

$$\left(\frac{2\,\pi}{P}\right)^2\, a^3 = \mu \tag{2.12}$$

This is **Kepler's third law** that links the square of the orbital period to the third power of the orbital separation. The hodograph of the elliptical motion is shown in the right panel of Fig. 2.3. As pointed out above, the hodograph is a circle and one notices that the center of force F lies inside this circle.

2.3 The case of the parabola

If $\mathcal{E} = 0$ and $e = 1$ (see above), we immediately obtain

$$r = \frac{2\,r_P}{1+\cos\phi}$$

as well as $h = \sqrt{2\,\mu\,r_P}$ and

$$\dot{r}^2 + r^2\,\dot{\phi}^2 = \frac{2\,\mu}{r}$$

This then yields

$$r^2\,\dot{r}^2 - 2\,\mu\,r + 2\,\mu\,r_P = 0$$

Introducing an ancillary variable s such that $r = r_P\,(1+s^2)$, we then obtain $\dot{r} = 2\,r_P\,s\,\dot{s}$ and

$$4\,r_P^4\,(1+s^2)^2\,s^2\dot{s}^2 = 2\,\mu\,r_P\,s^2$$

$$(1+s^2)\dot{s} = \sqrt{\frac{\mu}{2\,r_P^3}}$$

Hence,

$$s + \frac{s^3}{3} = \sqrt{\frac{\mu}{2\,r_P^3}}\,(t - t_0) \tag{2.13}$$

Equation 2.13 plays the same role as Kepler's equation (eq. 2.11) for the motion on a parabolic orbit.

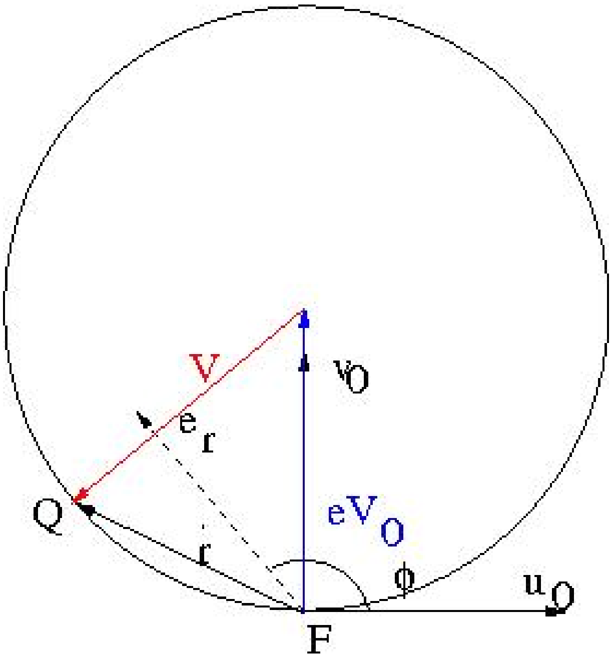


Figure 2.4: Hodograph of the Keplerian motion in the case of a parabola ($e = 1$).

The corresponding hodograph is shown in Fig. 2.4 and one notices that the center of force F lies on the circle. We note that for $\phi \to \pi$, $r \to \infty$ and the velocity tends to zero.

2.4 The case of the hyperbola

For a hyperbola, $\mathcal{E} > 0$ and $e > 1$. We now have that $r_P = a\,(e - 1)$ leading to

$$r = \frac{a\,(e^2 - 1)}{1 + e\,\cos\phi}$$

as well as $h = \sqrt{\mu\,a\,(e^2 - 1)}$ and $\mathcal{E} = \frac{\mu}{2\,a}$. The energy equation then becomes

$$\dot{r}^2 + r^2\,\dot{\phi}^2 = \frac{2\,\mu}{r} + \frac{\mu}{a}$$

This then yields

$$r^2\,\dot{r}^2 = \frac{\mu}{a}\,(r^2 + 2\,r\,a + a^2) - \mu\,a\,e^2$$

Introducing an ancillary variable F such that $\cosh F = \frac{r+a}{a\,e}$, we then obtain

$$e\,\sinh F - F = \sqrt{\frac{\mu}{a^3}}\,(t - t_0) \tag{2.14}$$

Again, equation 2.14 plays the same role as Kepler's equation (eq. 2.11) for the motion on a hyperbolic orbit. Note that $\dot{F} = \frac{a}{r}\,\dot{M}$, where $M = \frac{2\,\pi}{P}\,(t - t_0)$ is the mean anomaly as in the case of the elliptical motion.

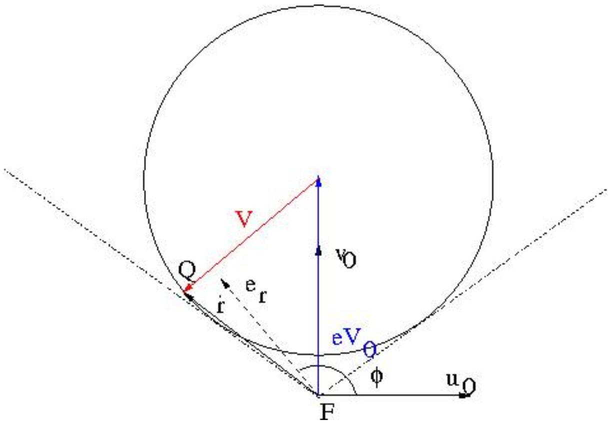


Figure 2.5: Hodograph of the Keplerian motion in the case of a hyperbola ($e > 1$).

For an hyperbola, only values of ϕ in the range $]\arccos(-1/e), 2\pi - \arccos(-1/e)[$ are acceptable. When ϕ approaches these asymptotic values, $r \to \infty$. The hodograph is illustrated in Fig. 2.5. One notices that the center of force F lies outside of the circle and that only an arc of the circle between $\arccos(-1/e)$ and $2\pi - \arccos(-1/e)$ is actually covered. Note that the velocity does not tend to zero when r tends to infinity.

2.5 The elements of the orbit

As we have shown above, any Keplerian motion can be described by 6 + 1 parameters. These parameters are $(\vec{h}, \vec{l}, t_0, \mu)$. This are actually 8 parameters (two vectors and two scalars), but the two vectors are not independent since they must satisfy the relation $\vec{h} \cdot \vec{l} = 0$. Usually, these parameters are replaced by a combination of equivalent numbers that we call the elements of the orbit. In the following, we will mainly consider the case of an elliptical motion. In this case, the elements of the orbit are $(\Omega, i, \omega, a, e, t_0, \mu)$. The line of nodes NN' is defined by the intersection between the orbital plane and the plane xy. The ascending node N is the node where the moving mass crosses the xy plane from a negative towards a positive latitude. Ω is the longitude of the ascending node measured from the x direction which is a fixed direction in space (e.g. the direction of the vernal point at equinox J2000). i is the inclination of the orbit with respect to the plane of reference xy. If $i < \pi/2$, the orbit is prograde (the longitude increases with time), whereas it is retrograde if $i > \pi/2$. ω is the argument of the pericenter (also called the longitude of the pericenter). $a = \frac{h^2}{\mu\,(1-e^2)}$ and $e = |\vec{l}|$ are respectively the semi-major axis and the eccentricity of the orbit.

The latitude θ_0 and the longitude λ_0 of the pericenter are given by $\sin\theta_0 = \sin i \, \sin\omega$ and $\tan(\lambda_0 - \Omega) = \cos i \, \tan\omega$.

When the inclination is either very close to 0 or π, the line of nodes is only poorly defined. In that case one rather uses $\varpi = \Omega + \omega$ instead of Ω and ω. Note that ϖ is the sum of two angles that are not coplanar if i is not strictly equal to zero. If e is very small or null, $\vec{u_0}$ is no longer defined and hence ω or ϖ are also no longer defined. If both the inclination and the eccentricity are close to zero, it is preferable to use the quantities $u = e\,\cos\varpi$, $v = e\,\sin\varpi$, $q = \cos\Omega\,\sin(i/2)$ and $p = \sin\Omega\,\sin(i/2)$.

Using the spherical law of cosines, one can relate the x, y and z coordinates of the moving mass to the elements of the orbit:

$$x = r\,[\cos\Omega\,\cos(\omega+\phi) - \sin\Omega\,\sin(\omega+\phi)\,\cos i] \tag{2.15}$$

$$y = r\,[\sin\Omega\,\cos(\omega+\phi) + \cos\Omega\,\sin(\omega+\phi)\,\cos i] \tag{2.16}$$

$$z = r\,[\sin i\,\sin(\omega+\phi)] \tag{2.17}$$

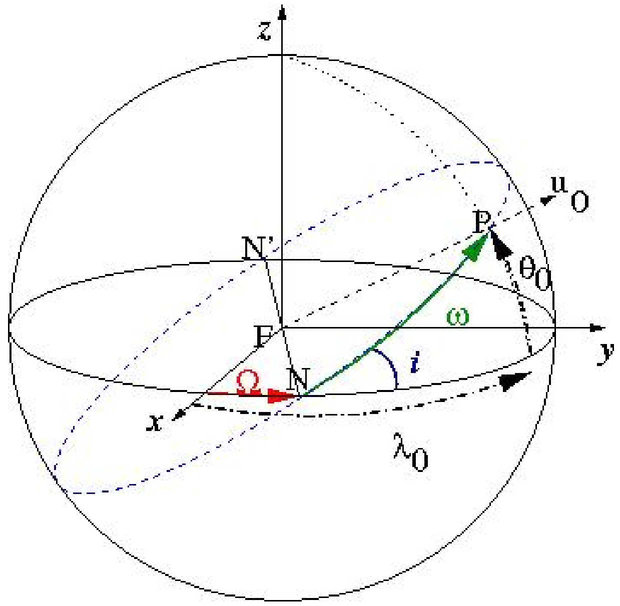


Figure 2.6: Definition of the angular elements of the orbit.

The velocities are obtained by deriving the above relations:

$$\begin{aligned}
\dot{x} &= \dot{r}\,[\cos\Omega\,\cos(\omega+\phi) - \sin\Omega\,\sin(\omega+\phi)\,\cos i] - \frac{h}{r}\,[\cos\Omega\,\sin(\omega+\phi) + \sin\Omega\,\cos(\omega+\phi)\,\cos i] \\
\dot{y} &= \dot{r}\,[\sin\Omega\,\cos(\omega+\phi) + \cos\Omega\,\sin(\omega+\phi)\,\cos i] - \frac{h}{r}\,[\sin\Omega\,\sin(\omega+\phi) - \cos\Omega\,\cos(\omega+\phi)\,\cos i] \\
\dot{z} &= \dot{r}\,[\sin i\,\sin(\omega+\phi)] + \frac{h}{r}\,[\sin i\,\cos(\omega+\phi)]
\end{aligned}$$

From these relations it becomes clear that if we can determine $\vec{r} = (x, y, z)$ and $\dot{\vec{r}} = (\dot{x}, \dot{y}, \dot{z})$ at one point in time, we are able to evaluate all the orbital elements. Indeed

$$\vec{h} = \vec{r} \wedge \dot{\vec{r}} = h\,(\sin i\,\sin\Omega, -\sin i\,\cos\Omega, \cos i)$$

Hence i and Ω can be determined along with h. The total energy is evaluated from $\mathcal{E} = |\dot{\vec{r}}|^2/2 - \mu/r$ which yields the nature of the orbit (ellipse, parabola or hyperbola). Let us assume that the orbit is an ellipse. The value of $\mathcal{E} = -\mu/(2\,a)$ then yields a. Combining this result with $h = \sqrt{\mu\,a\,(1-e^2)}$ we then obtain e. The definition of the eccentric anomaly allows to write

$$e\,\cos E = 1 - r/a$$

and

$$e\,\sin E = \frac{r\,\dot{r}}{\sqrt{\mu\,a}}$$

which provide the value of E. Finally, Kepler's equation is used to obtain M and hence the time of pericenter passage t_0.

The orbital parameters of a number of planets of the Solar System are given in the Table below. The plane of the orbital motion of the Earth around the Sun is called the **ecliptic**. The Astronomical Unit (AU) is defined as the semi-major axis of the Earth's orbit around the Sun (149 598 000 km). In the table, we list the longitude $L_0 = \varpi + n\,(t'_0 - t_0)$ where t'_0 was arbitrarily taken to be J2000 (i.e. 1st January 2000 at 12:00 UT) and n is the angular velocity (expressed here in arcsec per day). The mean longitude ($L = \varpi + M$) for any given date is then

Table 2.1: Main characteristics of the heliocentric orbits of the planets (and the dwarf planet Pluto) of the Solar System.

Planet	a (AU)	e	i (°)	Ω (°)	ϖ (°)	L_0 (°)	n (arcsec day^{-1})
Mercury	0.3871	0.2056	7.00	48.33	77.46	252.25	14732.42
Venus	0.7233	0.0068	3.39	76.68	131.56	181.98	5767.67
Earth	1.0000	0.0167	–	–	102.94	100.47	3548.19
Mars	1.5237	0.0934	1.85	49.56	336.06	355.43	1886.52
Jupiter	5.2028	0.0485	1.30	100.46	14.33	34.35	299.128
Saturn	9.5388	0.0555	2.49	113.66	93.06	50.08	120.455
Uranus	19.182	0.0463	0.77	74.01	173.00	314.05	42.231
Neptune	30.058	0.0090	1.77	131.78	48.12	304.39	21.534
Pluto	39.44	0.2485	17.33	110.7	224.6	237.7	14.3

obtained from $L = L_0 + n\,(t - t_0')$. For instance, we find that the Earth reaches its perihelion (i.e. $L = \varpi$) on 4 January.

2.6 Appendix: the main laws of spherical trigonometry

Consider a spherical triangle on a unit sphere of center O (see left panel of Fig. 2.7).

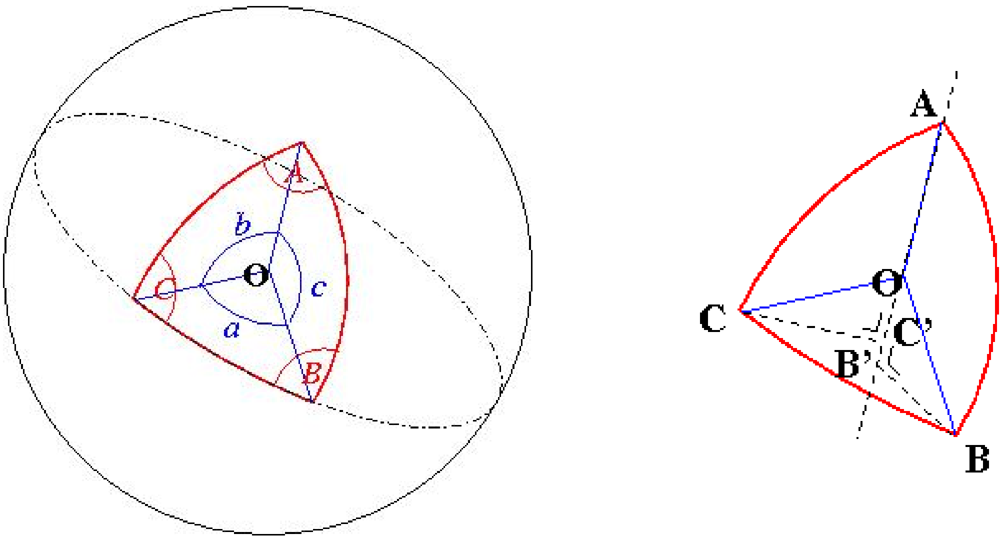


Figure 2.7: Left: the spherical triangle ABC on top of the unit sphere centred on O. Right: the points B' and C' are the projections of B and C onto the straight line OA.

The angles A, B, C and a, b, c are related to each other through

$$\cos a = \cos b\,\cos c + \sin b\,\sin c\,\cos A \tag{2.18}$$

$$\cos b = \cos c\,\cos a + \sin c\,\sin a\,\cos B \tag{2.19}$$

$$\cos c = \cos a\,\cos b + \sin a\,\sin b\,\cos C \tag{2.20}$$

and

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \tag{2.21}$$

These are the spherical laws of cosines and sines respectively.

These relations can be established easily by computing the values of $\vec{OB} \cdot \vec{OC}$ and $\vec{OA} \cdot (\vec{OB} \wedge \vec{OC})$ in two different ways.
Let us start by noting that for a unit sphere ($|OA| = |OB| = |OC| = 1$)

$$\vec{OB} \cdot \vec{OC} = \cos a$$

We then introduce the points B' and C' which are the projections of B and C onto the straight line OA (see Fig. 2.7). By construction, and since we are dealing with a unit sphere, we have that

$$\begin{aligned} |OB'| &= \cos c \\ |BB'| &= \sin c \\ |OC'| &= \cos b \\ |CC'| &= \sin b \end{aligned}$$

With these definitions, we can write

$$\begin{aligned} \cos a &= (\vec{OB'} + \vec{B'B}) \cdot (\vec{OC'} + \vec{C'C}) \\ &= \vec{OB'} \cdot \vec{OC'} + \vec{OB'} \cdot \vec{C'C} + \vec{B'B} \cdot \vec{OC'} + \vec{B'B} \cdot \vec{C'C} \\ &= \vec{OB'} \cdot \vec{OC'} + \vec{B'B} \cdot \vec{C'C} + \vec{B'B} \cdot \vec{OC'} + \vec{OB'} \cdot \vec{C'C} \end{aligned}$$

The last two scalar products are both equal to zero because $\vec{OC'} \parallel \vec{OA}$ and $\vec{B'B} \perp \vec{OA}$ and $\vec{OB'} \parallel \vec{OA}$ and $\vec{C'C} \perp \vec{OA}$. Since $\vec{B'B}$ and $\vec{C'C}$ are both perpendicular to $\vec{OA}$, the angle between $\vec{B'B}$ and $\vec{C'C}$ is equal to A. Hence

$$\vec{B'B} \cdot \vec{C'C} = |B'B|\,|C'C|\,\cos A = \sin c\,\sin b\,\cos A$$

and we finally obtain that

$$\cos a = \cos b\,\cos c + \sin b\,\sin c\,\cos A$$

Let us now turn our attention to the sine rule. We start by expressing

$$\begin{aligned} \vec{OA} \cdot (\vec{OB} \wedge \vec{OC}) &= \vec{OA} \cdot \left[(\vec{OB'} + \vec{B'B}) \wedge (\vec{OC'} + \vec{C'C})\right] \\ &= \vec{OA} \cdot \left[\vec{OB'} \wedge \vec{OC'} + \vec{OB'} \wedge \vec{C'C} + \vec{B'B} \wedge \vec{OC'} + \vec{B'B} \wedge \vec{C'C})\right] \\ &= \vec{OA} \cdot (\vec{B'B} \wedge \vec{C'C}) \end{aligned}$$

where the last relation stems from the facts that $\vec{OB'} \wedge \vec{OC'} = \vec{0}$, $\vec{OA} \cdot (\vec{OB'} \wedge \vec{C'C}) = 0$, and $\vec{OA} \cdot (\vec{B'B} \wedge \vec{OC'}) = 0$ because $\vec{OB'} \parallel \vec{OC'} \parallel \vec{OA}$. Now since $\vec{B'B} \perp \vec{OA}$ and $\vec{C'C} \perp \vec{OA}$, we have that

$$\vec{B'B} \wedge \vec{C'C} = |B'B|\,|C'C|\,\sin A\,\vec{OA}$$

Thus, we obtain that

$$\begin{aligned} \vec{OA} \cdot (\vec{OB} \wedge \vec{OC}) &= \sin c\,\sin b\,\sin A \\ = \vec{OB} \cdot (\vec{OC} \wedge \vec{OA}) &= \sin a\,\sin c\,\sin B \\ = \vec{OC} \cdot (\vec{OA} \wedge \vec{OB}) &= \sin a\,\sin b\,\sin C \end{aligned}$$

where we have used the property that the scalar triple product is invariant under a circular shift of its operands. From these relations, we then infer that

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

2.7 Appendix: the Roche limit and planetary rings

Consider a celestial body of mass m held together only by its own gravity and orbiting another, more massive ($M \gg m$), body. The *Roche limit*, named after the French astronomer Edouard Roche (1820 – 1883), is the theoretical distance within which the smaller of the two bodies will disintegrate under the effect of the tidal forces of the more massive body that exceed the internal gravitational forces of the smaller body.
In practice, real satellites, either natural or artificial, frequently orbit inside the Roche limit of their host planet because they are held together by other forces in addition to gravity. This is the case for instance of the moons Phobos around Mars, Pan around Saturn, Cordelia around Uranus, Naïade and Thalassa around Neptune.

It is thought that the Roche limit also explains the existence of rings around some planets. These rings form either when a moon or another body moves in too close to the planet and gets disrupted by the action of the tidal forces, or because the tidal forces prevent debris of the protoplanetary disk from coalescing to form a larger body. With the remarkable exception of the E rings of Saturn, most rings around giant planets in the Solar System are indeed located inside the Roche limit. A situation where a small body was disrupted by the tidal attraction of a giant planet occured in July 1992 when comet Shoemaker-Levy 9 was fragmented into smaller pieces while passing within the Roche limit of planet Jupiter.

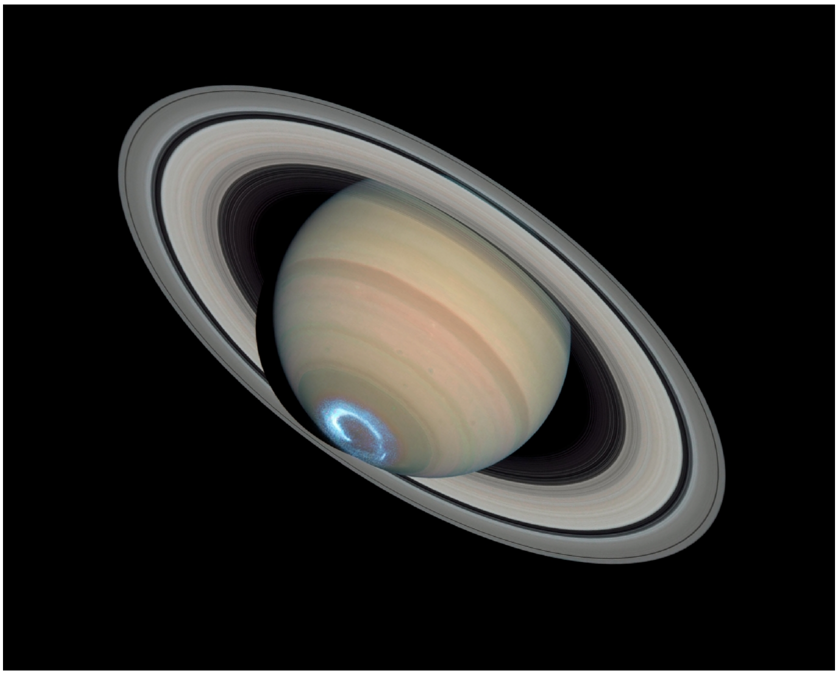


Figure 2.8: The rings of the giant planets in the Solar System, with the exception of the E rings of Saturn, lie within the Roche limits of their host planets.

The mathematical expression of the Roche limit depends upon the properties (mostly the rigidity) of the satellite. A strictly rigid satellite will maintain its shape until the tidal forces break it apart. A highly fluid satellite on the other hand gradually elongates into an ellipsoidal shape which then amplifies the importance of the tidal forces. Thus a fluid satellite will break up more easily and the radius of the Roche limit will then be larger in the fluid case. Generally speaking, the reader can find a number of different mathematical formulations of the Roche limit in the literature (e.g. Holsapple & Michel 2006). Sometimes the reasoning that leads to these expressions is not straightforward. This is why we restrict ourself to a rather simple case here, respecting the general laws of classical mechanics.

2.7.1 Rigid satellites

In this approximation, the smaller body is assumed to hold together by its own self-gravity and to maintain its (spherical) shape.
We model this situation by considering a small mass δm at the surface of the satellite of mass m facing the main body (i.e. we assume a synchronous rotation and revolution). The satellite orbits the main body of mass M and radius R at a distance d (see Fig. 2.9). Let $\vec{x}$ be the position vector of the small mass $\delta\, m$ with respect to the center of the mass M. We can write immediately that $\vec{x} = (d - r)\, \vec{e_r}$. In the rotating frame of reference, we can express the acceleration of the small mass as

$$\frac{\delta^2 \vec{x}}{\delta\, t^2} = \ddot{\vec{x}} - (2\, \vec{\omega} \wedge \frac{\delta \vec{x}}{\delta\, t} + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{x}))$$

where $\ddot{\vec{x}}$, $2\, \vec{\omega} \wedge \frac{\delta \vec{x}}{\delta\, t}$ and $-\vec{\omega} \wedge (\vec{\omega} \wedge \vec{x})$ are respectively the absolute acceleration (in an inertial frame of reference), the Coriolis and centrifugal accelerations with $\omega^2 = \frac{G\, M}{d^3}$. As long as the small mass remains attached to the satellite, both the relative acceleration and the relative velocity are zero.

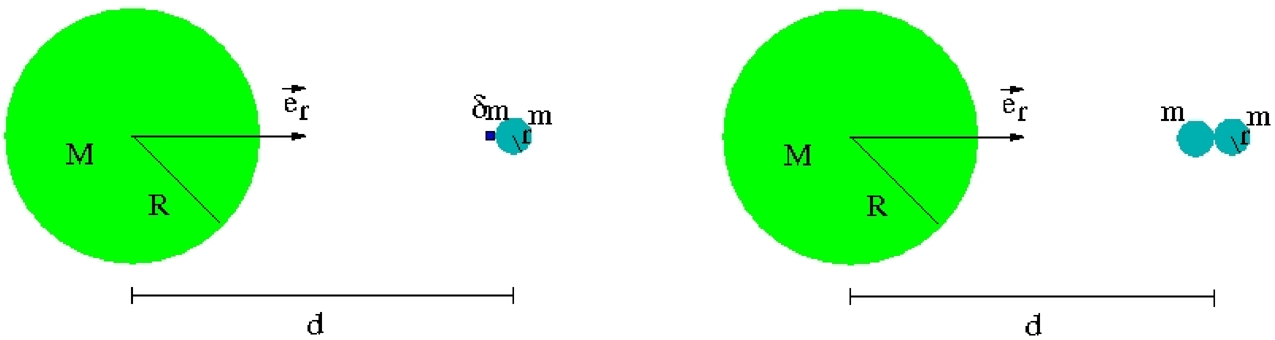


Figure 2.9: Left: illustration of the geometry assumed for a rigid satellite. Right: same for the satellite consisting of two equal spheres.

Newton's equation of the small mass can then be written:

$$\delta\, m\, \ddot{\vec{x}} = -\frac{G\, M\, \delta\, m}{d^3}\, \vec{x} = -\frac{G\, M\, \delta\, m}{x^2}\, \vec{e_r} + \frac{G\, m\, \delta\, m}{r^2}\, \vec{e_r} + \vec{N} \tag{2.22}$$

Here $\vec{N} = -N\, \vec{e_r}$ is the contact force between the small mass and the satellite. The latter can be expressed using equation 2.22:

$$\begin{aligned} \frac{N}{G\, \delta\, m} &= \frac{M}{d^3}\, (d - r) - \frac{M}{(d - r)^2} + \frac{m}{r^2} \\ &\simeq \frac{M}{d^2}\, (1 - \frac{r}{d}) - \frac{M}{d^2}\, (1 + 2\, \frac{r}{d}) + \frac{m}{r^2} \\ &= -\frac{3\, M\, r}{d^3} + \frac{m}{r^2} \end{aligned} \tag{2.23}$$

Hence the contact force becomes zero, i.e. the small mass detaches from the satellite, when

$$\frac{d}{r} = \left(\frac{3\,M}{m}\right)^{1/3}$$

Therefore, the Roche limit in this case is given by

$$d_{\rm RL} = \left(\frac{3\,\rho_M}{\rho_m}\right)^{1/3} R = 1.44\,\left(\frac{\rho_M}{\rho_m}\right)^{1/3} R \tag{2.24}$$

where ρ_M and ρ_m are the densities of the main mass and the satellite respectively.

2.7.2 The fluid case

We consider a satellite that has already undergone some deformation due to the tidal forces. For this purpose, we model the satellite as consisting of two equal spheres of radius r and mass m each. The reasoning is essentially the same as above. We again assume that the satellite is in synchronous rotation and is oriented as shown in Fig. 2.9. The distance between the center of mass of M and of the satellite is d. $\vec{x}$ is the position vector of the center of the sphere that is closest to the main mass with respect to the main mass: $\vec{x} = (d-r)\,\vec{e_r}$.
For the sphere closest to the main body, Newton's equation becomes:

$$m\,\ddot{\vec{x}} = -\frac{G\,M\,m}{d^3}\,\vec{x} = -\frac{G\,M\,m}{x^2}\,\vec{e_r} + \frac{G\,m\,m}{4\,r^2}\,\vec{e_r} + \vec{N} \tag{2.25}$$

Here $\vec{N} = -N\,\vec{e_r}$ is the contact force between the two spheres. With the same reasoning as above, but this time using equation 2.25, we obtain:

$$\begin{aligned}
\frac{N}{G\,m} &= \frac{M}{d^3}\,(d-r) - \frac{M}{(d-r)^2} + \frac{m}{4\,r^2} \\
&\simeq \frac{M}{d^2}\,(1-\frac{r}{d}) - \frac{M}{d^2}\,(1+2\,\frac{r}{d}) + \frac{m}{4\,r^2} \\
&= -\frac{3\,M\,r}{d^3} + \frac{m}{4\,r^2}
\end{aligned} \tag{2.26}$$

Therefore, the Roche limit in this case is given by

$$d_{\rm RL} = \left(\frac{12\,\rho_M}{\rho_m}\right)^{1/3} R = 2.29\,\left(\frac{\rho_M}{\rho_m}\right)^{1/3} R \tag{2.27}$$

where ρ_M and ρ_m are again the densities of the main body and of the satellite respectively.
A more complex mathematical treatment leads to a slightly different value of the numerical constant for the fluid case: 2.423 instead of 2.29. This is the value of the Roche limit that is usually quoted for a fluid satellite.

2.8 Exercises

2.1 A comet is observed at a distance r from the Sun and traveling at a speed v. Demonstrate that the orbit is elliptical, parabolic or hyperbolic depending on whether $\frac{v^2\,r}{v_♁^2\,a_♁}$ is less than, equal to, or greater than 2, respectively. Here $v_♁$ and $a_♁$ stand for the mean orbital velocity of the Earth and its mean distance from the Sun, respectively. (Adapted from Fitzpatrick 2012).

2.2 Consider a comet on a hyperbolic orbit around the Sun. The impact parameter λ is defined as the minimum distance between the Sun and the comet if there were no gravitational interaction between them. Show that $\lambda = \frac{h}{\sqrt{2\,\varepsilon}}$ where h and ε are the angular momentum and the energy (both per unit mass) of the comet. Demonstrate that the relation between λ and the actual perihelion distance r_p can be written $r_p = \frac{2\,\lambda}{\alpha+\sqrt{\alpha^2+4}}$ where $\alpha = GM_\odot/(\varepsilon\,\lambda)$ with $M_\odot$ the mass of the Sun. Show that the condition to avoid a collision with the Sun is $\alpha < \lambda/R_\odot - R_\odot/\lambda$ (if $\lambda > R_\odot$) with $R_\odot$ the radius of the Sun.
Suggestion: make use of the relations of section 2.4 and the fact that the mathematical equation of an hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with $b^2 = a^2\,(e^2 - 1)$ and the Sun being located at position $(a\,e, 0)$.

2.3 In the two-body problem, the motion on a parabolic trajectory follows equation 2.13:

$$s + \frac{s^3}{3} = \sqrt{\frac{\mu}{2\,r_P^3}}\,(t - t_0) = M$$

where r_P is the distance at pericenter, and t_0 is the time of pericenter passage. Demonstrate that the above equation admits the analytic solution: $s = \frac{1}{2}\,q^{1/3} - 2\,q^{-1/3}$ with $q = 12\,M + 4\,\sqrt{4 + 9\,M^2}$. (Adapted from Fitzpatrick 2012).

2.4 A comet moves on a parabolic orbit inside the plane of the ecliptic. Assuming that the Earth's orbit is circular with radius $a_♁$, show that the intersections between the comet's trajectory and the Earth's orbit occur for $\cos\phi = -1 + \frac{2\,r_p}{a_♁}$ where r_p is the perihelion distance of the comet and $\phi = 0$ at perihelion. Demonstrate that the time interval (in years) during which the comet remains within less than one astronomical unit from the Sun is given by

$$\frac{\sqrt{2}}{3\,\pi}\,(1 + \frac{2\,r_p}{a_♁})\,\sqrt{1 - \frac{r_p}{a_♁}}$$

Show that this duration is maximum for $r_p = \frac{a_♁}{2}$ and establish the value of this maximum duration. (Adapted from Fitzpatrick 2012).

2.5 Consider an object moving on an elliptical orbit about the Sun. At time t_1, the object is located in point P_1 which has a distance r_1 from the Sun. Similarly, at time t_2, it is located in point P_2 at a distance r_2 from the Sun. Let s be the length of the vector $\vec{P_1 P_2}$. Demonstrate Lambert's theorem, i.e. show that the time required to move from P_1 to P_2 is equal to

$$t_2 - t_1 = \frac{P_{\rm orb}}{2\,\pi}\,[(\alpha - \sin\alpha) - (\beta - \sin\beta)]$$

with

$$\sin\frac{\alpha}{2} = \frac{1}{2}\left(\frac{r_1 + r_2 + s}{a}\right)^{1/2}$$

$$\sin\frac{\beta}{2} = \frac{1}{2}\left(\frac{r_1 + r_2 - s}{a}\right)^{1/2}$$

where $P_{\rm orb}$ and a stand for the orbital period and semi-major axis, respectively.
Suggestion: use Kepler's equation (equation 2.11) and define $\alpha - \beta = E_2 - E_1$ and $\cos\frac{\alpha+\beta}{2} = e\,\cos\frac{E_1+E_2}{2}$ where E is the eccentric anomaly and e is the eccentricity of the orbit.

Chapter 3

The two-body problem in Lagrangian and Hamiltonian mechanics

3.1 Lagrangian and Hamiltonian mechanics

Lagrangian and Hamiltonian mechanics are re-formulation of the equations of classical mechanics that are sometimes advantageous for a deeper understanding of some complex problems. Here we recall some of the fundamental concepts and results of both formalisms[1].
Consider a system of masses m_i whose positions and velocities can be described by a set of **generalized coordinates** q_j and $\dot{q}_j$ (where $j = 1, ..., n$) respectively. Let us assume that the forces that act on each mass m_i derive from a scalar potential U. We define the Lagrangian of this system as $\mathcal{L} = T - U$ where T and U are respectively the kinetic and potential energies. The principle of virtual work then leads to Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial\,\mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial\,\mathcal{L}}{\partial q_j} = 0 \tag{3.1}$$

which are second-order differential constraints on an n-dimensional coordinate space.

In the Hamiltonian formalism, we re-formulate the problem in terms of first-order differential equations on a $2\,n$ dimensional phase space. For this purpose, we introduce the generalized momenta

$$p_j = \frac{\partial\,\mathcal{L}}{\partial \dot{q}_j} \tag{3.2}$$

with $(j = 1, ..., n)$, such that

$$\frac{d}{dt}\,p_j = \frac{\partial\,\mathcal{L}}{\partial q_j} \tag{3.3}$$

q_j and p_j $(j = 1, ..., n)$ are conjugated variables that provide a set of $2\,n$ independent variables allowing to describe the system. The Hamiltonian of the system is then defined as

$$\mathcal{H} = \sum_{j=1}^{n} p_j\,\dot{q}_j - \mathcal{L}$$

[1] For a full demonstration and extensive discussion of these results we refer to the lectures on *Analytical Mechanics* by P. Dauby.

From this definition, one derives the canonical equations of Hamilton:

$$\frac{\partial \mathcal{H}}{\partial p_j} = \dot{q_j} \tag{3.4}$$

$$\frac{\partial \mathcal{H}}{\partial q_j} = -\dot{p_j} \tag{3.5}$$

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \tag{3.6}$$

Since T is a quadratic form of $\dot{q_j}$ and if the potential U does not depend on $\dot{q_j}$, we find that $\mathcal{H} = T + U$.
If the Hamiltonian does not depend explicitly on certain variables, the integration of the canonical equations relative to their conjugated variables is straightforward.

There exist a number of rules allowing us to change the set of variables used to describe the system whilst preserving the canonical form of the equations. Consider the transformations:

$$q_j = f_j(x_1, ..., x_n, y_1, ..., y_n, t)$$

$$p_j = g_i(x_1, ..., x_n, y_1, ..., y_n, t)$$

The new set of variables x_j and y_j are said to be canonical if $\frac{\partial \mathcal{H}'}{\partial y_j} = \dot{x_j}$ and $\frac{\partial \mathcal{H}'}{\partial x_j} = -\dot{y_j}$. This can be shown to be the case provided that

$$[x_j, y_k] = -[y_j, x_k] = \delta_{jk}$$

$$[x_j, x_k] = [y_j, y_k] = 0$$

and provided that there exists a function $F^*(x_j, y_j, t)$ such that $[t, \alpha] = -\frac{\partial F^*}{\partial \alpha} \quad \forall \alpha$.
Here, the Lagrange bracket is defined as

$$[u, v] = \sum_{i=1}^{n} \left(\frac{\partial f_i}{\partial u} \frac{\partial g_i}{\partial v} - \frac{\partial f_i}{\partial v} \frac{\partial g_i}{\partial u} \right)$$

The new Hamiltonian can then be written as $\mathcal{H}'(x_i, y_i, t) = \mathcal{H} + F^*$.
For a transformation to be canonical, the function F^* must be such that one can define a generating function $G(q_j, y_j, t)$ such that

$$\sum_{j=1}^{n} (p_j \, dq_j + x_j \, dy_j) + F^* \, dt = dG$$

which is equivalent to

$$\sum_{j=1}^{n} (p_j \, dq_j + x_j \, dy_j) + (\mathcal{H}' - \mathcal{H}) \, dt = dG$$

For some applications it is advantageous to perform a transformation such that the new Hamiltonian $\mathcal{H}' = 0$. The Hamilton-Jacobi method then consists in the resolution of the equations

$$\mathcal{H}(q_j, \frac{\partial G}{\partial q_j}, t) + \frac{\partial G}{\partial t} = 0 \tag{3.7}$$

$$\frac{\partial G}{\partial q_j} = p_j \tag{3.8}$$

$$\frac{\partial G}{\partial y_j} = x_j \tag{3.9}$$

where $x_j = \alpha_j$ and $y_j = \beta_j$ are constants. In fact, $\frac{\partial \mathcal{H}'}{\partial y_j} = \dot{x_j} = 0$ and $\frac{\partial \mathcal{H}'}{\partial x_j} = -\dot{y_j} = 0$ because $\mathcal{H}' = 0$. Thus the choice $\mathcal{H}' = 0$ simultaneously implies that the canonical variables are constant (i.e. not variable with time).

3.2 Application to the two body problem

Let us consider the **instantaneous** plane of the orbit Π, defined as the plane that contains the points O (the origin of the axes) and P (the instantaneous position of the moving mass) and the instantaneous velocity vector. Π intersects the xy plane along the X' axis. The axis perpendicular to the Π plane is called Z' and we have that $\cos\gamma = \vec{e_{Z'}} \cdot \vec{e_z}$, $\cos\theta = \vec{e_{X'}} \cdot \vec{e_x}$ (see Fig. 3.1). In the Π plane, the position of P is specified by the polar coordinates (r, ψ).

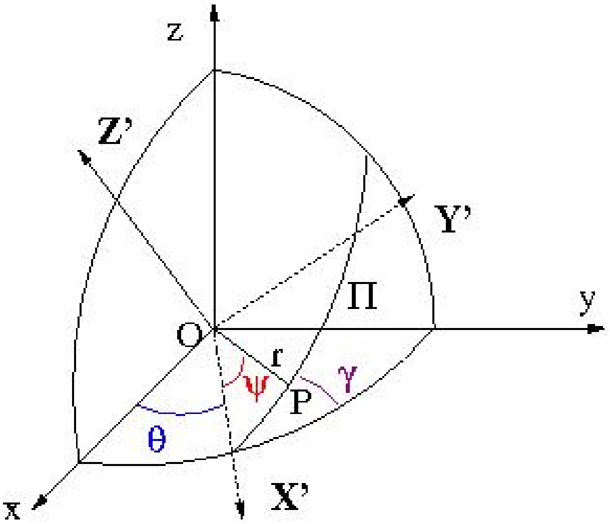


Figure 3.1: Definition of the angles θ, γ and ψ that specify the position of the point P with respect to the inertial frame of reference $Oxyz$.

Hence, the absolute position of P is specified by $(r, \theta, \gamma, \psi)$ where $r = |OP|$. The velocity is $\dot{\vec{r}} = \dot{r}\,\vec{e_r} + r\,(\vec{\Omega} \wedge \vec{e_r})$ where $\vec{\Omega}$ is the angular velocity vector of the axes $(\vec{e_r}, \vec{e_\psi}, \vec{e_{Z'}})$ with respect to the inertial frame of reference $(\vec{e_x}, \vec{e_y}, \vec{e_z})$:

$$\vec{\Omega} = \dot{\theta}\,\vec{e_z} + \dot{\gamma}\,\vec{e_{X'}} + \dot{\psi}\,\vec{e_{Z'}} = \dot{\gamma}\,\vec{e_{X'}} + \dot{\theta}\,\sin\gamma\,\vec{e_{Y'}} + (\dot{\theta}\,\cos\gamma + \dot{\psi})\,\vec{e_{Z'}}$$

This then leads to

$$\dot{\vec{r}} = \dot{r}\,\vec{e_r} + r\,(\dot{\theta}\,\cos\gamma + \dot{\psi})\,\vec{e_\psi} + r\,(\dot{\gamma}\,\sin\psi - \dot{\theta}\,\sin\gamma\,\cos\psi)\,\vec{e_{Z'}}$$

The kinetic energy is then equal to

$$T = \frac{1}{2}\left[\dot{r}^2 + r^2\,(\dot{\theta}\,\cos\gamma + \dot{\psi})^2 + r^2\,(\dot{\gamma}\,\sin\psi - \dot{\theta}\,\sin\gamma\,\cos\psi)^2\right]$$

The fact that the plane Π by definition contains the instantaneous velocity vector leads to the constraint $\dot{\gamma}\,\sin\psi - \dot{\theta}\,\sin\gamma\,\cos\psi = 0$ between the angular velocities $\dot{\gamma}$ and $\dot{\theta}$. The last term in the expression of the kinetic energy is thus zero because of the constraint.

The Lagrangian is given by $\mathcal{L} = T + \frac{\mu}{r}$. This then leads to the following equations:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) - \frac{\partial\mathcal{L}}{\partial r} &= 0 \\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}}\right) - \frac{\partial\mathcal{L}}{\partial \psi} &= 0 \\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial \theta} &= 0 \\
\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\gamma}}\right) - \frac{\partial\mathcal{L}}{\partial \gamma} &= 0
\end{aligned}$$

We can now introduce the canonical variables:

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \dot{r} = R$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = r^2\,(\dot{\psi} + \dot{\theta}\,\cos\gamma) = \Psi$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = r^2\,(\dot{\theta}\,\cos^2\gamma + \dot{\psi}\,\cos\gamma) = \Theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\gamma}} = 0 = \Gamma$$

The Hamiltonian then becomes

$$\mathcal{H} = \frac{1}{2}\left[R^2 + \frac{\Psi^2}{r^2}\right] - \frac{\mu}{r}$$

and the corresponding canonical equations are

$$\begin{aligned}
\frac{d\,r}{dt} &= \frac{\partial\,\mathcal{H}}{\partial R} = R \\
\frac{d\,R}{dt} &= -\frac{\partial\,\mathcal{H}}{\partial r} = \frac{\Psi^2}{r^3} - \frac{\mu}{r^2} \\
\frac{d\,\psi}{dt} &= \frac{\partial\,\mathcal{H}}{\partial \Psi} = \frac{\Psi}{r^2} \\
\frac{d\,\Psi}{dt} &= -\frac{\partial\,\mathcal{H}}{\partial \psi} = 0 \\
\frac{d\,\theta}{dt} &= \frac{\partial\,\mathcal{H}}{\partial \Theta} = 0 \\
\frac{d\,\Theta}{dt} &= -\frac{\partial\,\mathcal{H}}{\partial \theta} = 0 \\
\frac{d\,\gamma}{dt} &= \frac{\partial\,\mathcal{H}}{\partial \Gamma} = 0 \\
\frac{d\,\Gamma}{dt} &= -\frac{\partial\,\mathcal{H}}{\partial \gamma} = 0
\end{aligned}$$

Given the definition of the Π plane, we know that $\Gamma = 0$. From the canonical equations, we further infer $\frac{d\gamma}{dt} = 0$. Hence $\gamma = i = Const$. The constraint that stems from the definition of Π and the canonical equations then lead to $\dot{\theta} = 0$, and thus $\theta = \Omega = Const$. Hence, $\Theta = \Psi\,\cos\gamma$. On the other hand, the fact that the Hamiltonian does not depend upon ψ, θ nor γ leads successively to $\Psi = r^2\,\dot{\psi} = Const = h$ (conservation of the angular momentum), $\Theta = Const = h'$ and $\Gamma = 0$ (already known).
The two variables γ (given by $\cos\gamma = \Theta/\Psi$) and $\Gamma = 0$ are not needed to describe the status of the system and can thus be eliminated.
The canonical variables are therefore $(r, \psi, \theta, R, \Psi, \Theta)$, where three of these variables are constant: $\theta = \Omega$, $\Psi = r^2\,\dot{\psi} = h$ and $\Theta = h' = h\,\cos i$.
The Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left[R^2 + \frac{\Psi^2}{r^2}\right] - \frac{\mu}{r}$$

does not explicitly depend on t and is thus constant as a function of time $\mathcal{H} = \varepsilon = Const(t)$.

3.2.1 The Delaunay canonical elements

Using the Hamilton-Jacobi equations, we seek now a new set of canonical variables $(q_1, q_2, q_3, p_1, p_2, p_3)$ such that all parameters are constant, except for one that we choose to be the time $q_1 = t$. This condition is achieved if the new Hamiltonian $\mathcal{H}' = p_1 = \varepsilon$. Indeed, in that case $\dot{q_1} = \frac{\partial \mathcal{H}'}{\partial p_1} = 1$.
The generating function $G(r, \psi, \theta, p_1, p_2, p_3)$ thus has to obey to the following equation:

$$dG = R\,dr + \Psi\,d\psi + \Theta\,d\theta + q_1\,dp_1 + q_2\,dp_2 + q_3\,dp_3$$

with

$$p_1 = \mathcal{H} = \frac{1}{2}\left[\left(\frac{\partial G}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial G}{\partial \psi}\right)^2\right] - \frac{\mu}{r}$$

Since ψ, θ and Θ do not appear in the expression of $\mathcal{H}$, their conjugated variables Ψ, Θ and θ are constant. We can thus choose G in such a way to preserve these properties: $q_3 = \frac{\partial G}{\partial p_3} = \theta = \Omega$, $p_2 = \frac{\partial G}{\partial \psi} = \Psi = h$ and $p_3 = \frac{\partial G}{\partial \theta} = \Theta = h'$. An easy way to fulfil these conditions is to adopt

$$G = h\,\psi + h'\,\Omega + G'(r, -, -, \varepsilon, h, -)$$

where $G'(r, -, -, \varepsilon, h, -)$ must satisfy the equation

$$\varepsilon - \frac{1}{2}\left(\frac{\partial G'}{\partial r}\right)^2 - \frac{h^2}{2\,r^2} + \frac{\mu}{r} = 0$$

The new variables are thus $(q_1, q_2, \Omega, \varepsilon, h, h')$.
We find that

$$\frac{\partial G'}{\partial r} = \pm\sqrt{2\,\varepsilon - \frac{h^2}{r^2} + \frac{2\,\mu}{r}}$$

which yields:

$$G'(r, -, -, \varepsilon, h, -) = \pm\int_{r_0}^{r}\sqrt{2\,\varepsilon + \frac{2\,\mu}{s} - \frac{h^2}{s^2}}\,ds \tag{3.10}$$

Since we know that $q_1 = \frac{\partial G}{\partial p_1} = \frac{\partial G'}{\partial \varepsilon} = t - t_0$, we obtain

$$\frac{\partial G'}{\partial \varepsilon} = \pm\int_{r_0}^{r}\frac{ds}{\sqrt{2\,\varepsilon + \frac{2\mu}{s} - \frac{h^2}{s^2}}} \mp \frac{\partial\, r_0}{\partial \varepsilon}\sqrt{2\,\varepsilon + \frac{2\,\mu}{r_0} - \frac{h^2}{r_0^2}} = t - t_0$$

t_0 corresponds to the time when $r = r_0$ provided that $\sqrt{2\,\varepsilon + \frac{2\,\mu}{r_0} - \frac{h^2}{r_0^2}} = 0$ which implies that

$$r_0 = \frac{h^2/\mu}{1 \pm \sqrt{1 + 2\,\varepsilon\, h^2/\mu^2}}$$

Moreover, since $\dot{r} = R = \frac{\partial G'}{\partial r} = \sqrt{2\,\varepsilon + \frac{2\mu}{r} - \frac{h^2}{r^2}}$, we find that $\dot{r} = 0$ for $r = r_0$. This implies that t_0 corresponds to an extremum of r and in the following we shall adopt

$$r_0 = \frac{h^2/\mu}{1 + \sqrt{1 + 2\,\varepsilon\, h^2/\mu^2}} \tag{3.11}$$

i.e. r_0 is the pericenter of the orbit and t_0 is hence the time of pericenter passage.

On the other hand, $\frac{\partial \mathcal{H}'}{\partial p_2} = \frac{\partial \mathcal{H}'}{\partial h} = \dot{q_2} = 0$, which implies that $q_2 = \omega = Cst$. As a result, we can write, using eq. 3.10:

$$q_2 = \frac{\partial G}{\partial p_2} = \psi + \frac{\partial G'}{\partial h} = \psi \mp \int_{r_0}^{r} \frac{h\, ds}{s^2\sqrt{2\,\varepsilon + \frac{2\,\mu}{s} - \frac{h^2}{s^2}}} \mp \frac{\partial\, r_0}{\partial h}\sqrt{2\,\varepsilon + \frac{2\,\mu}{r_0} - \frac{h^2}{r_0^2}} = \omega$$

Since $\sqrt{2\,\varepsilon + \frac{2\,\mu}{r_0} - \frac{h^2}{r_0^2}} = 0$, the term that multiplies $\frac{\partial\, r_0}{\partial h}$ vanishes and we thus obtain that $\psi = \omega$ at $t = t_0$ and

$$\psi - \omega = \pm \int_{r_0}^{r} \frac{h\, ds}{s^2\sqrt{2\,\varepsilon + \frac{2\,\mu}{s} - \frac{h^2}{s^2}}}$$

which (by means of an ancillary variable $u = 1/r$) yields

$$r = \frac{h^2/\mu}{1 + \sqrt{1 + \frac{2\,\varepsilon\, h^2}{\mu^2}}\,\cos\left(\psi - \omega\right)} \tag{3.12}$$

We recover here the result that the trajectory is a conic of eccentricity $e = \sqrt{1 + 2\,\varepsilon\, h^2/\mu^2}$ as found in the previous chapter. The canonical elements of the orbit are therefore $(t - t_0, \omega, \Omega, \varepsilon, h, h')$. The last five quantities are all constant.

What remains to be done is to establish the link between $t - t_0$ and r. For this purpose, we consider the case of the elliptical motion, which implies $\varepsilon = -\frac{\mu}{2\,a}$ and $h = \sqrt{\mu\, a\,(1 - e^2)}$ (since $r_0 = a\,(1 - e)$). Therefore,

$$q_1 = t - t_0 = \sqrt{\frac{a}{\mu}} \int_{r_0}^{r} \frac{s\, ds}{\sqrt{e^2\, a^2 - (s - a)^2}}$$

Defining $\cos E = \frac{a - r}{a\, e}$, we recover Kepler's equation:

$$M = \sqrt{\frac{\mu}{a^3}}\,(t - t_0) = E - e\,\sin E \tag{3.13}$$

Finally, we can substitute $t - t_0$ by the mean anomaly M, provided that the conjugated variable ε is replaced by $L = \sqrt{\mu\, a}$. We thus obtain the Delaunay canonical elements:

$$(l, g, \theta, L, G, \Theta) = (M, \omega, \Omega, \sqrt{\mu\, a}, \sqrt{\mu\, a\,(1 - e^2)}, \sqrt{\mu\, a\,(1 - e^2)}\,\cos i) \tag{3.14}$$

3.2.2 The Poincaré canonical elements

The main problem of the Delaunay canonical elements concerns situations where either the eccentricity or the orbital inclination or both are small. In these cases, it is more advantageous to use the set of canonical variables introduced by Poincaré. These are

$$\begin{pmatrix} \Lambda \\ \xi \\ p \\ \lambda \\ \eta \\ q \end{pmatrix} = \begin{pmatrix} L \\ \sqrt{2\,(L - G)}\,\cos\left(g + \theta\right) \\ \sqrt{2\,(G - \Theta)}\,\cos\theta \\ l + g + \theta \\ -\sqrt{2\,(L - G)}\,\sin\left(g + \theta\right) \\ -\sqrt{2\,(G - \Theta)}\,\sin\theta \end{pmatrix} \tag{3.15}$$

Chapter 4

Expanding the elliptical motion in series of e

The goal of this chapter is to establish some useful expressions of important quantities of the elliptical solution of the two-body problem in series of increasing powers of e. These formulae will be extremely useful when studying the impact of various perturbations on the motion.
First of all, we start by noting that the important quantities $x' = r\cos\phi = a(\cos E - e)$, $y' = r\sin\phi = a\sqrt{1-e^2}\sin E$, $r = \frac{a(1-e^2)}{1+e\cos\phi} = a(1 - e\cos E)$, $M = E - e\sin E$ as well as $\sin p E$ and $\cos p E$ are all periodic functions of ϕ, E and M. Hence these quantities can all be expressed by Fourier series.

4.1 Expanding the elliptical motion in Fourier series

Any periodic function $f(u)$ of period 2π can be expressed by a Fourier series:

$$f(u) = \frac{a_0}{2} + \sum_{k=1}^{+\infty}(a_k\cos(k\,u) + b_k\sin(k\,u))$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\cos(k\,u)\,du \qquad k = 0, 1, 2, ...$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\sin(k\,u)\,du \qquad k = 1, 2, ...$$

If $f(u)$ is

- an even function of u, then $b_k = 0 \quad \forall k = 1, 2, ...$;
- an odd function of u, then $a_k = 0 \quad \forall k = 0, 1, 2, ...$

For instance, the quantity $\frac{a}{r} = \frac{dE}{dM} = \frac{1}{1-e\cos E}$ is an even function of E, M and ϕ. Hence

$$\begin{aligned}\frac{a}{r} &= \frac{1}{2\pi}\int_0^{2\pi}\frac{a}{r}\,dM' + \sum_{k=1}^{+\infty}\left(\frac{2}{\pi}\int_0^{\pi}\frac{a}{r}\cos(k\,M')\,dM'\right)\cos(k\,M)\\ &= 1 + \sum_{k=1}^{+\infty}\left(\frac{2}{\pi}\int_0^{\pi}\cos[k\,(E - e\sin E)]\,dE\right)\cos(k\,M)\end{aligned}$$

By virtue of eq. 4.34, the coefficients of the Fourier development are equal to $J_k(k\,e)$, the Bessel function of the first kind.

$$\frac{a}{r} = 1 + \sum_{k=1}^{+\infty} 2\,J_k(k\,e)\,\cos(k\,M) \tag{4.1}$$

From $dE = \frac{a}{r}\,dM$, it follows immediately that

$$E = M + \sum_{k=1}^{+\infty} \frac{2}{k}\,J_k(k\,e)\,\sin(k\,M) \tag{4.2}$$

and Kepler's equation then yields

$$\sin E = \frac{E-M}{e} = \sum_{k=1}^{+\infty} \frac{2}{k\,e}\,J_k(k\,e)\,\sin(k\,M) \tag{4.3}$$

We can generalize the latter result for $\cos(p\,E)$ and $\sin(p\,E)$ where p is a non-zero integer number. For this purpose, we start by noting that[1] $\exp(j\,p\,E) = \cos(p\,E) + j\,\sin(p\,E)$ and

$$dM = (1 - e\,\cos E)\,dE = (1 - e\,\frac{\exp(j\,E) + \exp(-j\,E)}{2})\,dE$$

where $j^2 = -1$. Hence, the complex function $\exp(j\,p\,E)$ can be expressed as a Fourier series like

$$\exp(j\,p\,E) = \sum_{k=-\infty}^{+\infty} c_k\,\exp(j\,k\,M)$$

where the coefficients are complex numbers given by

$$c_k = \frac{1}{2\,\pi}\int_{-\pi}^{\pi} \exp(j\,p\,E)\,\exp(-j\,k\,M')\,dM' \qquad k = 0, 1, 2, ...$$

$$\begin{aligned}
c_k &= \frac{1}{2\,\pi}\int_{-\pi}^{\pi} \exp(j\,p\,E)\exp[-j\,k\,(E - e\,\sin E)]\,(1 - \frac{e}{2}\,(\exp(j\,E) + \exp(-j\,E))\,dE \\
&= \frac{1}{2\,\pi}\int_{-\pi}^{\pi} \exp(j[k\,e\,\sin E - (k-p)\,E])\,dE - \frac{e}{4\,\pi}\int_{-\pi}^{\pi} \exp(j[(p+1-k)\,E + k\,e\,\sin E])\,dE \\
&\quad - \frac{e}{4\,\pi}\int_{-\pi}^{\pi} \exp(j[(p-1-k)\,E + k\,e\,\sin E])\,dE \\
&= J_{k-p}(k\,e) - \frac{e}{2}(J_{k-p-1}(k\,e) + J_{k-p+1}(k\,e))
\end{aligned}$$

This then yields $c_0 = J_{-p}(0) - \frac{e}{2}(J_{-p-1}(0) + J_{1-p}(0))$, which implies that for integer values of p, $c_0 = \delta_{p0} - \frac{e}{2}\,\delta_{p1}$. On the other hand, for $k \neq 0$, one finds using eq. 4.32 that

$$c_k = J_{k-p}(k\,e) - \frac{k-p}{k} J_{k-p}(k\,e) = \frac{p}{k} J_{k-p}(k\,e)$$

Hence,

$$\exp(j\,p\,E) = \delta_{p0} - \frac{e}{2}\,\delta_{p1} + \sum_{k=1}^{+\infty}\left(\frac{p}{k}\,J_{k-p}(k\,e)\,\exp(j\,k\,M) - \frac{p}{k}\,J_{-k-p}(-k\,e)\,\exp(-j\,k\,M)\right)$$

[1] To avoid confusion, we use the notation i for the inclination angle, whilst j as part of the argument of an exponential function stands for the imaginary unit ($j^2 = -1$).

which finally leads to

$$\cos(p\,E) = \delta_{p0} - \frac{e}{2}\,\delta_{p1} + \sum_{k=1}^{+\infty} \frac{p}{k}(J_{k-p}(k\,e) - J_{k+p}(k\,e))\,\cos(k\,M) \tag{4.4}$$

$$\sin(p\,E) = \sum_{k=1}^{+\infty} \frac{p}{k}(J_{k-p}(k\,e) + J_{k+p}(k\,e))\,\sin(k\,M) \tag{4.5}$$

In particular, using eqs. 4.33 and 4.32, we derive the following results from the above relations:

$$\frac{r}{a}\,\cos\phi = \cos E - e = -\frac{3\,e}{2} + \sum_{k=1}^{+\infty} \frac{1}{k}(J_{k-1}(k\,e) - J_{k+1}(k\,e))\,\cos(k\,M)$$

$$\frac{r}{a}\,\cos\phi = -\frac{3\,e}{2} + \sum_{k=1}^{+\infty} \frac{2}{k}\,J'_k(k\,e)\,\cos(k\,M) \tag{4.6}$$

$$\frac{r}{a}\,\sin\phi = \sqrt{1-e^2}\,\sin E = \sqrt{1-e^2}\sum_{k=1}^{+\infty} \frac{1}{k}(J_{k-1}(k\,e) + J_{k+1}(k\,e))\,\sin(k\,M)$$

$$\frac{r}{a}\,\sin\phi = \sqrt{1-e^2}\sum_{k=1}^{+\infty} \frac{2}{k\,e}\,J_k(k\,e)\,\sin(k\,M) \tag{4.7}$$

and eventually,

$$\frac{r}{a} = 1 - e\,\cos E = 1 + \frac{e^2}{2} - 2\,e\sum_{k=1}^{+\infty} \frac{1}{k}\,J'_k(k\,e)\,\cos(k\,M) \tag{4.8}$$

The latter result shows that the average value of r/a over a full orbital cycle is equal to $1 + e^2/2$ rather than being 1. Therefore a should not be interpreted as the average value of r for an eccentric orbit.

In some applications, it is important to evaluate the constant term a_0 of the Fourier expansion. For instance, if we consider $\left(\frac{a}{r}\right)^3\,\cos(p\,\phi)$, we find that in this case

$$a_0 = \frac{1}{\pi}\int_0^{2\,\pi} \left(\frac{a}{r}\right)^3\,\cos(p\,\phi)\,dM$$

Since $\frac{d\,M}{dE} = \frac{r}{a}$ and $\frac{d\,E}{d\phi} = \frac{r}{a\,\sqrt{1-e^2}}$, we obtain

$$a_0 = \frac{1}{\pi}\int_0^{2\,\pi} \frac{a}{r\,\sqrt{1-e^2}}\,\cos(p\,\phi)\,d\phi$$

hence

$$a_0 = \frac{1}{\pi}\int_0^{2\,\pi} \frac{1 + e\,\cos\phi}{(1-e^2)^{3/2}}\,\cos(p\,\phi)\,d\phi = \frac{1}{(1-e^2)^{3/2}}\,(2\,\delta_{p0} + e\,\delta_{p1})$$

These constant terms play an important role in the perturbation theory that we will introduce in the forthcoming chapters.

4.2 The d'Alembert characteristics

The Fourier series established in the previous section are absolutely convergent. In addition, the majority of the orbits of major bodies in the Solar System as well as the orbits of the majority of the artificial satellites have rather modest eccentricities. As a result, we can consider that e is often small and one can wonder whether it is possible to express the relations found in the previous section as convergent series of e by expanding the Bessel functions appearing in these relations into series of e.

For this purpose, we first stress that the value of $J_k(x)$ is of the order of $x^{|k|}$ as becomes clear from eq. 4.29. In the expressions of $\cos{(p\,E)}$ and $\sin{(p\,E)}$ inferred in the previous section, we found that the terms in $\cos{(k\,M)}$ and $\sin{(k\,M)}$ are multiplied by Bessel functions of index $k \pm p$. Hence, these terms are of order $\min{(|k-p|, |k+p|)}$ in e and the dominant term in all these series is always the one corresponding to J_0, which means the one for which $k = p$; the neighbouring terms $k = p \pm 1$, $p \pm 2$,... being of order 1, 2, ...

A Fourier series is said to possess the d'Alembert characteristics of degree p, if the dominant term of this series corresponds to the pth harmonics. This is the case of the Fourier developments of $\cos{(p\,E)}$ and $\sin{(p\,E)}$ presented above. Such a series can then be written as

$$S_p(e, M) = \sum_{k=-\infty}^{+\infty} e^{|p-k|}\, s_k(e)\, \exp{(j\,k\,M)}$$

where $s_k(e)$ is a coefficient of order 0 in e.

In the forthcoming section, we will thus expand the Bessel functions into Taylor series of e. Inserting these results into the Fourier series and making use of the d'Alembert characteristics, we then obtain a new formulation in terms of a series of increasing powers of e. Generally speaking, the new expansions obtained in this way are asymptotic series that are absolutely converging only for values of e below a threshold of about 0.66.

4.3 Development into asymptotic series of e

As stressed above, if e is small, we can develop the Bessel functions into series of e and insert these developments into the Fourier expansions introduced in section 4.1. If e is small enough, these asymptotic series can be truncated while still preserving a good accuracy.

For this purpose, let us first establish the expressions of the Bessel functions of the first kind and their derivatives. Here we restrict ourselves to the expression of order 6 for $J_s(s\,e)$ and order 5 for $J'_s(s\,e)$:

$$J_s(s\,e) = \sum_{i=0}^{+\infty} \frac{(-1)^i}{i!\,(i+s)!} \left(\frac{s\,e}{2}\right)^{2i+s} \tag{4.9}$$

$$J_1(e) = \frac{e}{2} - \frac{e^3}{16} + \frac{e^5}{384} + \mathcal{O}(e^7) \tag{4.10}$$

$$J_2(2\,e) = \frac{e^2}{2} - \frac{e^4}{6} + \frac{e^6}{48} + \mathcal{O}(e^8) \tag{4.11}$$

$$J_3(3\,e) = \frac{9\,e^3}{16} - \frac{81\,e^5}{256} + \mathcal{O}(e^7) \tag{4.12}$$

$$J_4(4\,e) = \frac{2\,e^4}{3} - \frac{8\,e^6}{15} + \mathcal{O}(e^8) \tag{4.13}$$

$$J_5(5\,e) = \frac{625\,e^5}{768} + \mathcal{O}(e^7) \tag{4.14}$$

$$J_6(6\,e) = \frac{81\,e^6}{80} + \mathcal{O}(e^8) \tag{4.15}$$

$$
\begin{aligned}
J'_s(s\,e) &= \sum_{i=0}^{+\infty} \frac{(-1)^i\,(2\,i+s)}{2\,(i!\,(i+s)!)} \left(\frac{s\,e}{2}\right)^{2i+s-1} && (4.16)\\
J'_1(e) &= \frac{1}{2} - \frac{3\,e^2}{16} + \frac{5\,e^4}{384} + \mathcal{O}(e^6) && (4.17)\\
J'_2(2\,e) &= \frac{e}{2} - \frac{e^3}{3} + \frac{e^5}{16} + \mathcal{O}(e^7) && (4.18)\\
J'_3(3\,e) &= \frac{9\,e^2}{16} - \frac{135\,e^4}{256} + \mathcal{O}(e^6) && (4.19)\\
J'_4(4\,e) &= \frac{2\,e^3}{3} - \frac{4\,e^5}{5} + \mathcal{O}(e^7) && (4.20)\\
J'_5(5\,e) &= \frac{625\,e^4}{768} + \mathcal{O}(e^6) && (4.21)\\
J'_6(6\,e) &= \frac{81\,e^5}{80} + \mathcal{O}(e^7) && (4.22)
\end{aligned}
$$

Another useful expansion concerns $\sqrt{1-e^2}$ which can be expressed as

$$\sqrt{1-e^2} = 1 - \frac{e^2}{2} - \frac{e^4}{8} - \frac{e^6}{16} - \frac{5\,e^8}{128} + \mathcal{O}(e^{10}) \tag{4.23}$$

Combining the above results with the Fourier expansions established in section 4.1, we then obtain the following asymptotic series:

$$
\begin{aligned}
\frac{r}{a} &= 1 + \frac{e^2}{2} - (e - \frac{3\,e^3}{8} + \frac{5\,e^5}{192})\,\cos M - (\frac{e^2}{2} - \frac{e^4}{3})\cos{(2\,M)} - (\frac{3\,e^3}{8} - \frac{45\,e^5}{128})\,\cos{(3\,M)}\\
&\quad - \frac{e^4}{3}\,\cos{(4\,M)} - \frac{125\,e^5}{384}\,\cos{(5\,M)} + \mathcal{O}(e^6) && (4.24)
\end{aligned}
$$

$$
\begin{aligned}
\frac{a}{r} &= 1 + (e - \frac{e^3}{8} + \frac{e^5}{192})\,\cos M + (e^2 - \frac{e^4}{3})\cos{(2\,M)} + (\frac{9\,e^3}{8} - \frac{81\,e^5}{128})\,\cos{(3\,M)}\\
&\quad + \frac{4\,e^4}{3}\,\cos{(4\,M)} + \frac{625\,e^5}{384}\,\cos{(5\,M)} + \mathcal{O}(e^6) && (4.25)
\end{aligned}
$$

$$
\begin{aligned}
\frac{r}{a}\,\cos\phi &= -\frac{3\,e}{2} + (1 - \frac{3\,e^2}{8} + \frac{5\,e^4}{192})\,\cos M + (\frac{e}{2} - \frac{e^3}{3} + \frac{e^5}{16})\cos{(2\,M)} + (\frac{3\,e^2}{8} - \frac{45\,e^4}{128})\,\cos{(3\,M)}\\
&\quad + (\frac{e^3}{3} - \frac{2\,e^5}{5})\,\cos{(4\,M)} + \frac{125\,e^4}{384}\,\cos{(5\,M)} + \frac{27\,e^5}{80}\,\cos{(6\,M)} + \mathcal{O}(e^6) && (4.26)
\end{aligned}
$$

$$
\begin{aligned}
\frac{r}{a}\,\sin\phi &= (1 - \frac{5\,e^2}{8} - \frac{11\,e^4}{192})\,\sin M + (\frac{e}{2} - \frac{5\,e^3}{12} + \frac{e^5}{24})\sin{(2\,M)} + (\frac{3\,e^2}{8} - \frac{51\,e^4}{128})\,\sin{(3\,M)}\\
&\quad + (\frac{e^3}{3} - \frac{13\,e^5}{30})\,\sin{(4\,M)} + \frac{125\,e^4}{384}\,\sin{(5\,M)} + \frac{27\,e^5}{80}\,\sin{(6\,M)} + \mathcal{O}(e^6) && (4.27)
\end{aligned}
$$

4.4 Appendix: the Bessel functions of the first kind

The Bessel functions of the first kind $J_\alpha(x)$ are solutions of Bessel's differential equation

$$x^2\,\frac{d^2\,f(x)}{dx^2} + x\,\frac{d\,f(x)}{dx} + (x^2 - \alpha^2)\,f(x) = 0 \tag{4.28}$$

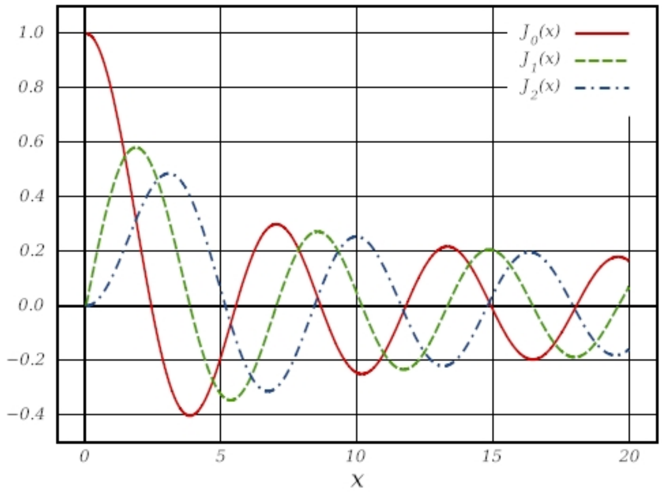


Figure 4.1: Plot of the Bessel functions of the first kind for increasing orders s = 0, 1 and 2.

that are finite at $x = 0$ for non-negative integer values of α, where α is called the order of the Bessel function. For integer order $\alpha = s$, the Bessel functions can be written as

$$J_s(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+s)!} \left(\frac{x}{2}\right)^{2m+s} \tag{4.29}$$

The generating function of the Bessel functions of the first kind is

$$\exp\,[\frac{x}{2}\,(t - \frac{1}{t})] = \sum_{s=-\infty}^{+\infty} J_s(x)\,t^s$$

This generating function can be used to derive some interesting and useful properties of the Bessel functions of the first kind:

$$J_s(x) = (-1)^s\,J_{-s}(x) \tag{4.30}$$

$$J_s(x) = (-1)^s\,J_s(-x) \tag{4.31}$$

$$s\,J_s(x) = [J_{s-1}(x) + J_{s+1}(x)]\,\frac{x}{2} \tag{4.32}$$

$$J_s'(x) = \frac{d\,J_s(x)}{dx} = \frac{1}{2}\,(J_{s-1}(x) - J_{s+1}(x)) \tag{4.33}$$

Another important result can be obtained by setting $t = \exp\,(j\,\psi)$ in the generating function of the Bessel functions of the first kind. In fact, in this way, we find that $t - 1/t = 2\,j\,\sin\psi$. Hence, for p an integer, we find that

$$\int_0^{2\,\pi} \exp\,(j\,x\,\sin\psi)\,\exp\,(-j\,p\,\psi)\,d\psi = \sum_{s=-\infty}^{+\infty} J_s(x)\int_0^{2\,\pi} \exp\,(j\,s\,\psi)\,\exp\,(-j\,p\,\psi)\,d\psi$$

$$\int_0^{2\,\pi} \exp\,[j(\,x\,\sin\psi - p\,\psi)]\,d\psi = \sum_{s=-\infty}^{+\infty} J_s(x)\,2\,\pi\delta_{sp}$$

This result yields

$$J_s(x) = \frac{1}{2\,\pi}\int_0^{2\,\pi} \exp\,[j(\,x\,\sin\psi - s\,\psi)]\,d\psi$$

which can finally be transformed into

$$J_s(x) = \frac{1}{\pi}\int_0^{\pi} \cos\,(s\,\psi - x\,\sin\psi)\,d\psi \tag{4.34}$$

4.5 Exercises

4.1 Establish that the Fourier series in M, limited to e^1, of $\left(\frac{a}{r}\right)^3 \sin 2\,\phi$ is given by

$$\sin 2\,M + e\,(\frac{7}{2}\,\sin 3\,M - \frac{1}{2}\,\sin M) + \mathcal{O}(e^2)$$

Suggestion: express $\left(\frac{a}{r}\right)$, $\sin\phi$ and $\cos\phi$ in terms of the eccentric anomaly, then expand $(1 - e\,\cos E)^{-5}$ up to order e^1. Finally, take into account that $\sin a\,\cos b = \frac{1}{2}\,(\sin{(a+b)} + \sin{(a-b)})$, and use equation 4.5 and the expressions of the Bessel functions 4.10 – 4.15.

4.2 Demonstrate that the independent term of the Fourier series in M of the function $\left(\frac{a}{r}\right)^4 \cos 2\,\phi$ is given by:

$$\frac{a_0}{2} = \frac{e^2}{4\,(1-e^2)^{5/2}}$$

4.3 Calculate the value of

$$\frac{1}{\pi}\int_0^{2\,\pi}\left(\frac{a}{r}\right)^5 \cos p\,\phi\,dM$$

where ϕ and M are the true and mean anomalies of an elliptical trajectory, respectively, and p is an integer number that is positive or zero.
Suggestion: make use of the relation $\frac{d\,M}{d\phi} = \frac{r^2}{a^2\,(1-e^2)^{1/2}}$ and perform the integration with respect to the variable ϕ.

Chapter 5

The Forces acting on a body in space

The forces that impact the motion of a body in space are the gravitational attraction (by a planet or other body that cannot be considered as a point-like mass), the radiation pressure (mainly from the Sun) and the drag force due to the residual atmosphere (for an artificial satellite orbiting a planet or moon with an atmosphere). In most applications, the fact that the planet is not a point-like mass, the atmospheric drag and the radiation pressure effectively lead to forces that are small compared to the main term of the gravitational force and can hence be treated as perturbations.

5.1 The gravitational potential

As highlighted in the previous chapters, the gravitational potential at a point P in space, produced by a point-like mass m located at O is simply expressed as

$$U(P) = -\frac{G\,m}{|\vec{OP}|}$$

For a number of point-like masses m_i located at O_i, the potential becomes

$$U(P) = -\sum_i \frac{G\,m_i}{|\vec{O_iP}|}$$

The acceleration, that a test mass positioned in $P \neq O_i$ undergoes, is given by

$$\vec{g} = -\vec{\nabla}\,U(P)$$

For a single mass m, $\vec{g} = -\frac{G\,m}{r^3}\,\vec{r}$ where $\vec{r} = \vec{OP}$. As a result,

$$\vec{\nabla}\cdot\vec{g} = 0$$

We thus obtain the Laplace equation:

$$\Delta\,U(P) = 0 \quad \forall P \neq O \tag{5.1}$$

which can be generalized in the case of several point-like masses:

$$\Delta\,U(P) = 0 \quad \forall P \neq O_i \tag{5.2}$$

Now, if we consider the situation at the points O_i, we can generalize these results to:

$$\Delta\,U(O_i) = -4\,\pi\,G\,m_i \tag{5.3}$$

Let us now consider a continuous distribution of mass, characterized by a density ρ such that in any point Q within the volume occupied by this distribution of matter $dm = \rho(Q)\,dV$. It can be shown that in this case, $\Delta\,U$ follows Poisson's equation

$$\Delta\,U(P) = -4\,\pi\,G\,\rho(P) \tag{5.4}$$

where $\rho(P) = 0$ outside the volume occupied by the distribution of mass. If we consider the special case of a spherical distribution S of mass, we find that, by symmetry, the acceleration $\vec{g}(P) = -\int_{Q\in S} \frac{G\,\vec{QP}}{|\vec{QP}|^3}\,dm$ is a radial vector with a norm equal to $g(r) = \frac{G\,M}{r^2}$ with M being the mass contained inside the sphere of radius r. This is fully equivalent to the situation of a point-like mass.

We now consider the situation for a body having a non-spherical distribution of mass. To address this topic, we express the potential $U(P)$ in spherical coordinates (θ being the latitude and λ the longitude) as:

$$U(P) = U(r,\theta,\lambda) = \frac{-G\,M}{r}\sum_{n=0}^{\infty}\frac{W_n(\theta,\lambda)}{r^n}$$

It is obvious that the deviations from a spherical symmetry should not impact on the potential for distances that are much larger than the dimensions of the body itself. Hence, at very large distances, we must recover the result valid for a point-like mass. This implies that $W_0 = 1$. To satisfy Poisson's equation at any place outside the distribution of mass, one must have

$$\Delta\left(\frac{W_n(\theta,\lambda)}{r^{n+1}}\right) = 0$$

The Laplace operator in spherical coordinates[1] can be written as

$$\Delta\,U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\,U}{\partial r}\right) + \frac{1}{r^2\,\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial\,U}{\partial\theta}\right) + \frac{1}{r^2\,\cos^2\theta}\frac{\partial^2\,U}{\partial\lambda^2}$$

and this leads to the following equation:

$$n\,(n+1)\,W_n(\theta,\lambda) + \frac{1}{\cos^2\theta}\frac{\partial^2\,W_n(\theta,\lambda)}{\partial\lambda^2} + \frac{\partial^2\,W_n(\theta,\lambda)}{\partial\theta^2} - \tan\theta\,\frac{\partial\,W_n(\theta,\lambda)}{\partial\theta} = 0 \tag{5.5}$$

Whatever the shape of the mass distribution, $W_n(\theta,\lambda)$ is a function of period $2\,\pi$ of the longitude (λ) and can thus be written as

$$W_n(\theta,\lambda) = \sum_{p=-\infty}^{+\infty} Q_n^{(p)}(\theta)\,\exp\left(j\,p\,\lambda\right)$$

If we insert this expression into equation 5.5, we obtain the differential equation

$$\sum_{p=-\infty}^{+\infty}\left\{[n\,(n+1) - \frac{p^2}{\cos^2\theta}]\,Q_n^{(p)}(\theta) + \frac{\partial^2\,Q_n^{(p)}(\theta)}{\partial\theta^2} - \tan\theta\,\frac{\partial\,Q_n^{(p)}(\theta)}{\partial\theta}\right\}\exp\left(j\,p\,\lambda\right) = 0 \tag{5.6}$$

Since $\exp\left(j\,p\,\lambda\right)$ are orthogonal functions for different values of p, the term between curly braces in equation 5.6 must be equal to zero whatever the value of p.

$$[n\,(n+1) - \frac{p^2}{\cos^2\theta}]\,Q_n^{(p)}(\theta) + \frac{\partial^2\,Q_n^{(p)}(\theta)}{\partial\theta^2} - \tan\theta\,\frac{\partial\,Q_n^{(p)}(\theta)}{\partial\theta} = 0 \tag{5.7}$$

[1] We remind the reader that θ stands for the **latitude** not the co-latitude.

If we introduce the ancillary variable $s = \sin\theta$, the equation becomes:

$$[n\,(n+1) - \frac{p^2}{1-s^2}]\,Q_n^{(p)}(s) - 2\,s\,\frac{\partial\,Q_n^{(p)}(s)}{\partial s} + (1-s^2)\,\frac{\partial^2\,Q_n^{(p)}(s)}{\partial s^2} = 0 \tag{5.8}$$

This is the general Legendre differential equation, whose solutions are the associated Legendre functions (see Sect. 5.4): $Q_n^{(p)}(\theta) = P_n^{(p)}(s)$.
The potential at any position outside the body can thus be written as

$$U = -\frac{G\,M}{r}\sum_{n=0}^{+\infty}\sum_{p=0}^{n}\frac{1}{r^n}P_n^{(p)}(\sin\theta)\,[c_{np}\,\cos{(p\,\lambda)} + s_{np}\,\sin{(p\,\lambda)}] \tag{5.9}$$

where the c_{np} and s_{np} coefficients depend on the distribution of the mass inside the body. Obviously to recover the potential of a spherical mass distribution at large distances r, one must have $c_{00} = 1$ and $s_{00} = 0$. In addition, it can be shown that $c_{1p} = s_{1p} = 0$ if the origin of the axes is chosen to be the center of gravity of the mass distribution. Hence, the most commonly used expression of the potential is

$$U = -\frac{G\,M}{r}\left(1 + \sum_{n=2}^{+\infty}\left(\frac{R_e}{r}\right)^n\left\{-J_n\,P_n(\sin\theta) + \sum_{p=1}^{n}P_n^{(p)}(\sin\theta)\,[c_{np}\,\cos{(p\,\lambda)} + s_{np}\,\sin{(p\,\lambda)}]\right\}\right) \tag{5.10}$$

where R_e is the equatorial radius of the planet. In principle, the different coefficients in this development can all be calculated analytically provided that the distribution of mass is accurately known (see Sect. 5.5). In practice, such a detailed knowledge of the mass distribution is not available and one rather determines the values of the coefficients indirectly from the observation of their effects on artificial satellites (see the next chapter).
The terms corresponding to $p = 0$ (coefficients J_n) and $p = n$ (coefficients c_{nn} and s_{nn}) are called zonal and sectoral terms respectively, whilst those for $0 < p < n$ are called tesseral.
The dominating non-spherical term in this development is J_2. For a solid spheroid rotating about its axis of symmetry, one can show that $J_2 = \frac{2\,(R_e - R_p)}{3\,R_e} - \frac{\omega^2\,R_e^3}{3\,G\,M}$, where R_e, R_p and ω are respectively the equatorial radius, the polar radius and the angular rotational velocity. J_2 is thus directly related to the equatorial flattening of the rotating body.
The values of the most important coefficients are given in Table 5.1. Note that the values of J_2 are quite large for Jupiter and Saturn as a result of their important equatorial flattening.
For some minor bodies (asteroids or comets) with rather complex shapes, the above expansion of the potential sometimes fails to represent the actual potential for small values of r (i.e. near the surface of the object). It should be stressed that the Earth is not a solid body and changes its shape periodically as a result of the tides produced by the differential attraction of the Sun and the Moon. Note also that the tidal deformation of the surface of the Earth alters the position of ground stations and this needs to be accounted for in the tracking of artificial satellites.
From these expansions of the potential, we can draw some important conclusions:

- J_2 being usually much smaller than 1, at large distances, the potential of a planet is very well represented by the potential of a point-like mass;

- the force acting on a test mass $\vec{\nabla}U = \left(\frac{\partial\,U}{\partial r}, \frac{1}{r\,\cos\theta}\,\frac{\partial\,U}{\partial\lambda}, \frac{1}{r}\,\frac{\partial\,U}{\partial\theta}\right)$ is no longer radial. A satellite orbiting a non-spherical body in a low orbit will thus experience a series of perturbations in its motion compared to a pure Keplerian orbit.

Table 5.1: The most important coefficients in the expansion of the potential of some important planets of the Solar System and the Moon.

	Earth	Mars	Jupiter	Saturn	Moon
GM (m^3 s^{-2})	$3.986\,10^{14}$	$4.283\,10^{13}$	$1.267\,10^{17}$	$3.793\,10^{16}$	$4.903\,10^{12}$
R_e (km)	6378	3397	71398	60000	1738
J_2	0.001083	0.001964	0.01475	0.01645	0.000203
c_{22}	$1.57\,10^{-6}$	$-5.5\,10^{-5}$			$2.23\,10^{-5}$
s_{22}	$-0.90\,10^{-6}$	$3.1\,10^{-5}$			
J_3	$-2.53\,10^{-6}$	$3.6\,10^{-5}$			$6\,10^{-6}$
c_{31}	$2.19\,10^{-6}$				$2.9\,10^{-5}$
s_{31}	$0.27\,10^{-6}$	$2.6\,10^{-5}$			$4\,10^{-6}$
J_4	$-1.62\,10^{-6}$		$-5.8\,10^{-4}$	$-1.0\,10^{-3}$	

5.2 The drag force due to the residual atmosphere

An object moving across the rarefied layers of the upper atmosphere of a planet experiences collisions with atoms and molecules that make up this atmosphere (see Fig. 5.1). The mean free path of the atmospheric particles is usually much longer than the typical dimensions of the moving object.
Let us consider an object of mass m moving at velocity $\dot{\vec{r}}$ across a medium of density ρ moving itself at a bulk velocity $\vec{v_p}$. Let $\vec{v_r} = \dot{\vec{r}} - \vec{v_p}$ be the relative velocity. If $S(t)$ is the cross-section of the moving body perpendicular to the body's velocity relative to the atmosphere, then the volume of the atmosphere crossed by the object in a time interval dt is given by $dV = S(t)\,|\vec{v_r}|\,dt$. The collisions between the particles and the object alter the momentum of both the particles and the object, but preserve the total momentum:

$$m\,d\dot{\vec{r}} = -\sum_p m_p\,d\vec{v_p}$$

In the latter relation, the sum encompasses all the particles inside the volume dV. The change in the particle velocity $d\vec{v_p}$ depends upon the nature of the surface of the body and upon the angle of incidence of the particles in the collision. If the particles arrive at an angle α (relative to the direction of the normal to $S(t)$) and are simply reflected off the surface of the body, then $|d\vec{v_p}| = |\vec{v_r}|\,(1 + \cos(2\,\alpha))$. Note that the angle α is not the same all over the body, especially if it has a somewhat complex shape (e.g. Fig 5.1).
As a result, the drag force experienced by the body can be expressed as

$$\vec{F_D} = -\frac{C_D}{2}\,S(t)\,\rho\,|\vec{v_r}|\,\vec{v_r} \tag{5.11}$$

where C_D is the drag coefficient that characterises the aerodynamic properties of the body in the direction of $\vec{v_r}$. Since $\vec{v_p}$ is often very small compared to $\dot{\vec{r}}$, it is quite common to consider that the atmospheric drag acts as a force opposed to the motion of the body. Obviously the atmospheric density plays a key role. For most planetary atmospheres ρ and thus also F_D decrease exponentially with altitude. In the case of the Earth, a spacecraft orbiting at an altitude of 250 km experiences a 1000 times larger atmospheric drag than the same spacecraft orbiting at 800 km altitude.
In practice, the use of formula 5.11 to predict the motion of a body is not straightforward: the drag coefficient is usually not well known and $S(t)$ depends on the orientation of the body with respect to the atmosphere and, last but not least, ρ is usually only poorly known, depends on many external parameters and changes in a rather complex and unpredictable way with the Solar activity.

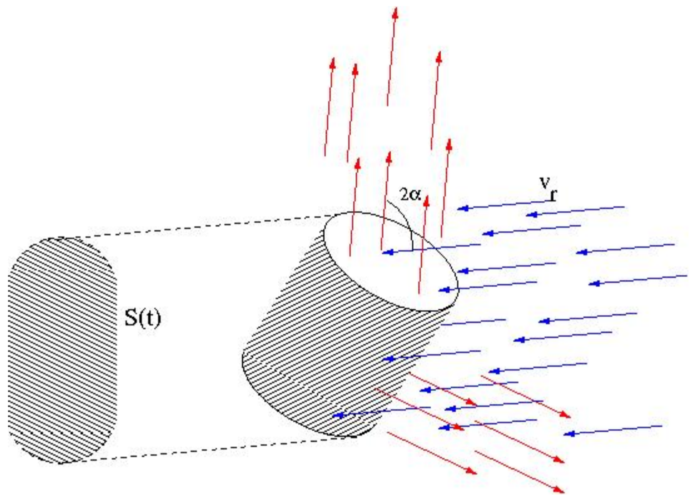


Figure 5.1: Schematic representation of the interaction between a solid body and the particles of a rarefied atmosphere.

If we compare the effect of the drag force to the gravitational force, we find that

$$\frac{|\vec{F_D}|\, r^2}{G\,M\,m} = \frac{C_D}{2\,G\,M}\frac{S}{m}\, r^2\,\rho\,|\vec{v_r}|^2$$

If we assume a circular orbit and if we neglect $\vec{v_p}$ compared to $\dot{\vec{r}}$, then $|\vec{v_r}|^2 = G\,M/r$, hence

$$\frac{|\vec{F_D}|\, r^2}{G\,M\,m} = \frac{C_D}{2}\frac{S}{m}\, r\,\rho$$

We conclude that the ratio S/m is an important quantity. A compact object with a small area and a large mass experiences much less drag than a low-density body.

5.3 The radiation pressure

From a similar reasoning as for the atmospheric drag, we find that the force due to radiation pressure can be expressed as

$$\vec{F_R} = -\frac{C_R\,I}{c}\,S\,\vec{e_R} \tag{5.12}$$

Here, C_R is a coefficient that accounts for the reflectivity of the body, I is the intensity (the power per unit area) of the light received from a direction $\vec{e_R}$ (i.e. the direction from the satellite towards the light source), c is the speed of light and S the surface that is lit by the light source. Note that the light reflected off the surface of the planet must also be accounted for (usually using an albedo coefficient).

For orbits of spacecraft about a minor body of the Solar System (typically an asteroid), the gravitational attraction by the central body (which has frequently a complex shape, deviating strongly from spherical symmetry) is rather low and can be of comparable importance to the effect of radiation pressure (especially in the case of an asteroid revolving around the Sun within 1 AU). The corresponding orbits are complex and deviate from simple Keplerian orbits. Their stability strongly depends upon the initial conditions (the initial orbit radius about the asteroid must be within a specific range to ensure stability). However, it has to be stressed that solar radiation pressure actually helps to stabilize the orbit! Indeed, in these cases, the radiation pressure tends to force the orbital plane to remain perpendicular to the direction between the Sun and the asteroid, thereby producing a kind of heliosynchronous orbit.

Finally, we emphasize that there exist some additional perturbation forces that affect the orbits of artificial satellites. These are

- electrostatic forces, especially for satellites crossing the radiation belts.
- the differential attraction by the Moon, the Sun and other planets. We will address this issue in the chapter dealing with the N-body problem.
- relativistic effects;
- ...

Except for the differential attraction by other major bodies, most of the effects listed above are actually very small compared to those discussed in this chapter.

5.4 Appendix: the associated Legendre functions

The solutions of the general Legendre differential equation

$$[n\,(n+1) - \frac{p^2}{1-x^2}]\,P_n^{(p)}(x) - 2\,x\,\frac{d\,P_n^{(p)}(x)}{dx} + (1-x^2)\,\frac{d^2\,P_n^{(p)}(x)}{dx^2} = 0 \tag{5.13}$$

for $p \geq 0$ are the so-called associated Legendre functions

$$P_n^{(p)}(x) = \frac{(1-x^2)^{p/2}}{2^n\,n!}\,\frac{d^{n+p}}{dx^{n+p}}(x^2-1)^n \tag{5.14}$$

These functions are obviously zero if $p > n$. For $p = 0$, the associated Legendre functions become actually associated Legendre polynomials $P_n(x)$.

$$P_n(x) = \frac{1}{2^n\,n!}\,\frac{d^n}{dx^n}(x^2-1)^n \tag{5.15}$$

The generating function of the Legendre polynomials can be expressed as

$$\frac{1}{\sqrt{1-2\,x\,t+t^2}} = \sum_{n=0}^{+\infty} P_n(x)\,t^n \tag{5.16}$$

A remarkable property of the Legendre polynomials is that they are mutually orthogonal:

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \frac{2\,\delta_{nm}}{2\,n+1} \tag{5.17}$$

One can verify that there exist two reccurrence relations that allow to compute the higher order associated Legendre functions:

$$(n+1-p)\,P_{n+1}^{(p)}(x) - (2\,n+1)\,x\,P_n^{(p)}(x) + (n+p)\,P_{n-1}^{(p)}(x) = 0 \tag{5.18}$$

$$P_n^{(p+2)}(x) - \frac{2\,(p+1)\,x}{\sqrt{1-x^2}}\,P_n^{(p+1)}(x) + (n-p)\,(n+p+1)\,P_n^{(p)}(x) = 0 \tag{5.19}$$

Using these recurrence relations, one finds that the first associated Legendre functions are:

$P_0^{(0)}(x) = 1$
$P_1^{(0)}(x) = x \qquad P_1^{(1)}(x) = \sqrt{1-x^2}$
$P_2^{(0)}(x) = \frac{3\,x^2}{2} - \frac{1}{2} \qquad P_2^{(1)}(x) = 3\,x\,\sqrt{1-x^2} \qquad P_2^{(2)} = 3\,(1-x^2)$
$P_3^{(0)}(x) = \frac{5\,x^3}{2} - \frac{3\,x}{2} \qquad P_3^{(1)}(x) = (\frac{15\,x^2}{2} - \frac{3}{2})\,\sqrt{1-x^2} \qquad P_3^{(2)} = 15\,x\,(1-x^2) \qquad P_3^{(3)}(x) = 15\,(1-x^2)^{3/2}$

5.5 Appendix: the relation between J_n, c_{np}, s_{np} and the moments of inertia

In this chapter, we have shown that the potential at any position outside the body can be written as

$$U = -\frac{G\,M}{r}\left(1 + \sum_{n=2}^{+\infty}\left(\frac{R_e}{r}\right)^n\left\{-J_n\,P_n(\sin\theta) + \sum_{p=1}^{n} P_n^{(p)}(\sin\theta)\,[c_{np}\,\cos{(p\,\lambda)} + s_{np}\,\sin{(p\,\lambda)}]\right\}\right) \tag{5.20}$$

where the J_n, c_{np} and s_{np} coefficients depend on the shape of the body and the distribution of the mass in its interior.
We can express some of these coefficients in terms of the moments of inertia of the body (see also the forthcoming Chapter 8). These moments are the elements of a matrix $\mathcal{I}$ of dimension 3×3 defined by $\mathcal{I}_{ij} = -\int(x_i\,x_j)\,\rho\,dV$ for $i,j=1,2,3$ and $i\neq j$ and $\mathcal{I}_{ii} = \sum_{j\neq i,j=1}^{3}\int x_j^2\,\rho\,dV$.
Let us start by noting that the potential can be expressed by

$$U = -G\int_{V'}\frac{\rho\,dV'}{|\vec{r}-\vec{r'}|} \tag{5.21}$$

with

$$|\vec{r}-\vec{r'}| = (r^2 + r'^2 - 2\,r\,r'\,\cos\delta)^{1/2} \tag{5.22}$$

and

$$\frac{1}{|\vec{r}-\vec{r'}|} = \frac{1}{r}\sum_{n=0}^{+\infty} P_n(\cos\delta)\,\left(\frac{r'}{r}\right)^n \tag{5.23}$$

If we limit the development to $n=2$, we obtain

$$\frac{1}{|\vec{r}-\vec{r'}|} = \frac{1}{r}\left(1 + \frac{r'}{r}\,\cos\delta + \left(\frac{r'}{r}\right)^2\left(\frac{3}{2}\,\cos^2\delta - \frac{1}{2}\right)\right) \tag{5.24}$$

In spherical coordinates centred on the center of mass and where θ is the latitude, we can write

$$\vec{r} = r\,(\cos\theta\,\cos\lambda, \cos\theta\,\sin\lambda, \sin\theta)$$

whilst the coordinates of point P' are (x', y', z') in a conventional cartesian frame of reference. Since $\cos\delta = \frac{\vec{r}\cdot\vec{r'}}{r\,r'}$, the potential U becomes

$$\begin{aligned} U &= -\frac{G}{r}\left[\int_{V'}\rho\,dV' + \frac{1}{r^2}\,\vec{r}\cdot\int_{V'}\rho\,\vec{r'}\,dV' + \int_{V'}\rho\,\left(\frac{r'}{r}\right)^2\left(\frac{3}{2}\,\cos^2\delta - \frac{1}{2}\right)\,dV'\right] \\ &= -\frac{G\,M}{r} - \frac{G}{r}\int_{V'}\rho\left[\frac{3\,(x'^2\,\cos^2\theta\,\cos^2\lambda + y'^2\,\cos^2\theta\,\sin^2\lambda + z'^2\,\sin^2\theta) - (x'^2+y'^2+z'^2)}{2\,r^2}\right. \\ &\quad \left. + \frac{3\,(x'\,y'\,\cos^2\theta\,\cos\lambda\,\sin\lambda + x'\,z'\,\sin\theta\,\cos\theta\,\cos\lambda + y'\,z'\,\sin\theta\,\cos\theta\,\sin\lambda)}{r^2}\right]\,dV' \end{aligned} \tag{5.25}$$

$$
\begin{aligned}
&= -\frac{G\,M}{r} + \frac{G}{r^3}\,\frac{3\,\sin^2\theta - 1}{4}\int_{V'}\rho\,(x'^2 + y'^2 - 2\,z'^2)\,dV' - \frac{G}{r^3}\,3\,\cos^2\theta\,\cos(2\,\lambda)\int_{V'}\rho\,\frac{x'^2 - y'^2}{4}\,dV' \\
&\quad -\frac{G}{r^3}\,3\,\cos^2\theta\,\sin(2\,\lambda)\int_{V'}\rho\,\frac{x'\,y'}{2}\,dV' - \frac{G}{r^3}\,\frac{3}{2}\,\sin 2\,\theta\left[\cos\lambda\int_{V'}\rho\,x'\,z'\,dV' + \sin\lambda\int_{V'}\rho\,y'\,z'\,dV'\right] \\
&= -\frac{G\,M}{r} - \frac{G}{r^3}\left[P_2(\sin\theta)\,\frac{\mathcal{I}_{xx} + \mathcal{I}_{yy} - 2\,\mathcal{I}_{zz}}{2} + P_2^{(1)}(\sin\theta)\,(\mathcal{I}_{xz}\,\cos\lambda + \mathcal{I}_{yz}\,\sin\lambda)\right. \\
&\quad \left. + P_2^{(2)}(\sin\theta)\left(\frac{\mathcal{I}_{yy} - \mathcal{I}_{xx}}{4}\,\cos(2\,\lambda) + \frac{\mathcal{I}_{xy}}{2}\,\sin(2\,\lambda)\right)\right] \qquad (5.26)
\end{aligned}
$$

Now, comparing expressions 5.20 and 5.26 of U, we obtain the expressions of the coefficients of the potential as a function of the moments of inertia of the body of mass M:

$$J_2 = -\frac{1}{M\,R_e^2}\int_{V'}\rho\,\frac{2\,z'^2 - x'^2 - y'^2}{2}\,dV' = \frac{2\,\mathcal{I}_{zz} - \mathcal{I}_{xx} - \mathcal{I}_{yy}}{2\,M\,R_e^2} \qquad (5.27)$$

$$c_{21} = \frac{\mathcal{I}_{xz}}{M\,R_e^2} \qquad (5.28)$$

$$s_{21} = \frac{\mathcal{I}_{yz}}{M\,R_e^2} \qquad (5.29)$$

$$c_{22} = \frac{\mathcal{I}_{yy} - \mathcal{I}_{xx}}{4\,M\,R_e^2} \qquad (5.30)$$

$$s_{22} = \frac{\mathcal{I}_{xy}}{2\,M\,R_e^2} \qquad (5.31)$$

In addition, we directly obtain from this procedure that $c_{00} = 1$, $s_{00} = J_1 = c_{10} = s_{10} = c_{11} = s_{11} = 0$ if the origin of the axes is chosen (as we have done here) to be the center of mass.

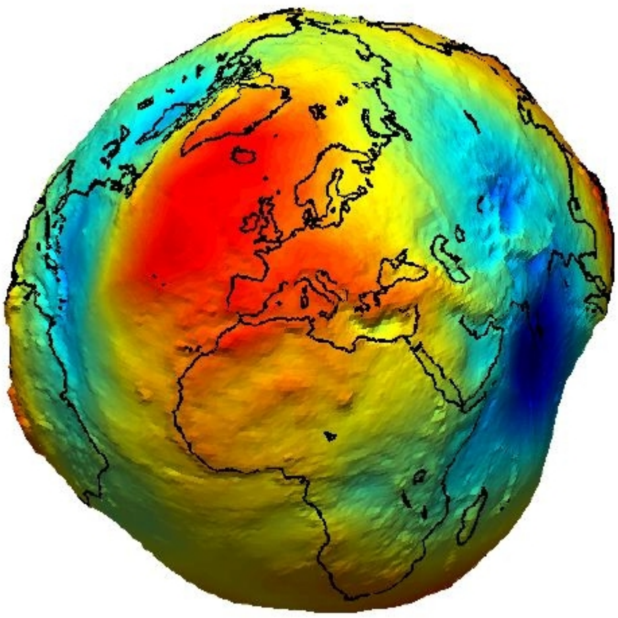


Figure 5.2: Schematic (highly exaggerated) representation of the elevation of the Earth's geoid with respect to the reference spheroid. The maximum deviations are -106 m (dark blue) and $+85$ m (red). Image credit: GOCE mission (ESA).

As shown by equation 5.21, the gravity field of a planet obviously depends upon the distribution of the material in the planet's interior. In the case of the Earth, the shape of the planet can be approximated to first order by a spheroid with an equatorial radius of 6378.137 km and a polar radius of 6356.752 km. However, for accurate computations in celestial mechanics, there are many terms of the spherical harmonics development that need to be taken into account. For instance, the Earth Gravity Model 1996 (EGM96) geoid contains coefficients to degree and order $n = 360$. Even higher frequency terms (due to mountains, trenches,...) are measured by the latest generation of Earth observation satellites such as Grace (NASA) and GOCE (ESA).

5.5.1 The potential of a spheroid

For a body with rotational symmetry about the z' axis, the c_{np} and s_{np} coefficients are zero, and equation 5.20 simplifies into

$$U = -\frac{G\,M}{r}\left(1 - \sum_{n=2}^{+\infty}\left(\frac{R_e}{r}\right)^n J_n\,P_n(\sin\theta)\right) \tag{5.32}$$

As we have seen in the previous section, the potential can also be expressed as a combination of equations 5.21 and 5.23:

$$U = -G\int_{V'}\frac{\rho\,dV'}{|\vec{r}-\vec{r'}|} = \frac{-G}{r}\int_{V'}\sum_{n=0}^{+\infty}P_n(\cos\delta)\left(\frac{r'}{r}\right)^n\rho\,dV' \tag{5.33}$$

Hence

$$\begin{aligned} -\frac{G\,M}{r}\left(1 - \sum_{n=2}^{+\infty}\left(\frac{R_e}{r}\right)^n J_n\,P_n(\sin\theta)\right) &= -\frac{G}{r}\int_{V'}\sum_{n=0}^{+\infty}P_n(\cos\delta)\left(\frac{r'}{r}\right)^n\rho\,dV' \\ \Rightarrow M\left(1 - \sum_{n=2}^{+\infty}\left(\frac{R_e}{r}\right)^n J_n\,P_n(\sin\theta)\right) &= \sum_{n=0}^{+\infty}\int_{V'}P_n(\cos\delta)\left(\frac{r'}{r}\right)^n\rho\,dV' \end{aligned} \tag{5.34}$$

In this relation, δ is the angle between $\vec{r}$, the position vector of the point where we evaluate the potential, and $\vec{r'}$, the position vector of an element of mass inside the body.
We can now evaluate relation 5.34 at an arbitrary position on the symmetry axis (i.e. on the z' axis), but outside the body. For such a position we have $\theta = \frac{\pi}{2}$ on the left of relation 5.34 and $\delta = \frac{\pi}{2} - \theta'$ on the right. Since $P_n(1) = 1$, and remembering that $dV' = r'^2\,\cos\theta'\,d\phi'\,d\theta'\,dr'$, we obtain

$$\begin{aligned} M\left(1 - \sum_{n=2}^{+\infty}\left(\frac{R_e}{r}\right)^n J_n\right) &= \sum_{n=0}^{+\infty}\int_{V'}P_n(\sin\theta')\left(\frac{r'}{r}\right)^n\rho\,dV' \\ &= 2\,\pi\sum_{n=0}^{+\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}P_n(\sin\theta')\,\cos\theta'\left(\int_0^{R(\theta')}\left(\frac{r'}{r}\right)^n\rho(r',\theta')\,r'^2\,dr'\right)d\theta' \end{aligned}$$

where $R(\theta')$ is the radius at latitude θ'. Since the different powers of $\frac{1}{r}$ are linearly independent, we must have

$$J_n = -\frac{2\,\pi\,R_e^3}{M}\int_{-\pi/2}^{\pi/2}P_n(\sin\theta')\,\cos\theta'\left(\int_0^{R(\theta')}\left(\frac{r'}{R_e}\right)^{2+n}\rho(r',\theta')\,\frac{dr'}{R_e}\right)d\theta' \tag{5.35}$$

If we call R_m the mean radius of the body, which differs from R_e by a quantity proportional to J_2, we can finally write:

$$J_n = -\frac{2\,\pi\,R_m^3}{M}\int_{-\pi/2}^{\pi/2}P_n(\sin\theta')\,\cos\theta'\left(\int_0^{R(\theta')}\left(\frac{r'}{R_m}\right)^{2+n}\rho(r',\theta')\,\frac{dr'}{R_m}\right)d\theta' \tag{5.36}$$

This is the general expression of the zonal coefficients. We note that equation 5.36 remains valid also for a body that does not have rotational symmetry. Indeed the same reasoning as above holds in this case since $P_n^{(p)}(1) = 0$, implying that a test position on the z' axis does not feel the influence of sectoral and tesseral terms, regardless of the actual values of the c_{np} and s_{np} coefficients.

For now, let us come back to the case of a celestial body with axial symmetry. A special case of such a configuration is the spheroid approximation. A spheroid is obtained by rotating an ellipse about its major or minor axis. The radius of the spheroid is given by

$$R(\theta) = R_m \left(1 - \frac{2\,\epsilon}{3}\, P_2(\sin\theta)\right) \tag{5.37}$$

where $\epsilon = \frac{R_e - R_p}{R_m}$ is the ellipticity, which we assume to be a small quantity. R_e, R_p and R_m are the equatorial, polar and mean radii, respectively. This leads to $R_p = R_m \left(1 - \frac{2\epsilon}{3}\right)$ and $R_e = R_m \left(1 + \frac{\epsilon}{3}\right)$. If $\epsilon > 0$, the spheroid is said to be oblate, whereas it is prolate if $\epsilon < 0$.
If we use the spherical coordinates as defined above with θ being the latitude, we obtain that

$$\frac{2\,z^2 - x^2 - y^2}{2} = r^2\, P_2(\sin\theta)$$

From equations 5.27 and 5.36, we thus derive that

$$\begin{aligned} J_2 &= -\frac{1}{M\,R_e^2} \int_V \rho\, \frac{2\,z^2 - x^2 - y^2}{2}\, dV \\ &= -\frac{2\,\pi}{M\,R_e^2} \int_{-\pi/2}^{\pi/2} P_2(\sin\theta)\, \cos\theta\, (\int_0^{R(\theta)} \rho\, r^4\, dr)\, d\theta \end{aligned} \tag{5.38}$$

If the spheroid has a uniform density, such that $\rho = \frac{3\,M}{4\,\pi\,R_m^3}$, the latter expression simplifies to

$$J_2 = -\frac{3}{10\,R_e^2\,R_m^3} \int_{-\pi/2}^{\pi/2} P_2(\sin\theta)\, \cos\theta\, R(\theta)^5\, d\theta$$

which can be expressed to first order in ϵ as

$$J_2 \simeq -\frac{3}{10} \int_{-\pi/2}^{\pi/2} P_2(\sin\theta)\, \cos\theta\, [1 - \frac{10}{3}\, \epsilon\, P_2(\sin\theta)]\, d\theta$$

Given the orthogonality of the Legendre polynomials (see equation 5.17), we eventually derive that

$$J_2 \simeq \frac{2\,\epsilon}{5} \tag{5.39}$$

Hence, we conclude that, to first order in ϵ, the potential created outside a spheroid of uniform density can be expressed as

$$U = -\frac{G\,M}{r} \left(1 - \frac{J_2\,R_e^2}{r^2}\, P_2(\sin\theta)\right) + \mathcal{O}(\epsilon^2) \tag{5.40}$$

The classical Keplerian term is also called the monopole gravitational potential, whilst the term in J_2 is often called the quadrupole gravitational potential.

If the spheroid does not rotate, then the potential on its surface can be expressed

$$U(R(\theta)) = -\frac{G\,M}{R(\theta)} \left(1 - \frac{J_2\,R_e^2}{R(\theta)^2}\, P_2(\sin\theta)\right) \simeq -\frac{G\,M}{R_m} \left[1 + \left(\frac{2\,\epsilon}{3} - J_2\right)\, P_2(\sin\theta)\right] \tag{5.41}$$

Note that this result is valid also if the density of the spheroid is not uniform. A non-rotating spheroidal fluid body of mass M is said to be self-gravitating if it is in equilibrium, which implies that $U(R(\theta))$ must be constant over the surface (otherwise there would be tangential forces acting on the surface). This condition implies that the surface potential must be independent of θ. From equation 5.41, we see that this equilibrium condition implies $\epsilon = 0$. Hence a uniform density self-gravitating fluid in equilibrium adopts a spherical shape.
For a body to behave as a fluid, the internal pressure needs to be larger than some threshold value. This condition translates into a limit on the radius and hence the mass of the body. Hence stars and planets have essentially spherical equilibrium shapes, whereas minor bodies (asteroids, comets, small moons,...) with mean radii of less than about 250 km can have very complex, non-spherical, shapes. Deviations from such equilibrium shapes can arise for different reasons such as additional internal forces (other than gravity), centrifugal forces due to rotation (see below), or tides due to an orbiting mass (see Sect. 8.5).

If the object is rotating as a solid body (i.e. with a uniform angular velocity ω), expression 5.41 needs to be corrected for the centrifugal term

$$-\frac{1}{2}\,\omega^2\,R(\theta)^2\,\cos^2(\theta) = -\frac{1}{3}\omega^2\,R(\theta)^2\,\left(1 - P_2(\sin\theta)\right)$$

The potential on the surface of a rotating spheroid hence becomes

$$U(R(\theta)) \simeq -\frac{G\,M}{R_m}\left[1 + \left(\frac{2\,\epsilon}{3} - J_2\right)\,P_2(\sin\theta)\right] - \frac{\omega^2\,R_m^2}{3}\,\left(1 - P_2(\sin\theta)\right) \tag{5.42}$$

Now, considering a self-gravitating rotating fluid spheroidal body in equilibrium, we again need to have a potential that is independent of θ. Hence, we obtain the condition

$$J_2 = \frac{2\,\epsilon}{3} - \frac{\omega^2\,R_m^3}{3\,G\,M} \tag{5.43}$$

5.6 Exercises

5.1 An object moves under the influence of a central force of the form

$$f(r) = -\frac{\mu}{r^2} + \frac{c}{r^3}$$

where μ and c are positive constants. Show that the orbit can be expressed as

$$r = \frac{a\,(1 - e^2)}{1 + e\,\cos\left(\lambda\phi + \phi_0\right)}$$

If $e < 1$, show that $\lambda = \sqrt{\frac{1-e^2}{1-e^2-\frac{c}{\mu\,a}}}$. (Adapted from Fitzpatrick 2012).

5.2 Two satellites are in orbit about the Earth. The first one moves on a circular orbit of radius r_1. We define a system of Cartesian axes, corotating with the first satellite, with the x axis pointing from the first satellite to the center of the Earth, the y axis being tangent to the first satellite's orbit and pointing in the opposite sense to the motion of this satellite. The z axis hence points along the direction of the $\vec{\omega_1}$ vector, i.e. the angular velocity vector of the first satellite. At time zero, the second satellite is located at $x = 0$, $y = A$, where $A << r_1$, and moves with a velocity that is aligned with the $\vec{e_x}$ vector. We neglect the gravitational interaction between the two satellites. Let (x_2, y_2) be the coordinates of the second satellite. Establish the differential equations of the motion:

$$\begin{aligned}\ddot{x}_2 &= 3\,\omega_1^2\,x_2 + 2\,\omega_1\,\dot{y}_2 \\ \ddot{y}_2 &= -2\,\omega_1\,\dot{x}_2\end{aligned}$$

Demonstrate that the second satellite moves on an elliptical orbit about the first satellite. Show that the motion is retrograde and that the major axis is aligned with the y direction and is twice as long as the minor axis. Discuss the implications of this result on the possibilities to perform formation flight (i.e. to have two satellites moving with a constant separation and fixed attitude) in low-Earth orbit.

5.3 Show that for a self-gravitating, rotating, fluid, spheroidal body whose density varies with radius as $r^{-\alpha}$ ($\alpha < 3$), the ellipticity is given by

$$\epsilon = \frac{5\,\omega^2\,R_m^3}{(4+2\,\alpha)\,G\,M}$$

What value of α is needed to explain the observed ellipticity of Jupiter ($\epsilon = 0.065$, $R_e = 7.1398\,10^7$ m, $\omega = 1.77\,10^{-4}\,\mathrm{rad\,s^{-1}}$, $G\,M_{♃} = 1.267\,10^{17}\,\mathrm{m^3\,s^{-2}}$)? (Adapted from Fitzpatrick 2012).

Chapter 6

Perturbations of the Keplerian motion

As we have seen in the previous chapter, there are a number of forces, other than the point-like gravitational interaction, that can influence the motion of a body in space.
Let us consider the equation of motion

$$\ddot{\vec{r}} = -\frac{\mu \vec{r}}{r^3} + \vec{P}(\vec{r}, \dot{\vec{r}}, t) \tag{6.1}$$

If the force $\vec{P} = \vec{0}$, we are left with a pure Keplerian motion. As discussed before, we can represent the Keplerian motion by the combination of a conical section and a hodograph in a 6 dimensional space consisting of $\vec{r}$ and $\dot{\vec{r}}$ and a seventh dimension actually consisting of the value of μ. Conversely, we have shown that any Keplerian orbit can also be described by the elements of the orbit: e.g. $(\vec{h}, \vec{l}, t_0, \mu)$. In the latter formulation, the orbit is actually represented by a single point in a 6 (+ 1) dimensional space.
If the force $\vec{P} \neq \vec{0}$, the trajectory is no longer Keplerian. However, at any given moment in time t, we can define the **osculating orbit** as the Keplerian orbit that is tangent to the actual orbit at time t. In this case, the orbital elements of the osculating orbit change with time and the representation of the elements of the orbit in the 6 dimensional space is no longer a single point.

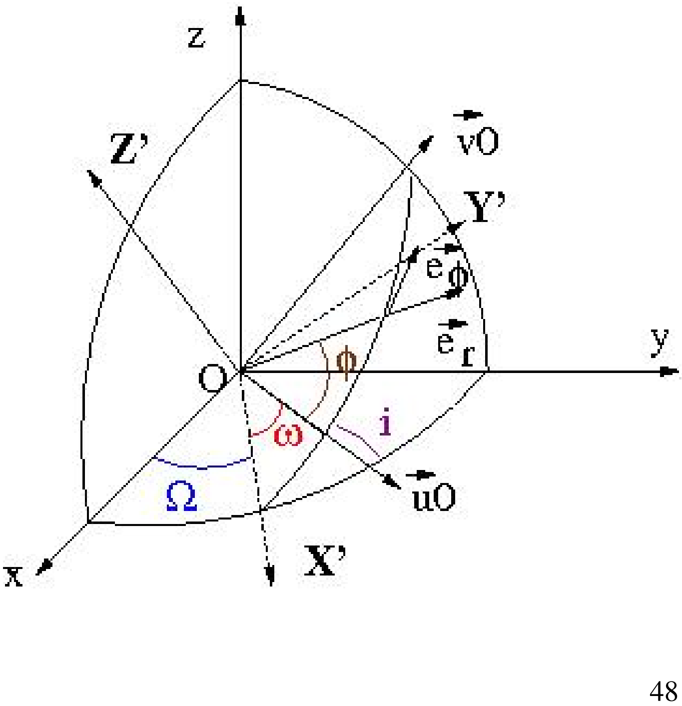


Figure 6.1: The various systems of axes and coordinates used in the formulations in this chapter.

6.1 The method of the variation of parameters and the Gauss equations

For a Keplerian motion, we can express the position and velocity[1] at any moment in time by relations of the kind

$$x_j = x_j(a, e, i, \Omega, \omega, t_0, t) \tag{6.2}$$

$$u_j = u_j(a, e, i, \Omega, \omega, t_0, t) \tag{6.3}$$

where $u_j = \frac{d\,x_j}{dt} = \dot{x}_j$ and $\dot{u}_j = -\mu \frac{x_j}{r^3} + P_j(x_j, u_j, t)$.
Now, if we deal with the osculating orbit, we can write:

$$\frac{d\,x_j}{dt} = \frac{\partial\,x_j}{\partial a}\frac{d\,a}{dt} + \frac{\partial\,x_j}{\partial e}\frac{d\,e}{dt} + ... + \frac{\partial\,x_j}{\partial t_0}\frac{d\,t_0}{dt} + \frac{\partial\,x_j}{\partial t} = u_j$$

$$\frac{d\,u_j}{dt} = \frac{\partial\,u_j}{\partial a}\frac{d\,a}{dt} + \frac{\partial\,u_j}{\partial e}\frac{d\,e}{dt} + ... + \frac{\partial\,u_j}{\partial t_0}\frac{d\,t_0}{dt} + \frac{\partial\,u_j}{\partial t} = -\frac{\mu\,x_j}{r^3} + P_j(x_j, u_j, t)$$

where $u_j = \frac{\partial x_j}{\partial t}$ and $\frac{\partial\,u_j}{\partial t} = -\frac{\mu\,x_j}{r^3}$ for the osculating orbit (that must satisfy the equations of the Keplerian motion). This then leads to

$$\frac{\partial\,x_j}{\partial a}\frac{d\,a}{dt} + \frac{\partial\,x_j}{\partial e}\frac{d\,e}{dt} + ... + \frac{\partial\,x_j}{\partial t_0}\frac{d\,t_0}{dt} = 0 \tag{6.4}$$

$$\frac{\partial\,u_j}{\partial a}\frac{d\,a}{dt} + \frac{\partial\,u_j}{\partial e}\frac{d\,e}{dt} + ... + \frac{\partial\,u_j}{\partial t_0}\frac{d\,t_0}{dt} = P_j(x_j, u_j, t) \tag{6.5}$$

The partial derivatives in these relations need to be calculated from the equations 6.2 and 6.3, valid for a Keplerian motion. The system of equations 6.4 and 6.5 is inverted to yield the derivatives of a, e,..., and t_0.

The same results can be obtained through a somewhat different approach. In fact, let $\frac{\delta\,X}{\delta t}$ be the variation of any quantity X as a result of the non-Keplerian part of the force acting on the mass. With this notation, any time derivative can be written as the sum of a Keplerian derivative plus a non-Keplerian component:

$$\frac{d\,X}{dt} = \left(\frac{d\,X}{dt}\right)_{\text{Kepler}} + \frac{\delta\,X}{\delta t}$$

Since the osculating motion is defined such that the instantaneous values of $\vec{r}$ and $\dot{\vec{r}}$ correspond to a tangential Keplerian motion, we find that $\frac{\delta\dot{\vec{r}}}{\delta t} = \vec{P}$ whilst $\frac{\delta\vec{r}}{\delta t} = \vec{0}$.
Let us now evaluate the impact of the non-Keplerian force on the angular momentum, the Laplace integral and the energy:

$$\frac{d\,\vec{h}}{dt} = \vec{r} \wedge \frac{\delta\dot{\vec{r}}}{\delta t} = \vec{r} \wedge \vec{P} \tag{6.6}$$

$$\mu\,\frac{d\,\vec{l}}{dt} = 2\,(\dot{\vec{r}} \cdot \vec{P})\,\vec{r} - (\dot{\vec{r}} \cdot \vec{r})\,\vec{P} - (\vec{r} \cdot \vec{P})\,\dot{\vec{r}} \tag{6.7}$$

$$\frac{d\,\varepsilon}{dt} = \dot{\vec{r}} \cdot \vec{P} \tag{6.8}$$

Note that these equations are valid whatever the nature (elliptical or hyperbolic) of the osculating Keplerian trajectory.

[1]Here the x_j and u_j, with $j = 1, 2$ or 3, stand for the cartesian components of the position and velocity vectors.

In Chapter 3, we have seen that the Poisson vector can be written

$$\vec{\Omega} = \dot{\Omega}\,\vec{e_z} + \frac{d\,i}{dt}\,\vec{e_{X'}} + \dot{\omega}\,\vec{e_{Z'}} = \frac{d\,i}{dt}\,\vec{e_{X'}} + \dot{\Omega}\,\sin i\,\vec{e_{Y'}} + (\dot{\Omega}\,\cos i + \dot{\omega})\,\vec{e_{Z'}}$$

and the derivatives of the angular momentum and the Laplace integral can also be expressed by

$$\frac{d\,\vec{h}}{dt} = \dot{h}\,\vec{e_{Z'}} + \vec{\Omega}\wedge\vec{h}$$

$$\frac{d\,\vec{l}}{dt} = \frac{d\,(e\,\vec{u_0})}{dt} = \dot{e}\,\vec{u_0} + \vec{\Omega}\wedge\vec{l}$$

This then leads to

$$\begin{aligned}
\dot{h} &= (\vec{r}\wedge\vec{P})\cdot\vec{e_{Z'}} = r\,\vec{e_\phi}\cdot\vec{P} && (6.9)\\
h\,\frac{d\,i}{dt} &= -(\vec{r}\wedge\vec{P})\cdot\vec{e_{Y'}} = r\,\cos(\phi+\omega)\,\vec{e_{Z'}}\cdot\vec{P} && (6.10)\\
h\,\dot{\Omega}\sin i &= (\vec{r}\wedge\vec{P})\cdot\vec{e_{X'}} = r\,\sin(\phi+\omega)\,\vec{e_{Z'}}\cdot\vec{P} && (6.11)\\
\mu\,\dot{e} &= 2\,(\dot{\vec{r}}\cdot\vec{P})\,(\vec{r}\cdot\vec{u_0}) - (\dot{\vec{r}}\cdot\vec{r})\,(\vec{P}\cdot\vec{u_0}) - (\vec{r}\cdot\vec{P})\,(\dot{\vec{r}}\cdot\vec{u_0}) && (6.12)\\
\mu\,e\,(\dot{\omega}+\dot{\Omega}\,\cos i) &= 2\,(\dot{\vec{r}}\cdot\vec{P})\,(\vec{r}\cdot\vec{v_0}) - (\dot{\vec{r}}\cdot\vec{r})\,(\vec{P}\cdot\vec{v_0}) - (\vec{r}\cdot\vec{P})\,(\dot{\vec{r}}\cdot\vec{v_0}) && (6.13)\\
\frac{\mu}{2\,a^2}\,\dot{a} &= \dot{\vec{r}}\cdot\vec{P} && (6.14)
\end{aligned}$$

Note that whilst equations 6.9 - 6.13 are valid regardless of the nature of the osculating orbit, in equation 6.14 we have considered the specific case of an elliptical osculating orbit, which we shall also consider in the following. Finally, the variation of M is somewhat more difficult to establish. It consists of two parts, the instantaneous Keplerian part $n(t)$ where

$$n(t) = n(t_0) - \int_{t_0}^{t}\frac{3}{2}\,\sqrt{\mu}\,a^{-5/2}\,\dot{a}\,dt' \tag{6.15}$$

and a non-Keplerian part which can be shown to be equal to

$$\frac{\delta\,M}{\delta t} = -\sqrt{1-e^2}\,(\dot{\omega}+\dot{\Omega}\,\cos i) - \frac{2\,r}{\sqrt{\mu\,a}}(\vec{P}\cdot\vec{e_r}) \tag{6.16}$$

Hence,

$$\dot{M} = n(t) - \sqrt{1-e^2}\,(\dot{\omega}+\dot{\Omega}\,\cos i) - \frac{2\,r}{\sqrt{\mu\,a}}(\vec{P}\cdot\vec{e_r}) \tag{6.17}$$

If we express the force $\vec{P}$ as $\vec{P} = R\,\vec{e_r} + T\,\vec{e_\phi} + W\,\vec{e_{Z'}}$ we can transform the above relations into

$$\begin{aligned}
\dot{a} &= \sqrt{\frac{a^3}{\mu}}\frac{2}{\sqrt{1-e^2}}\,[R\,(e\,\sin\phi) + T\,(1+e\,\cos\phi)] && (6.18)\\
\dot{e} &= \sqrt{\frac{a\,(1-e^2)}{\mu}}\,\{R\,\sin\phi + T\,[\cos\phi + \frac{r}{a\,(1-e^2)}\,(e+\cos\phi)]\} && (6.19)\\
\frac{d\,i}{dt} &= \frac{r\,\cos(\omega+\phi)}{\sqrt{a\,\mu\,(1-e^2)}}\,W && (6.20)\\
\dot{\Omega} &= \frac{r\,\sin(\omega+\phi)}{\sqrt{a\,\mu\,(1-e^2)}\,\sin i}\,W && (6.21)\\
\dot{\omega} &= \sqrt{\frac{a\,(1-e^2)}{\mu\,e^2}}\,[-R\,\cos\phi + T\,\sin\phi\,(1+\frac{r}{a\,(1-e^2)})] - \frac{r\,\sin(\omega+\phi)}{\sqrt{a\,\mu\,(1-e^2)}}\,\cot i\,W && (6.22)\\
\dot{M} &= n(t) - \frac{1-e^2}{e}\sqrt{\frac{a}{\mu}}\,[-R\,\cos\phi + T\,\sin\phi\,(1+\frac{r}{a\,(1-e^2)})] - \frac{2\,r}{\sqrt{a\,\mu}}\,R && (6.23)
\end{aligned}$$

These expressions are called the **Gauss equations for the osculating elements**. A first important conclusion is that only a force with a component perpendicular to the plane of the Keplerian orbit produces a variation of i or Ω.

6.1.1 The Gauss equations applied to the case of the atmospheric drag

As a first practical example, we consider the impact of a drag force as seen in the previous chapter. For such a force, we have $\vec{P} = -k\,\dot{\vec{r}}$, where $k = \frac{C_D\,S}{2\,m}\,\rho\,|\dot{\vec{r}}|$. This then leads to $R = -k\,\dot{r}$, $T = -k\,r\,\dot{\phi}$ and $W = 0$.
We conclude immediately that in this case $\frac{d\,i}{dt} = \dot{\Omega} = 0$. The fact that $\frac{d\,i}{dt} = 0$ has important consequences for debris from a spacecraft that fall back onto the Earth under the effect of atmospheric drag. Indeed, these debris will remain in the orbital plane of the original spacecraft and the latitudes that are potentially concerned by the debris are thus $-i \leq \theta \leq i$. For instance, the Chinese space station Tiangong 1 that fell back to Earth in 2018 had an orbital inclination of $42.8°$, implying that no debris could reach Belgium.
We further note that $|\dot{\vec{r}}| = \frac{n\,a}{\sqrt{1-e^2}}\,\sqrt{1+e^2+2\,e\,\cos\phi}$ with $n = \sqrt{\frac{\mu}{a^3}}$. This then leads to the following results:

$$\dot{a} = -C_D\,\frac{S}{m}\,\rho\,n\,a^2\,\left[\frac{1+e^2+2\,e\,\cos\phi}{1-e^2}\right]^{3/2} \tag{6.24}$$

$$\dot{e} = -C_D\,\frac{S}{m}\,\rho\,\frac{n\,a}{\sqrt{1-e^2}}\,\sqrt{1+e^2+2\,e\,\cos\phi}\,(e+\cos\phi) \tag{6.25}$$

$$\dot{\omega} = -C_D\,\frac{S}{m}\,\rho\frac{n\,a}{e\,\sqrt{1-e^2}}\,\sqrt{1+e^2+2\,e\,\cos\phi}\,\sin\phi \tag{6.26}$$

$$\dot{M} = n(t) + C_D\,\frac{S}{m}\,\rho\,n\,a\sqrt{1+e^2+2\,e\,\cos\phi}\,\frac{\sin\phi}{e}\frac{1+e^2+e\,\cos\phi}{1+e\,\cos\phi} \tag{6.27}$$

The net effect of the drag force is therefore to reduce the value of the semi-major axis and to reduce the eccentricity of the orbit.
It has to be stressed that the density ρ of the atmosphere varies with time and with altitude. In the case of a satellite orbiting the Earth, the time dependence of ρ stems from the Earth's rotation (due to the heating by the solar radiation, a maximum is observed roughly two hours after local noon and a minimum about two hours after local midnight) and from the solar activity (the latter results in an irregular variability due to flares, a roughly periodic modulation due to the solar rotation period and the visibility of active regions at the solar surface as well as a long-term modulation with the solar cycle). The dependence of the density on the altitude can be expressed as $\rho = \rho_0\,\exp\left(\frac{h_0-h}{H}\right)$, where h is the altitude and H is the scale height at the altitude h_0.

6.1.2 Application to a conservative force

A conservative force can be expressed as the gradient of a potential U. In this section, we consider the case of the first non-Keplerian term in the gravitational potential of a non-spherical body. As we have seen before, this part of the potential is dominated by the J_2 term and we hence consider $\vec{P} = -\vec{\nabla}U'$ where

$$U' = J_2\,\mu\frac{R_e^2}{r^3}\,(\frac{3}{2}\,\sin^2\theta - \frac{1}{2})$$

In the spherical coordinates (r, λ, θ), centred on the center of mass of the non-spherical body (see Fig. 6.2), this yields

$$\begin{aligned} \vec{P} &= -(\frac{\partial U'}{\partial r}, \frac{1}{r\,\cos\theta}\frac{\partial U'}{\partial\lambda}, \frac{1}{r}\frac{\partial U'}{\partial\theta}) \\ &= (J_2\,\mu\frac{R_e^2}{r^4}\,(\frac{9}{2}\,\sin^2\theta - \frac{3}{2}), 0, -3\,J_2\,\mu\frac{R_e^2}{r^4}\,\sin\theta\,\cos\theta) \end{aligned} \tag{6.28}$$

If we wish to express $\vec{P}$ in the axes of the osculating motion, we find that

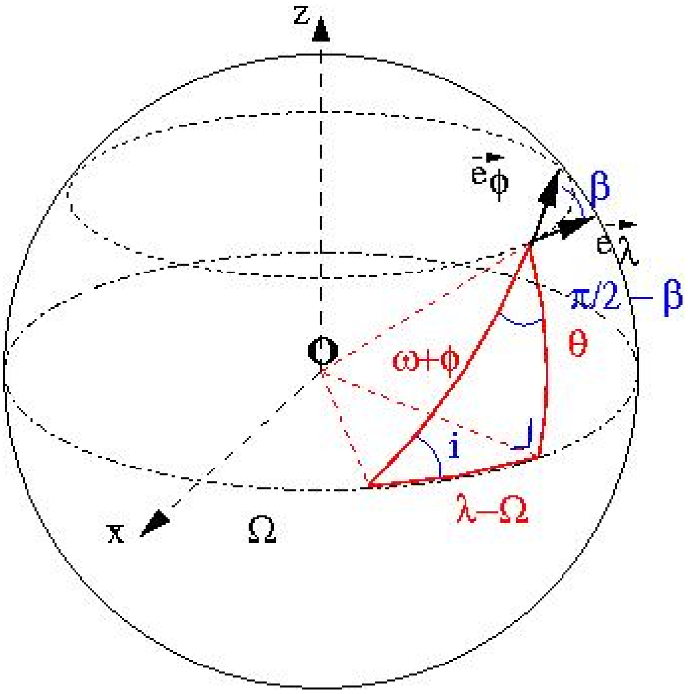


Figure 6.2: Definition of the angle β between the instantaneous vectors $\vec{e_\phi}$ and $\vec{e_\lambda}$. The former refers to the polar angle in the osculating plane of the motion, whilst the latter corresponds to the longitude in the spherical coordinates centred on the center of mass O of the non-spherical distribution of matter.

$$R = J_2\,\mu \frac{R_e^2}{r^4}\,(\frac{9}{2}\,\sin^2\theta - \frac{3}{2})$$

$$T = -3\,J_2\,\mu \frac{R_e^2}{r^4}\,\sin\theta\,\cos\theta\,\sin\beta$$

$$W = -3\,J_2\,\mu \frac{R_e^2}{r^4}\,\sin\theta\,\cos\theta\,\cos\beta$$

where β is the angle between $\vec{e_\lambda}$ and $\vec{e_\phi}$ (see Fig. 6.2). Before using these expressions in the Gauss equations, we need to convert θ and β into the elements of the osculating orbit. This can be done by noting that

$$\sin\theta = \sin i\,\sin(\omega + \phi)$$

$$\sin\theta\,\cos\theta\,\sin\beta = \sin^2 i\,\sin(\omega+\phi)\,\cos(\omega+\phi)$$

$$\sin\theta\,\cos\theta\,\cos\beta = \sin i\,\cos i\,\sin(\omega+\phi)$$

which eventually yields

$$R = \frac{3}{2}\,J_2\,\mu \frac{R_e^2}{r^4}\,(3\,\sin^2 i\,\sin^2(\omega+\phi) - 1)$$

$$T = -3\,J_2\,\mu \frac{R_e^2}{r^4}\,\sin^2 i\,\sin(\omega+\phi)\,\cos(\omega+\phi)$$

$$W = -3\,J_2\,\mu \frac{R_e^2}{r^4}\,\sin i\,\cos i\,\sin(\omega+\phi)$$

We can now insert the expressions of R, T and W in the Gauss equations. In this way, we obtain:

$$
\begin{aligned}
\dot{a} &= \frac{3\,J_2\,n\,a}{4\,\sqrt{1-e^2}}\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^4\Big[e\,\sin^2 i\,(6\,\sin\phi - 5\,\sin(2\,\omega+3\,\phi)+\sin(2\,\omega+\phi)) \\
&\quad -4\,e\,\sin\phi - 4\,\sin^2 i\,\sin(2\,\omega+2\,\phi)\Big] && (6.29)\\
\dot{e} &= \frac{3}{8}\,J_2\,n\,\sqrt{1-e^2}\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^4\Big[6\,\sin^2 i\,\sin\phi - 5\,\sin^2 i\,\sin(2\,\omega+3\,\phi)+\sin^2 i\,\sin(2\,\omega+\phi) - 4\,\sin\phi\Big] \\
&\quad -\frac{3}{4}\frac{J_2\,n}{\sqrt{1-e^2}}\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^3\sin^2 i\,\left[2\,e\,\sin(2\,\omega+2\,\phi)+\sin(2\,\omega+\phi)+\sin(2\,\omega+3\,\phi)\right] && (6.30)\\
\frac{d\,i}{dt} &= \frac{-3\,J_2\,n}{2\,\sqrt{1-e^2}}\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^3\sin i\,\cos i\,\sin(2\,\omega+2\,\phi) && (6.31)\\
\dot{\Omega} &= \frac{-3\,J_2\,n}{2\,\sqrt{1-e^2}}\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^3\cos i\,(1-\cos(2\,\omega+2\,\phi)) && (6.32)\\
\dot{\omega} &= \frac{3}{8}\,J_2\,n\,\frac{\sqrt{1-e^2}}{e}\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^4\Big[4\,\cos\phi - \sin^2 i\,(6\,\cos\phi - 5\,\cos(2\,\omega+3\,\phi)-\cos(2\,\omega+\phi))\Big] \\
&\quad +\frac{3}{4}\frac{J_2\,n}{e\,\sqrt{1-e^2}}\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^3\sin^2 i\,\left[\cos(2\,\omega+3\,\phi)-\cos(2\,\omega+\phi)\right]-\dot{\Omega}\,\cos i && (6.33)\\
\dot{M} &= n(t) - \sqrt{1-e^2}\,(\dot{\omega}+\dot{\Omega}\,\cos i) + 3\,J_2\,n\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^3\left[1-\frac{3}{2}\,\sin^2 i\,(1-\cos(2\,\omega+2\,\phi))\right] && (6.34)
\end{aligned}
$$

In all the equations above $n = \sqrt{\frac{\mu}{a^3}}$.

6.2 The Lagrange equations

An alternative formulation of the perturbation of the elements of the osculating orbit can be obtained through the Hamiltonian formalism that we have introduced in Chapter 3.
Indeed, we have seen that the Hamiltonian represents the total energy per unit mass of the moving body. For an elliptical motion due to a point-like mass, the Hamiltonian can thus be expressed as

$$\mathcal{H} = -\frac{\mu^2}{2\,L^2}$$

where $L = \sqrt{\mu\,a}$. In those cases where the perturbating force is conservative (i.e. can be expressed as the gradient of a potential U'), the total energy and hence the Hamiltonian become

$$\mathcal{H}' = -\frac{\mu^2}{2\,L^2} + U'(l, g, \theta, L, G, \Theta)$$

In the latter expression, we have used the Delaunay canonical elements

$$(l, g, \theta, L, G, \Theta) = (M, \omega, \Omega, \sqrt{\mu\,a}, \sqrt{\mu\,a\,(1-e^2)}, \sqrt{\mu\,a\,(1-e^2)}\,\cos i)$$

to express the potential as a function of the elements of the osculating orbit. The canonical equations of Hamilton then yield

$$\frac{d\,L}{dt} = -\frac{\partial\,U'}{\partial l} \qquad (6.35)$$

$$\frac{d\,G}{dt} = -\frac{\partial\,U'}{\partial g} \tag{6.36}$$
$$\frac{d\,\Theta}{dt} = -\frac{\partial\,U'}{\partial \theta} \tag{6.37}$$
$$\frac{d\,l}{dt} = \frac{\mu^2}{L^3} + \frac{\partial\,U'}{\partial L} \tag{6.38}$$
$$\frac{d\,g}{dt} = \frac{\partial\,U'}{\partial G} \tag{6.39}$$
$$\frac{d\,\theta}{dt} = \frac{\partial\,U'}{\partial \Theta} \tag{6.40}$$

For an elliptical osculating orbit, we can express the relation between the conjugated Delaunay moments and the elements a, e and i:

$$\mu\,a = L^2 \quad \Rightarrow \quad \mu\,da = 2\,L\,dL$$
$$e^2 = 1 - \frac{G^2}{L^2} \quad \Rightarrow \quad e\,de = \frac{G^2}{L^3}\,dL - \frac{G}{L^2}\,dG$$
$$\cos i = \frac{\Theta}{G} \quad \Rightarrow \quad \sin i\,di = \frac{\Theta}{G^2}\,dG - \frac{1}{G}\,d\Theta$$

From these relations and the canonical equations of Hamilton, we infer thus

$$\frac{d\,a}{dt} = \frac{2\,L}{\mu}\frac{d\,L}{dt} = -\frac{2}{n\,a}\frac{\partial\,U'}{\partial M} \tag{6.41}$$
$$\frac{d\,e}{dt} = \frac{1}{e}\left[\frac{h^2}{(\mu\,a)^{3/2}}\frac{d\,L}{dt} - \frac{h}{\mu\,a}\frac{d\,G}{dt}\right] = \frac{1}{e}\left[\frac{e^2-1}{n\,a^2}\frac{\partial\,U'}{\partial M} + \frac{\sqrt{1-e^2}}{n\,a^2}\frac{\partial\,U'}{\partial \omega}\right] \tag{6.42}$$
$$\frac{d\,i}{dt} = \frac{1}{n\,a^2\sqrt{1-e^2}\,\sin i}\left[\frac{\partial\,U'}{\partial \Omega} - \cos i\,\frac{\partial\,U'}{\partial \omega}\right] \tag{6.43}$$
$$\frac{d\,\Omega}{dt} = \frac{-1}{n\,a^2\sqrt{1-e^2}\,\sin i}\,\frac{\partial\,U'}{\partial i} \tag{6.44}$$
$$\frac{d\,\omega}{dt} = \frac{\partial\,U'}{\partial i}\frac{\partial\,i}{\partial G} + \frac{\partial\,U'}{\partial e}\frac{\partial\,e}{\partial G} = \frac{\cos i}{n\,a^2\sqrt{1-e^2}\,\sin i}\frac{\partial\,U'}{\partial i} - \frac{\sqrt{1-e^2}}{n\,a^2\,e}\frac{\partial\,U'}{\partial e} \tag{6.45}$$
$$\frac{d\,M}{dt} = \frac{\mu^2}{(\mu\,a)^{3/2}} + \frac{\partial\,U'}{\partial a}\frac{\partial\,a}{\partial L} + \frac{\partial\,U'}{\partial e}\frac{\partial\,e}{\partial L} = n(t) + \frac{2}{n\,a}\frac{\partial\,U'}{\partial a} + \frac{1-e^2}{n\,a^2\,e}\frac{\partial\,U'}{\partial e} \tag{6.46}$$

These **Lagrange equations** can be expressed in a more compact form:

$$\begin{bmatrix} \frac{d\,a}{dt} \\ \frac{d\,M}{dt} \\ \frac{d\,e}{dt} \\ \frac{d\,\omega}{dt} \\ \frac{d\,i}{dt} \\ \frac{d\,\Omega}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ n(t) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{n\,a^2}\begin{bmatrix} 0 & -2\,a & 0 & 0 & 0 & 0 \\ 2\,a & 0 & \frac{1-e^2}{e} & 0 & 0 & 0 \\ 0 & -\frac{1-e^2}{e} & 0 & \frac{\sqrt{1-e^2}}{e} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{1-e^2}}{e} & 0 & \frac{\cos i}{\sqrt{1-e^2}\,\sin i} & 0 \\ 0 & 0 & 0 & -\frac{\cos i}{\sqrt{1-e^2}\,\sin i} & 0 & \frac{1}{\sqrt{1-e^2}\,\sin i} \\ 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{1-e^2}\,\sin i} & 0 \end{bmatrix}\begin{bmatrix} \frac{\partial\,U'}{\partial a} \\ \frac{\partial\,U'}{\partial M} \\ \frac{\partial\,U'}{\partial e} \\ \frac{\partial\,U'}{\partial \omega} \\ \frac{\partial\,U'}{\partial i} \\ \frac{\partial\,U'}{\partial \Omega} \end{bmatrix} \tag{6.47}$$

6.3 Resolution of a differential equation depending on a small parameter

Before we discuss the impact of the different terms in the Gauss or Lagrange equations on the osculating orbit, we first need to consider the resolution of a non-linear differential equation involving a small parameter.

Let us thus consider a non-linear differential equation

$$\frac{d\,x}{dt} = f(x, \eta) \tag{6.48}$$

that depends on a small parameter η. The solution of this differential equation of the first order implies the knowledge of a single constant of integration that we shall call α.
We assume that we know the solution of this equation for $\eta = 0$ which is given by $x_0(t, \alpha_0)$ where α_0 is a constant of integration. We are looking for a solution of the differential equation of the form

$$x(t, \eta) = x_0(t) + \eta\, x_1(t) + \eta^2\, x_2(t) + ... \tag{6.49}$$

This leads to

$$\begin{aligned}
\frac{d\,x_0}{dt} + \eta\,\frac{d\,x_1}{dt} + \eta^2\,\frac{d\,x_2}{dt} + ... &= f(x_0 + \eta\, x_1 + \eta^2\, x_2 + ..., t, 0 + \eta) \\
&= f(x_0, t, 0) + \frac{\partial\, f}{\partial x_0}(\eta\, x_1 + \eta^2\, x_2 + ...) + \frac{1}{2}\frac{\partial^2\, f}{\partial x_0^2}(\eta\, x_1 + \eta^2\, x_2 + ...)^2 \\
&\quad + \left.\frac{\partial\, f}{\partial \eta}\right)_{\eta=0} \eta + \frac{\partial^2\, f}{\partial x_0\,\partial \eta}(\eta\, x_1 + \eta^2\, x_2 + ...)\,\eta + \left.\frac{1}{2}\frac{\partial^2\, f}{\partial \eta^2}\right)_{\eta=0} \eta^2 \\
&\quad + ...
\end{aligned} \tag{6.50}$$

Where $\frac{\partial\, f}{\partial x_0}$ stands for the partial derivative of f with respect to x evaluated at $x = x_0$. By comparing the coefficients of the various powers of η, we obtain that

$$\frac{d\,x_0}{dt} = f(x_0, t, 0) \tag{6.51}$$

$$\frac{d\,x_1}{dt} = \frac{\partial\, f}{\partial x_0}\, x_1 + \left.\frac{\partial\, f}{\partial \eta}\right)_{\eta=0} \tag{6.52}$$

$$\frac{d\,x_2}{dt} = \frac{\partial\, f}{\partial x_0}\, x_2 + \frac{1}{2}\frac{\partial^2\, f}{\partial x_0^2}\, x_1^2 + \frac{\partial^2\, f}{\partial x_0\,\partial \eta}\, x_1 + \left.\frac{1}{2}\frac{\partial^2\, f}{\partial \eta^2}\right)_{\eta=0} \tag{6.53}$$

$$...$$

The equations hereabove yield successively $x_0(t, \alpha_0)$, $x_1(t, \alpha_1)$, $x_2(t, \alpha_2)$,... where the α_j are constants of integration. If we restrict ourselves to the nth power of η, we can now express the solution of the problem as

$$x(t, \eta) = x_0(t, \alpha_0) + \eta\, x_1(t, \alpha_1) + \eta^2\, x_2(t, \alpha_2) + ... + \eta^n\, x_n(t, \alpha_n) + \mathcal{O}(\eta^{n+1})$$

Expressed in this way, the solution involves a total of $n + 1$ constants of integration which is of course too much given that the differential equation is of the first order. To solve this issue, we can adopt either of two approaches:

- in the first method, the constant of integration α_0 is determined for $x_0(t, \alpha_0)$ by the condition that at time t_0, $x_0(t_0, \alpha_0) = X_0 = x(t_0)$. For the subsequent functions, one then adopts $x_j(t_0, \alpha_j) = 0 \quad \forall j = 1, ..., n$. In this way all the α_j can be determined consistently.

- in the second technique, a constant of integration is added only in the resolution of the very first equation, whilst no such constants are added for the subsequent equations. Hence,

  $$x(t) = x_0(t, \alpha_0) + \eta\, x_1(t) + \eta^2\, x_2(t) + ... + \eta^n\, x_n(t) + \mathcal{O}(\eta^{n+1})$$

  The value of α_0 is then determined from the condition

  $$x(t_0) = X_0 = x_0(t_0, \alpha_0) + \eta\, x_1(t_0) + \eta^2\, x_2(t_0) + ... + \eta^n\, x_n(t_0) + \mathcal{O}(\eta^{n+1})$$

6.4 Secular, periodic and mixed terms

In the Lagrange equations, we can distinguish between metric and angular elements of the osculating orbit. The former are a, e and i, that we shall designate in the following by the generic notation α, whilst the latter are ω, Ω and M which we designate by β in this section.
Let us assume that the potential U' depends on a small parameter η and can be expressed as

$$U' = \eta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[U_{jkl}^{(1)}(a,e,i) + \eta\, U_{jkl}^{(2)}(a,e,i) + ...\right] \cos\left(j\,\Omega + k\,\omega + l\,M\right)$$

For the metric parameters, the perturbations can be expressed through the Lagrange equation as

$$\dot{\alpha} = \eta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[A_{jkl}^{(1)}(a,e,i) + \eta\, A_{jkl}^{(2)}(a,e,i) + ...\right] \sin\left(j\,\Omega + k\,\omega + l\,M\right) \tag{6.54}$$

Conversely, for the angular elements, the perturbations can be expressed through the Lagrange equation as

$$\dot{\beta} = n(t)\,\delta_{\beta M} + \eta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[B_{jkl}^{(1)}(a,e,i) + \eta\, B_{jkl}^{(2)}(a,e,i) + ...\right] \cos\left(j\,\Omega + k\,\omega + l\,M\right) \tag{6.55}$$

where $n(t) = n_0(a_0) + \frac{d\,n}{da_0}\,(a_1\,\eta + a_2\,\eta^2 + ...) + \frac{1}{2}\,\frac{d^2\,n}{da_0^2}\,(a_1\,\eta + a_2\,\eta^2 + ...)^2 + ...$
It has to be stressed that whilst there does not exist a term $A_{000}^{(p)}$ (because $\sin 0 = 0$), there is the possibility to have a term $B_{000}^{(p)}$ different from zero.
Since we are looking for solutions of the type $\alpha(t,\eta) = \alpha_0 + \eta\,\alpha_1(t) + \eta^2\,\alpha_2(t) + ...$ and $\beta(t,\eta) = \beta_0 + \eta\,\beta_1(t) + \eta^2\,\beta_2(t) + ...$, we can now use the method outlined in Sect. 6.3 to find that at order 0 in η:

$$\dot{\alpha_0} = 0 \tag{6.56}$$
$$\dot{\beta_0} = n_0\,\delta_{\beta M} \tag{6.57}$$

This implies that, at order 0 in η, the metric variables are constants and the angular variables are constants or linear functions of time, in the case of the mean anomaly $M = n_0\,t + m_0$.
At first order in η, we can write for the metric variables:

$$\dot{\alpha_1} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} A_{jkl}^{(1)}(a_0,e_0,i_0)\ \sin\left(j\,\Omega_0 + k\,\omega_0 + l\,M_0\right) \tag{6.58}$$

which can be integrated into

$$\alpha_1 = -\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \frac{A_{jkl}^{(1)}(a_0,e_0,i_0)\ \cos\left(j\,\Omega_0 + k\,\omega_0 + l\,M_0\right)}{l\,n_0} \tag{6.59}$$

where we use the second approach of Sect. 6.3 for the determination of the constants of integration.
Note that the Lagrange equation of $\dot{a}$ involves $\frac{\partial U'}{\partial M}$ and hence there is no term corresponding to $l = 0$ in the sum. For $\dot{e}$ and $\frac{d\,i}{dt}$, the absence of $l = 0$ is not guaranteed and this can bring up a double sum of secular terms (see below for the definition) of the kind $[\sin\left(j\,\Omega_0 + k\,\omega_0\right)]\,t$. We note however that for artificial satellites revolving a planet, these terms do not exist.

For the angular variables, we obtain

$$\dot{\beta_1} = \frac{d\,n}{da_0}\, a_1\, \delta_{\beta M} + B_{000}^{(1)}(a_0, e_0, i_0) + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} B_{jkl}^{(1)}(a_0, e_0, i_0)\, \cos\left(j\, \Omega_0 + k\, \omega_0 + l\, M_0\right)$$

Given the expression of a_1 that can be derived from equation 6.59, the first term of the right hand side expression can be merged into the triple sum of cosines yielding

$$\dot{\beta_1} = B_{000}^{(1)}(a_0, e_0, i_0) + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} B_{jkl}^{\prime(1)}(a_0, e_0, i_0)\, \cos\left(j\, \Omega_0 + k\, \omega_0 + l\, M_0\right) \tag{6.60}$$

which can be integrated into

$$\beta_1 = B_{000}^{(1)}(a_0, e_0, i_0)\, t + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \frac{B_{jkl}^{\prime(1)}(a_0, e_0, i_0)\, \sin\left(j\, \Omega_0 + k\, \omega_0 + l\, M_0\right)}{l\, n_0} \tag{6.61}$$

It is worth comparing expressions 6.59 and 6.61. In the first case, we find only **periodic** terms (t appears only through trigonometric functions) whilst for the angular elements, there is also a **secular** term, where the time appears as a factor. At first order in η, the metric variables hence undergo periodic changes, whilst the angular variables change linearly with time.

Finally, for the second order in η, we obtain for the metric variables:

$$\begin{aligned}
\dot{\alpha_2} \;=\; & \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[A_{jkl}^{(2)}(a_0, e_0, i_0) + \frac{\partial\, A_{jkl}^{(1)}}{\partial a_0}\, a_1 + \frac{\partial\, A_{jkl}^{(1)}}{\partial e_0}\, e_1 + \frac{\partial\, A_{jkl}^{(1)}}{\partial i_0}\, i_1 \right] \sin\left(j\, \Omega_0 + k\, \omega_0 + l\, M_0\right) \\
& + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} A_{jkl}^{(1)}(a_0, e_0, i_0)\left(j\, \Omega_1 + k\, \omega_1 + l\, M_1\right)\, \cos\left(j\, \Omega_0 + k\, \omega_0 + l\, M_0\right)
\end{aligned} \tag{6.62}$$

Using the expressions 6.59 and 6.61 we can then transform this relation into

$$\begin{aligned}
\dot{\alpha_2} \;=\; & \sum_{j''=-\infty}^{\infty} \sum_{k''=-\infty}^{\infty} \sum_{l''=-\infty}^{\infty} A''_{j''k''l''}(a_0, e_0, i_0)\, \sin\left(j''\, \Omega_0 + k''\, \omega_0 + l''\, M_0\right) \\
& + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} A'_{jkl}(a_0, e_0, i_0)\, t\, \cos\left(j\, \Omega_0 + k\, \omega_0 + l\, M_0\right)
\end{aligned} \tag{6.63}$$

which leads eventually to an expression of the type

$$\begin{aligned}
\alpha_2 \;=\; & -\sum_{j''=-\infty}^{\infty} \sum_{k''=-\infty}^{\infty} \sum_{l''=-\infty}^{\infty} \frac{A''_{j''k''l''}(a_0, e_0, i_0)}{l''\, n_0}\, \cos\left(j''\, \Omega_0 + k''\, \omega_0 + l''\, M_0\right) \\
& + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{A'_{jkl}(a_0, e_0, i_0)}{l\, n_0}\, t\, \sin\left(j\, \Omega_0 + k\, \omega_0 + l\, M_0\right) \\
& + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{A'_{jkl}(a_0, e_0, i_0)}{l^2\, n_0^2}\, \cos\left(j\, \Omega_0 + k\, \omega_0 + l\, M_0\right)
\end{aligned} \tag{6.64}$$

In addition to the periodic terms, we see that this expression now involves also **mixed** terms where the time appears as a factor of a sinusoidal function. With increasing powers of η^n, one obtains increasing powers of t^{n-1}. This illustrates that the development is valid only over a restricted interval of time and will not yield the correct answer for arbitrarily large values of t.

6.4.1 Periodic terms

One can avoid the existence of the mixed terms by expanding the osculating elements α and β in a somewhat different way. Indeed, let us consider an asymptotic development of α and β truncated at the Nth power of η.

$$\alpha = \alpha_0 + \sum_{n=1}^{N} \eta^n \alpha_n \tag{6.65}$$

$$\begin{aligned} \beta &= \beta_0 + \left(\sum_{n=1}^{N} \eta^n \overline{\beta_n} \right) t + \sum_{n=1}^{N} \eta^n \beta_n \\ &= \beta_0 + \eta \overline{\beta} t + \sum_{n=1}^{N} \eta^n \beta_n \end{aligned} \tag{6.66}$$

where the quantities $\overline{\beta}$ are constants to be determined. Note that these numbers have the dimensions of an angular velocity.

To simplify the notations in this section, we shall further introduce

$$\gamma^0_{jkl} = j\,\Omega_0 + k\,\omega_0 + l\,M_0 \tag{6.67}$$

$$\overline{\gamma}_{jkl} = j\,\overline{\Omega} + k\,\overline{\omega} + l\,\overline{M} \tag{6.68}$$

$$\gamma^n_{jkl} = j\,\Omega_n + k\,\omega_n + l\,M_n \tag{6.69}$$

Now, considering $N = 2$, we can write

$$\begin{aligned} \dot{\alpha_0} + \eta\,\dot{\alpha_1} + \eta^2\,\dot{\alpha_2} &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \left[\eta\, A^{(1)}_{jkl}(a_0, e_0, i_0) + \eta^2 \sum_{\alpha} \frac{\partial A^{(1)}_{jkl}}{\partial \alpha_0}\, \alpha_1 + \eta^2\, A^{(2)}_{jkl}(a_0, e_0, i_0) \right] \\ &\quad \times \sin\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\, t + \eta\,\gamma^1_{jkl}\right) \end{aligned} \tag{6.70}$$

$$\begin{aligned} &\dot{\beta_0} + \eta\,\overline{\beta_1} + \eta^2\,\overline{\beta_2} + \eta\,\dot{\beta_1} + \eta^2\,\dot{\beta_2} = n(a_0)\,\delta_{\beta M} + \frac{d\,n}{d a_0}\,(\eta\,a_1 + \eta^2\,a_2)\,\delta_{\beta M} + \frac{1}{2}\frac{d^2\,n}{d a_0^2}\,(\eta\,a_1 + \eta^2\,a_2)^2\,\delta_{\beta M} \\ &+ \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \left[\eta\, B^{(1)}_{jkl}(a_0, e_0, i_0) + \eta^2 \sum_{\alpha} \frac{\partial B^{(1)}_{jkl}}{\partial \alpha_0}\, \alpha_1 + \eta^2\, B^{(2)}_{jkl}(a_0, e_0, i_0) \right] \cos\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\, t + \eta\,\gamma^1_{jkl}\right) \end{aligned} \tag{6.71}$$

We note that one can write

$$\sin\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\, t + \eta\,\gamma^1_{jkl}\right) = \sin\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\, t\right) + \cos\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\, t\right) \eta\,\gamma^1_{jkl}$$

and

$$\cos\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\, t + \eta\,\gamma^1_{jkl}\right) = \cos\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\, t\right) - \sin\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\, t\right) \eta\,\gamma^1_{jkl}$$

Now, at order zero in η in equations 6.70 and 6.71, we recover expressions 6.56 and 6.57, i.e. the metric elements are constant and the angular elements are either constant (Ω and ω) or linear functions of the time (M).

At first order in η, we obtain for the metric parameters

$$\begin{aligned} \dot{\alpha_1} &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} A^{(1)}_{jkl}(a_0, e_0, i_0)\, \sin\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\, t\right) \\ \Rightarrow \alpha_1 &= -\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{A^{(1)}_{jkl}(a_0, e_0, i_0)}{n_0\, l + \eta\,\overline{\gamma}_{jkl}}\, \cos\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\, t\right) \end{aligned} \tag{6.72}$$

For the angular elements, we can write

$$\overline{\beta_1} + \dot{\beta_1} = \frac{d\,n}{da_0}\, a_1\, \delta_{\beta M} + B^{(1)}_{000}(a_0, e_0, i_0) + \sum_{jkl}{}' B^{(1)}_{jkl}(a_0, e_0, i_0)\, \cos\left(\gamma^0_{jkl} + \eta\, \overline{\gamma}_{jkl}\, t\right)$$

where $\sum'_{jkl}$ indicates the triple summation where we have excluded the term $j = k = l = 0$. Inserting the expression of a_1 taken from equation 6.72, this equation can be reformulated as

$$\begin{aligned}
\overline{\beta_1} + \dot{\beta_1} &= B^{(1)}_{000}(a_0, e_0, i_0) + \sum_{jkl}{}' B'^{(1)}_{jkl}(a_0, e_0, i_0)\, \cos\left(\gamma^0_{jkl} + \eta\, \overline{\gamma}_{jkl}\, t\right) \\
\Rightarrow \overline{\beta_1} &= B^{(1)}_{000}(a_0, e_0, i_0) && (6.73) \\
\beta_1 &= \sum_{jkl}{}' \frac{B'^{(1)}_{jkl}(a_0, e_0, i_0)}{n_0\, l + \eta\, \overline{\gamma}_{jkl}}\, \sin\left(\gamma^0_{jkl} + \eta\, \overline{\gamma}_{jkl}\, t\right) && (6.74)
\end{aligned}$$

Hence, from equation 6.73, we know $\overline{\beta}$ to first order in η.

Let us now consider the second order terms in η. We start again with the equation for the metric elements:

$$\begin{aligned}
\dot{\alpha_2} = {} & \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[\sum_{\alpha} \frac{\partial\, A^{(1)}_{jkl}}{\partial \alpha_0}\, \alpha_1 + A^{(2)}_{jkl}(a_0, e_0, i_0) \right] \sin\left(\gamma^0_{jkl} + \eta\, \overline{\gamma}_{jkl}\, t\right) \\
& + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} A^{(1)}_{jkl}(a_0, e_0, i_0)\, \cos\left(\gamma^0_{jkl} + \eta\, \overline{\gamma}_{jkl}\, t\right) \gamma^1_{jkl}
\end{aligned}$$

Now, since α_1 can be expressed as a sum of cosines (see equation 6.72) and γ^1_{jkl} is a sum of sines (see equation 6.74), we can reformulate the equation of $\dot{\alpha_2}$:

$$\begin{aligned}
\dot{\alpha_2} &= \sum_{j''k''l''} A'^{(2)}_{j''k''l''}\, \sin\left(\gamma^0_{j''k''l''} + \eta\, \overline{\gamma}_{j''k''l''}\, t\right) \\
\Rightarrow \alpha_2 &= -\sum_{j''k''l''} \frac{A'^{(2)}_{j''k''l''}}{n_0\, l'' + \eta\, \overline{\gamma}_{j''k''l''}}\, \cos\left(\gamma^0_{j''k''l''} + \eta\, \overline{\gamma}_{j''k''l''}\, t\right) && (6.75)
\end{aligned}$$

where j'', k'' and l'' are integer numbers (either negative, positive of zero).
On the other hand, for the angular elements, we obtain

$$\begin{aligned}
\overline{\beta_2} + \dot{\beta_2} = {} & \frac{d\,n}{da_0}\, a_2\, \delta_{\beta M} + \frac{1}{2} \frac{d^2\, n}{da_0^2}\, a_1^2\, \delta_{\beta M} + B^{(2)}_{000}(a_0, e_0, i_0) + \sum_{\alpha} \frac{\partial\, B^{(1)}_{000}}{\partial \alpha_0}\, \alpha_1 \\
& + \sum_{jkl}{}' \left[\sum_{\alpha} \frac{\partial\, B^{(1)}_{jkl}}{\partial \alpha_0}\, \alpha_1 + B^{(2)}_{jkl}(a_0, e_0, i_0) \right] \cos\left(\gamma^0_{jkl} + \eta\, \overline{\gamma}_{jkl}\, t\right) \\
& - \sum_{jkl}{}' B^{(1)}_{jkl}(a_0, e_0, i_0)\, \sin\left(\gamma^0_{jkl} + \eta\, \overline{\gamma}_{jkl}\, t\right) \gamma^1_{jkl}
\end{aligned}$$

Accounting for the expressions of a_1, a_2, α_1 and γ^1_{jkl}, we finally obtain:

$$\overline{\beta_2} + \dot{\beta_2} = B'^{(2)}_{000} + \sum_{j''k''l''}{}' B'^{(2)}_{j''k''l''}\, \cos\left(\gamma^0_{j''k''l''} + \eta\, \overline{\gamma}_{j''k''l''}\, t\right)$$

$$\Rightarrow \overline{\beta_2} = B'^{(2)}_{000}(a_0, e_0, i_0) \tag{6.76}$$

$$\Rightarrow \beta_2 = \sum_{j''k''l''} \frac{B'^{(2)}_{j''k''l''}}{n_0\, l'' + \eta\, \overline{\gamma}_{j''k''l''}} \sin\left(\gamma^0_{j''k''l''} + \eta\, \overline{\gamma}_{j''k''l''}\, t\right) \tag{6.77}$$

These results show that there is no longer a mixed term in the development of the orbital elements and we thus obtain a more accurate development (for increasing t) than in the previous section.
There are a number of periodic terms that appear in the development. The fundamental frequencies of these terms are $\eta\,\overline{\Omega}$, $\eta\,\overline{\omega}$, $l\,n_0 + \eta\,\overline{M}$. At this level, we can distinguish between long and short period terms. The **short period terms** result from the fundamental frequencies $l\,n_0 + \eta\,\overline{M}$ for $l \neq 0$. Indeed, in this case, the period is $P = \frac{2\pi}{l\,n_0} + \mathcal{O}(\eta)$, which is equal to the period of the osculating orbit ($l = 1$) or one of its harmonics. On the contrary, the **long period terms** result from $l = 0$ with either j or $k \neq 0$. These periods are then proportional to η^{-1} (which is a large number since η is small). It must be stressed that the long period terms of order η^n have amplitudes of order η^{n-1} (due to the denominator in the expressions of α_1, α_2, β_1 and β_2 when l or $l'' = 0$). The secular terms finally are 'slowly' changing with time since their amplitude is proportional to η.
Again, it must be stressed that these developments are valid only over a limited interval of time. The numerical resolution of the equations allows nowadays to achieve accuracies that are often much better than what could be done with the analytical developments limited to a certain power of η. However, as we shall illustrate in the next section, these analytical developments provide a much deeper insight into the physical meaning and the consequences of the various perturbations.

6.5 Perturbations due to J_2

In this section, we shall revisit the perturbation of the Keplerian motion by the J_2 term of the potential of a non-spherical mass distribution. Since this is the dominant non-Keplerian term, it is of fundamental importance for the understanding of the motion of artificial satellites.
Let us start by recalling that

$$U' = J_2\,\mu \frac{R_e^2}{a^3}\,\frac{a^3}{r^3}\,(\frac{3}{2}\,\sin^2\theta - \frac{1}{2})$$

with

$$\sin\theta = \sin i\,\sin\left(\omega + \phi\right)$$

as we have shown in Sect. 6.1.2. Thus, we obtain that

$$\begin{aligned} U' &= J_2\,\mu \frac{R_e^2}{a^3}\,\frac{a^3}{r^3}\,(\frac{3}{2}\,\sin^2 i\,\sin^2(\omega+\phi) - \frac{1}{2}) \\ &= J_2\,\mu \frac{R_e^2}{a^3}\,\frac{a^3}{r^3}\left(\frac{3}{4}\,\sin^2 i - \frac{1}{2} - \frac{3}{4}\,\sin^2 i\,\cos 2\,\omega\,\cos 2\,\phi + \frac{3}{4}\,\sin^2 i\,\sin 2\,\omega\,\sin 2\,\phi\right) \end{aligned} \tag{6.78}$$

The potential hence depends upon three variable quantities that are $\left(\frac{a}{r}\right)^3$, $\left(\frac{a}{r}\right)^3\cos 2\,\phi$ and $\left(\frac{a}{r}\right)^3\sin 2\,\phi$.
Our goal here is to characterize the perturbations due to J_2 by deriving the expression of U' as a function of the Delaunay elements and then injecting this expression into the Lagrange equations (see equation 6.47). We thus have to account for the dependence of the true anomaly ϕ as a function of M. For this purpose, we will now expand the three functions $\left(\frac{a}{r}\right)^3$, $\left(\frac{a}{r}\right)^3\cos 2\,\phi$ and $\left(\frac{a}{r}\right)^3\sin 2\,\phi$ into Fourier series of M following the formalism introduced in Chapter 4. In the latter chapter we have seen that the constant term of the Fourier expansion of $\left(\frac{a}{r}\right)^3\cos\left(p\,\phi\right)$ can be expressed as

$$\frac{1}{(1-e^2)^{3/2}}\,(2\,\delta_{p0} + e\,\delta_{p1})$$

Hence, the constant term of $\left(\frac{a}{r}\right)^3$ is $\frac{1}{(1-e^2)^{3/2}}$, whilst the constant terms of $\left(\frac{a}{r}\right)^3 \cos 2\,\phi$ and $\left(\frac{a}{r}\right)^3 \sin 2\,\phi$ are both zero (for the latter function this result stems from the fact that we are dealing with an odd function of M). These constant terms are important since they provide the secular perturbations as we have seen above.

On the other hand, to determine the periodic components of the perturbations, we must perform the Fourier development of U'. Assuming that e is a small quantity (though it can be much larger than $\eta = J_2$), we will expand each of the three functions above in a series of e. Since several of the Lagrange equations (6.47) involve the quantity $\frac{1}{e}\,\frac{\partial U'}{\partial e}$, a treatment of the perturbations up to the order of e^n implies a development of U' up to the order e^{n+2}. Hence, if we are interested e.g. in perturbations of the orbit up to the order of e, we actually have to develop U' up to e^3.

Let us start by noting that

$$\begin{aligned}
\left(\frac{a}{r}\right)^3 &= (1 - e\,\cos E)^{-3} = 1 + 3\,e\,\cos E + 6\,e^2\,\cos^2 E + 10\,e^3\,\cos^3 E + \mathcal{O}(e^4) \\
&= 1 + 3\,e^2 + (3\,e + \frac{15\,e^3}{2})\,\cos E + 3\,e^2\,\cos(2\,E) + \frac{5\,e^3}{2}\,\cos(3\,E) + \mathcal{O}(e^4)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{a}{r}\right)^3 \cos 2\,\phi &= \left(\frac{a}{r}\right)^3 (2\,\cos^2\phi - 1) = \left(\frac{a}{r}\right)^3 \left(\frac{2\,(\cos E - e)^2}{(1 - e\,\cos E)^2} - 1\right) \\
&= 2\left(\frac{a}{r}\right)^5 (\cos E - e)^2 - \left(\frac{a}{r}\right)^3 = 2\,(1 - e\,\cos E)^{-5}(\cos E - e)^2 - \left(\frac{a}{r}\right)^3 \\
&= 2\,(1 + 5\,e\,\cos E + 15\,e^2\,\cos^2 E + 35\,e^3\,\cos^3 E)(\cos^2 E - 2\,e\,\cos E + e^2) \\
&\quad -1 - 3\,e\,\cos E - 6\,e^2\,\cos^2 E - 10\,e^3\,\cos^3 E + \mathcal{O}(e^4) \\
&= \frac{e^2}{4} + \left(\frac{e}{2} + \frac{5\,e^3}{4}\right)\cos E + (1 + 2\,e^2)\,\cos(2\,E) + \left(\frac{5\,e}{2} + \frac{35\,e^3}{8}\right)\cos(3\,E) \\
&\quad + \frac{15\,e^2}{4}\,\cos(4\,E) + \frac{35\,e^3}{8}\,\cos(5\,E) + \mathcal{O}(e^4)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{a}{r}\right)^3 \sin 2\,\phi &= 2\left(\frac{a}{r}\right)^3 \sin\phi\,\cos\phi = 2\left(\frac{a}{r}\right)^3 \frac{\sqrt{1-e^2}\,\sin E}{1 - e\,\cos E}\,\frac{\cos E - e}{1 - e\,\cos E} \\
&= 2\sqrt{1-e^2}\,\sin E\,(\cos E - e)\,(1 - e\,\cos E)^{-5} \\
&= \sin E\,(2 - e^2)\,(1 + 5\,e\,\cos E + 15\,e^2\,\cos^2 E + 35\,e^3\,\cos^3 E)(\cos E - e) + \mathcal{O}(e^4) \\
&= \left(\frac{e}{2} + e^3\right)\sin E + (1 + 2\,e^2)\,\sin(2\,E) + \left(\frac{5\,e}{2} + \frac{35\,e^3}{8}\right)\sin(3\,E) \\
&\quad + \frac{15\,e^2}{4}\,\sin(4\,E) + \frac{35\,e^3}{8}\,\sin(5\,E) + \mathcal{O}(e^4)
\end{aligned}$$

We now need to develop the trigonometric functions of E into trigonometric functions of M. This can be done by means of equations 4.4 and 4.5, that we recall here.

$$\begin{aligned}
\cos(p\,E) &= \delta_{p0} - \frac{e}{2}\,\delta_{p1} + \sum_{k=1}^{+\infty} \frac{p}{k}(J_{k-p}(k\,e) - J_{k+p}(k\,e))\,\cos(k\,M) \\
\sin(p\,E) &= \sum_{k=1}^{+\infty} \frac{p}{k}(J_{k-p}(k\,e) + J_{k+p}(k\,e))\,\sin(k\,M)
\end{aligned}$$

We also remind that $J_s(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+s)!} \left(\frac{x}{2}\right)^{2m+s}$ (see equation 4.29).

Taking advantage of the d'Alembert characteristics and restricting the developments to powers of e such that they yield terms up to e^3 in the above expressions of $\left(\frac{a}{r}\right)^3$, $\left(\frac{a}{r}\right)^3 \cos 2\,\phi$ and $\left(\frac{a}{r}\right)^3 \sin 2\,\phi$, we find that

$$\cos E = \frac{-e}{2} + \left(1 - \frac{3\,e^2}{8}\right) \cos M + \frac{e}{2} \cos(2\,M) + \frac{3\,e^2}{8} \cos(3\,M) + \mathcal{O}(e^3) \quad (6.79)$$

$$\sin E = \left(1 - \frac{e^2}{8}\right) \sin M + \frac{e}{2} \sin(2\,M) + \frac{3\,e^2}{8} \sin(3\,M) + \mathcal{O}(e^3) \quad (6.80)$$

$$\cos(2E) = \left(-e + \frac{e^3}{12}\right) \cos M + (1 - e^2) \cos(2\,M) + \left(e - \frac{9\,e^3}{8}\right) \cos(3\,M) + e^2 \cos(4\,M) + \frac{25\,e^3}{24} \cos(5\,M) + \mathcal{O}(e^4) \quad (6.81)$$

$$\sin(2E) = \left(-e + \frac{e^3}{6}\right) \sin M + (1 - e^2) \sin(2\,M) + \left(e - \frac{9\,e^3}{8}\right) \sin(3\,M) + e^2 \sin(4\,M) + \frac{25\,e^3}{24} \sin(5\,M) + \mathcal{O}(e^4) \quad (6.82)$$

$$\cos(3E) = \frac{3\,e^2}{8} \cos M - \frac{3\,e}{2} \cos(2\,M) + \left(1 - \frac{9\,e^2}{4}\right) \cos(3\,M) + \frac{3\,e}{2} \cos(4\,M) + \frac{15\,e^2}{8} \cos(5\,M) + \mathcal{O}(e^3) \quad (6.83)$$

$$\sin(3E) = \frac{3\,e^2}{8} \sin M - \frac{3\,e}{2} \sin(2\,M) + \left(1 - \frac{9\,e^2}{4}\right) \sin(3\,M) + \frac{3\,e}{2} \sin(4\,M) + \frac{15\,e^2}{8} \sin(5\,M) + \mathcal{O}(e^3) \quad (6.84)$$

$$\cos(4\,E) = -2\,e \cos(3\,M) + \cos(4\,M) + 2\,e \cos(5\,M) + \mathcal{O}(e^2) \quad (6.85)$$

$$\sin(4\,E) = -2\,e \sin(3\,M) + \sin(4\,M) + 2\,e \sin(5\,M) + \mathcal{O}(e^2) \quad (6.86)$$

$$\cos(5\,E) = \cos(5\,M) + \mathcal{O}(e) \quad (6.87)$$

$$\sin(5\,E) = \sin(5\,M) + \mathcal{O}(e) \quad (6.88)$$

Substituting these results into the expressions of $\left(\frac{a}{r}\right)^3$, $\left(\frac{a}{r}\right)^3 \cos 2\,\phi$ and $\left(\frac{a}{r}\right)^3 \sin 2\,\phi$ then yields

$$\left(\frac{a}{r}\right)^3 = \frac{1}{(1-e^2)^{3/2}} + \left(3\,e + \frac{27\,e^3}{8}\right) \cos M + \frac{9\,e^2}{2} \cos(2\,M) + \frac{53\,e^3}{8} \cos(3\,M) + \mathcal{O}(e^4) \quad (6.89)$$

$$\left(\frac{a}{r}\right)^3 \cos(2\,\phi) = \left(-\frac{e}{2} + \frac{e^3}{12}\right) \cos M + \left(1 - \frac{5\,e^2}{2}\right) \cos(2\,M) + \left(\frac{7\,e}{2} - \frac{123\,e^3}{16}\right) \cos(3\,M) + \frac{17\,e^2}{2} \cos(4\,M) + \frac{845\,e^3}{48} \cos(5\,M) + \mathcal{O}(e^4) \quad (6.90)$$

$$\left(\frac{a}{r}\right)^3 \sin(2\,\phi) = \left(-\frac{e}{2} + \frac{e^3}{24}\right) \sin M + \left(1 - \frac{5\,e^2}{2}\right) \sin(2\,M) + \left(\frac{7\,e}{2} - \frac{123\,e^3}{16}\right) \sin(3\,M) + \frac{17\,e^2}{2} \sin(4\,M) + \frac{845\,e^3}{48} \sin(5\,M) + \mathcal{O}(e^4) \quad (6.91)$$

Inserting these results into the expression of U' finally yields

$$
\begin{aligned}
U' = {} & J_2\,\mu \frac{R_e^2}{a^3}\left(\frac{3}{4}\sin^2 i - \frac{1}{2}\right)\frac{1}{(1-e^2)^{3/2}} \\
& + J_2\,\mu \frac{R_e^2}{a^3}\left\{\left(\frac{3}{4}\sin^2 i - \frac{1}{2}\right)\left[\left(3\,e + \frac{27\,e^3}{8}\right)\cos M + \frac{9\,e^2}{2}\cos(2\,M) + \frac{53\,e^3}{8}\cos(3\,M)\right]\right. \\
& - \frac{3}{4}\sin^2 i\left[\frac{e^3}{48}\cos(2\,\omega - M) + \left(-\frac{e}{2} + \frac{e^3}{16}\right)\cos(2\,\omega + M) + \left(1 - \frac{5\,e^2}{2}\right)\cos(2\,\omega + 2\,M)\right. \\
& \left.\left. + \left(\frac{7\,e}{2} - \frac{123\,e^3}{16}\right)\cos(2\,\omega + 3\,M) + \frac{17\,e^2}{2}\cos(2\,\omega + 4\,M) + \frac{845\,e^3}{48}\cos(2\,\omega + 5\,M)\right]\right\} \\
& + \mathcal{O}(e^4)
\end{aligned}
\tag{6.92}
$$

Note that $\frac{\partial U'}{\partial \Omega} = 0$ in this case. If we apply the Lagrange equations (6.47) to U', we obtain at the first order[2] in e for the metric elements:

$$
\begin{aligned}
\frac{d\,a}{dt} = {} & 2\,\frac{n\,J_2\,R_e^2}{a}\left[\left(\frac{3}{4}\sin^2 i - \frac{1}{2}\right)3\,e\,\sin M - \frac{3}{4}\sin^2 i\left(\frac{-e}{2}\sin(2\,\omega + M) + 2\,\sin(2\,\omega + 2\,M)\right.\right. \\
& \left.\left. + \frac{21\,e}{2}\sin(2\,\omega + 3\,M)\right)\right] + \mathcal{O}(e^2)
\end{aligned}
\tag{6.93}
$$

$$
\begin{aligned}
\frac{d\,e}{dt} = {} & \frac{n\,J_2\,R_e^2}{a^2}\left[\left(\frac{3}{4}\sin^2 i - \frac{1}{2}\right)(3\,\sin M + 9\,e\,\sin(2\,M)) - \frac{3}{4}\sin^2 i\left(\frac{1}{2}\sin(2\,\omega + M)\right.\right. \\
& \left.\left. - e\,\sin(2\,\omega + 2\,M) + \frac{7}{2}\sin(2\,\omega + 3\,M) + 17\,e\,\sin(2\,\omega + 4\,M)\right)\right] + \mathcal{O}(e^2)
\end{aligned}
\tag{6.94}
$$

$$
\frac{d\,i}{dt} = \frac{3\,n\,J_2\,R_e^2}{4\,a^2}\sin i\,\cos i\,[e\,\sin(2\,\omega + M) - 2\,\sin(2\,\omega + 2\,M) - 7\,e\,\sin(2\,\omega + 3\,M)] + \mathcal{O}(e^2)
\tag{6.95}
$$

In a similar way, we obtain for the angular elements:

$$
\begin{aligned}
\frac{d\,\Omega}{dt} = {} & -\frac{3\,n\,J_2\,R_e^2\,\cos i}{2\,a^2}\left[\frac{1}{(1-e^2)^2} + 3\,e\,\cos M + \frac{e}{2}\cos(2\,\omega + M) - \cos(2\,\omega + 2\,M)\right. \\
& \left. - \frac{7\,e}{2}\cos(2\,\omega + 3\,M)\right] + \mathcal{O}(e^2)
\end{aligned}
\tag{6.96}
$$

$$
\begin{aligned}
\frac{d\,\omega}{dt} = {} & \frac{3\,n\,J_2\,R_e^2}{a^2\,(1-e^2)^2}\left(1 - \frac{5}{4}\sin^2 i\right) + \frac{n\,J_2\,R_e^2}{a^2}\left\{\left[\left(\frac{1}{2} - \frac{3}{4}\sin^2 i\right)\left(\frac{3}{e} + \frac{69\,e}{8}\right) + \frac{9\,e}{2}\cos^2 i\right]\cos M\right. \\
& + \left(\frac{1}{2} - \frac{3}{4}\sin^2 i\right)\left(9\,\cos(2\,M) + \frac{159\,e}{8}\cos(3\,M)\right) + \frac{3\,e}{64}\sin^2 i\,\cos(2\,\omega - M) \\
& + \left[\frac{3}{4}\sin^2 i\left(\frac{-1}{2\,e} + \frac{7\,e}{16}\right) + \frac{3\,e}{4}\cos^2 i\right]\cos(2\,\omega + M) - \frac{3}{2}\left(1 + \frac{3}{2}\sin^2 i\right)\cos(2\,\omega + 2\,M) \\
& + \left[\frac{3}{4}\sin^2 i\left(\frac{7}{2\,e} - \frac{397\,e}{16}\right) - \frac{21\,e}{4}\cos^2 i\right]\cos(2\,\omega + 3\,M) \\
& \left. + \frac{3}{4}\sin^2 i\left(17\,\cos(2\,\omega + 4\,M) + \frac{845\,e}{16}\cos(2\,\omega + 5\,M)\right)\right\} + \mathcal{O}(e^2)
\end{aligned}
\tag{6.97}
$$

$$
\frac{d\,M}{dt} = n + \frac{3\,n\,J_2\,R_e^2}{a^2\,(1-e^2)^{3/2}}\left(\frac{1}{2} - \frac{3}{4}\sin^2 i\right) + \frac{n\,J_2\,R_e^2}{a^2}\left\{\left(\frac{1}{2} - \frac{3}{4}\sin^2 i\right)\left[\left(\frac{-3}{e} + \frac{87\,e}{8}\right)\cos M\right.\right.
$$

[2] Remember that a priori a development of U' to e^3 allows to derive the expression of the perturbations to first order in e only.

$$-9\,\cos(2\,M) - \frac{159\,e}{8}\,\cos(3\,M)\Big] + \frac{3}{4}\,\sin^2 i\,\left[\frac{-e}{16}\,\cos(2\,\omega - M) + \left(\frac{1}{2\,e} - \frac{59\,e}{16}\right)\cos(2\,\omega + M)\right.$$
$$+11\,\cos(2\,\omega + 2\,M) + \left(\frac{-7}{2\,e} + \frac{761\,e}{16}\right)\cos(2\,\omega + 3\,M) - 17\,\cos(2\,\omega + 4\,M)$$
$$\left.-\frac{845\,e}{16}\,\cos(2\,\omega + 5\,M)\right]\Bigg\} + \mathcal{O}(e^2) \tag{6.98}$$

Note that the last two equations both involve terms that are proportional to $1/e$. As expected from Section 6.4, the time derivatives of the metric elements do not involve secular terms, but the angular terms contain a secular variation.

We will now apply the method outlined in Section 6.4 to the above relations. Inserting the expansions

$$\alpha = \alpha_0 + J_2\,\alpha_1 + (J_2)^2\,\alpha_2 + ... \tag{6.99}$$
$$\begin{aligned}\beta &= \beta_0 + (J_2\,\overline{\beta_1} + (J_2)^2\,\overline{\beta_2})\,t + J_2\,\beta_1 + (J_2)^2\,\beta_2 + ...\\ &= \beta_0 + J_2\,\overline{\beta}\,t + J_2\,\beta_1 + (J_2)^2\,\beta_2 + ...\end{aligned} \tag{6.100}$$

into the above set of equations, we find that at order zero in J_2,

$$\begin{aligned} a_0 &= Cst & \Omega_0 &= Cst\\ e_0 &= Cst & \omega_0 &= Cst\\ i_0 &= Cst & M_0 &= m_0 + n_0\,t \end{aligned}$$

where $n_0 = \sqrt{\frac{\mu}{a_0^3}}$. Restricting ourselves to the first order in J_2, we then obtain for the secular terms of the angular elements

$$J_2\,\overline{\Omega_1} = -\frac{3\,n_0\,J_2\,R_e^2\,\cos i_0}{2\,a_0^2\,(1-e_0^2)^2} \tag{6.101}$$
$$J_2\,\overline{\omega_1} = \frac{3\,n_0\,J_2\,R_e^2}{a_0^2\,(1-e_0^2)^2}\left(1 - \frac{5}{4}\,\sin^2 i_0\right) \tag{6.102}$$
$$J_2\,\overline{M_1} = \frac{3\,n_0\,J_2\,R_e^2}{a_0^2\,(1-e_0^2)^{3/2}}\left(\frac{1}{2} - \frac{3}{4}\,\sin^2 i_0\right) \tag{6.103}$$

The secular terms $J_2\,\overline{\Omega_1}$ and $J_2\,\overline{\omega_1}$ correspond respectively to a uniform precession of the line of nodes and a rotation of the pericenter in the plane of the orbit. Both effects are used for the control of the orbits of artificial satellites as we shall see below (see also lectures on *Space Exploration*).
Concerning the periodic terms in J_2, we find

$$\begin{aligned} J_2\,\frac{d\,a_1}{dt} &= \frac{n_0\,J_2\,R_e^2}{a_0}\left[\left(\frac{9}{2}\,\sin^2 i_0 - 3\right) e_0\,\sin(M_0 + J_2\,\overline{M}\,t)\right.\\ &+ \frac{3}{2}\,\sin^2 i_0\,\left(\frac{e_0}{2}\,\sin(2\,\omega_0 + M_0 + J_2\,(2\,\overline{\varpi} + \overline{M})\,t) - 2\,\sin(2\,\omega_0 + 2\,M_0 + 2\,J_2\,(\overline{\varpi} + \overline{M})\,t)\right.\\ &\left.\left.-\frac{21\,e_0}{2}\,\sin(2\,\omega_0 + 3\,M_0 + J_2\,(2\,\overline{\varpi} + 3\,\overline{M})\,t)\right)\right] \end{aligned} \tag{6.104}$$

$$...$$

$$\begin{aligned} J_2\,\frac{d\,M_1}{dt} &= \frac{-3\,n_0}{2\,a_0}\,J_2\,a_1 + \frac{n_0\,J_2\,R_e^2}{a_0^2}\,\left[\left(\frac{1}{2} - \frac{3}{4}\,\sin^2 i_0\right)\left(\frac{-3}{e_0}\,\cos(M_0 + J_2\,\overline{M}\,t)\right.\right.\\ &-9\,\cos(2\,M_0 + 2\,J_2\overline{M}\,t)\Big] + \frac{3}{4}\,\sin^2 i_0\,\left[\frac{1}{2\,e_0}\,\cos(2\,\omega_0 + M_0 + J_2\,(2\,\overline{\varpi} + \overline{M})\,t)\right. \end{aligned}$$

$$
\begin{aligned}
&+11\,\cos\left(2\,\omega_0+2\,M_0+2\,J_2\,(\overline{\varpi}+\overline{M})\,t\right)-\frac{7}{2\,e_0}\,\cos\left(2\,\omega_0+3\,M_0+J_2\,(2\,\overline{\varpi}+3\,\overline{M})\,t\right)\\
&-17\,\cos\left(2\,\omega_0+4\,M_0+2\,J_2\,(\overline{\varpi}+2\,\overline{M})\,t\right)\Big]\Big\}
\end{aligned}
\tag{6.105}
$$

Note that in the latter equation, we have omitted the terms in e since they are dominated by the $1/e$ and e^0 terms.

We can now integrate the derivative of the $J_2\,a_1$ term

$$
\begin{aligned}
J_2\,a_1 \;=\; & -\frac{n_0\,J_2\,R_e^2}{a_0}\left[\left(\frac{9}{2}\,\sin^2 i_0-3\right)\,e_0\,\frac{\cos\left(M_0+J_2\,\overline{M}\,t\right)}{n_0+J_2\,\overline{M}}\right.\\
&+\frac{3}{2}\,\sin^2 i_0\left(\frac{e_0}{2}\,\frac{\cos\left(2\,\omega_0+M_0+J_2\,(2\,\overline{\varpi}+\overline{M})\,t\right)}{n_0+J_2\,(\overline{M}+2\,\overline{\varpi})}-\frac{\cos\left(2\,\omega_0+2\,M_0+2\,J_2\,(\overline{\varpi}+\overline{M})\,t\right)}{n_0+J_2\,(\overline{M}+\overline{\varpi})}\right.\\
&\left.\left.-\frac{21\,e_0}{2}\,\frac{\cos\left(2\,\omega_0+3\,M_0+J_2\,(2\,\overline{\varpi}+3\,\overline{M})\,t\right)}{3\,n_0+J_2\,(3\,\overline{M}+2\,\overline{\varpi})}\right)\right]
\end{aligned}
$$

This expression can be further simplified. Indeed, since $\frac{J_2}{n_0+\beta\,J_2}=\frac{J_2}{n_0}+\mathcal{O}((J_2)^2)$, we can write to the first order in J_2:

$$
\begin{aligned}
J_2\,a_1 \;=\; & -\frac{J_2\,R_e^2}{a_0}\left[\left(\frac{9}{2}\,\sin^2 i_0-3\right)\,e_0\,\cos\left(M_0+J_2\,\overline{M}\,t\right)\right.\\
&+\frac{3}{2}\,\sin^2 i_0\left(\frac{e_0}{2}\,\cos\left(2\,\omega_0+M_0+J_2\,(2\,\overline{\varpi}+\overline{M})\,t\right)-\cos\left(2\,\omega_0+2\,M_0+2\,J_2\,(\overline{\varpi}+\overline{M})\,t\right)\right.\\
&\left.\left.-\frac{7\,e_0}{2}\,\cos\left(2\,\omega_0+3\,M_0+J_2\,(2\,\overline{\varpi}+3\,\overline{M})\,t\right)\right)\right]
\end{aligned}
\tag{6.106}
$$

Similar expressions can be derived for the first order periodic terms of the other elements. We note that the expression of a_1 only features short period terms. The same conclusion holds for the first order perturbations due to J_2 of all other elements.
In summary, the first order effect of J_2 on the angular elements is the appearance of a secular variation (that drops with distance as a_0^{-2}) and of short period variations. Conversely, the metric elements display only short term periodic variations.

6.5.1 Applications of the secular perturbations

An important application of the secular perturbation of Ω concerns the Sun-synchronous orbits that are (essentially circular) orbits of artificial satellites where the parameters (i_0 and a_0) are fine-tuned in such a way that $J_2\,\overline{\Omega_1}$ amounts exactly to one rotation of the line of nodes in one revolution period of the planet around the Sun. In such a way, the line of nodes keeps a constant orientation with respect to the direction between the Sun and the planet the satellite is revolving (see Fig. 6.3).
Another important application of the secular variations due to J_2 concerns the $J_2\,\overline{\omega_1}$ term. In fact, this term becomes zero if $\sin^2 i_0 = 0.8$. This trigonometric equation hence defines a critical inclination i_c of 63.4°. A satellite orbiting a planet will have its pericenter and apocenter at constant directions within the plane of the osculating orbit. Note that the value of the critical inclination does not depend upon the value of J_2. In other words, the value of the critical inclination is the same for different planets. Practical applications of this result are the highly eccentric Molniya and Tundra orbits that have orbital periods of 12 and 24 hours respectively. The Molniya satellites are used for telecommunications and for military observations. Their apogees occur alternatively (i.e. every second orbit) over the territories of Russia and North America. Since the satellites have highly eccentric

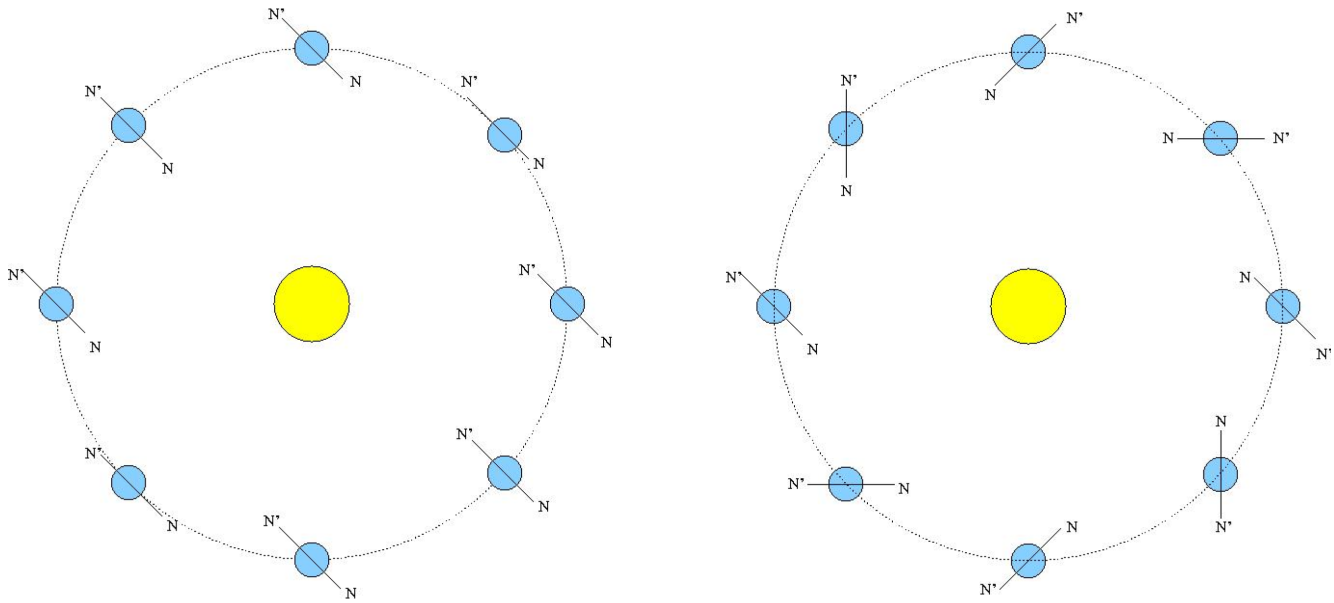


Figure 6.3: Left: for a satellite revolving a spherical planet, $J_2 = 0$ and the direction of the line of nodes is fixed in space (N and N' are respectively the ascending and the descending nodes). Right: for a Sun-synchronous orbit, the value of $J_2\,\overline{\Omega_1}$ is tailored in such a way that the line of nodes performs one rotation in the same time it takes the planet to revolve around the Sun. In this way, the line of nodes maintains a constant orientation with respect to the direction between the Sun and the planet.

orbits they move rather slowly when they are near apogee and the combination of their motion with the Earth's rotation leads to a very slow apparent motion as seen from the ground. Hence they remain visible from a chosen area of the Earth for a relatively large fraction of their orbital period (see Fig. 6.4).

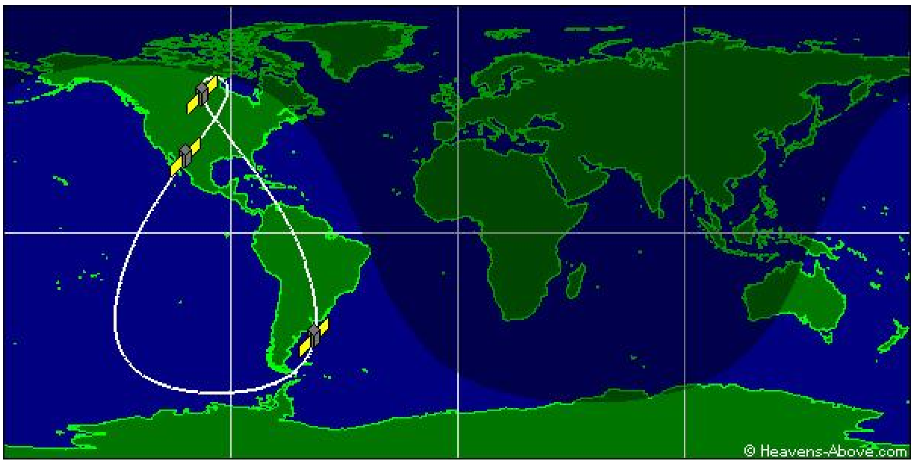


Figure 6.4: Ground track of a satellite revolving on a Tundra orbit.

For inclinations larger than i_c, the major axis will undergo a retrograde rotation, whilst it will be prograde for $i < i_c$.

6.5.2 Other effects of the potential of the Earth

The same kind of developments can be performed for the other zonal terms J_n with $n > 2$. Usually, the corresponding coefficients J_n are of order $(J_2)^2$ or smaller and therefore one has to simultaneously deal with the higher order perturbations due to J_2. The associated potentials also produce secular as well as periodic (both long and short term) variations.

A somewhat more complex situation is encountered for the sectoral and tesseral terms (c_{np} and s_{np}). They involve indeed the functions $\sin{(p\,\lambda)}$ and $\cos{(p\,\lambda)}$ where λ is the geographical longitude. If $\alpha(t)$ is the right ascension of the satellite at time t and t_* is the sidereal time along the Greenwich meridean at time t, we have

$$\lambda = \alpha(t) - t_*$$

with

$$t_* = \nu_{\oplus}\,(t - t_0)$$

where $\nu_{\oplus}$ is the frequency of the Earth rotation (corresponding to a period of 23h 56min 04sec). Therefore, these terms introduce diurnal perturbations into the motion of an artificial satellite.

We have to emphasize once more that all the perturbations of the motion of an artificial satellite due to the non-spherical shape of the Earth decrease with distance. At low altitudes (200 to 500 km above the ground), the most important perturbations stem from J_2 and from the drag force of the residual atmosphere.

Finally, we stress that the notion of osculating orbits is not restricted to perturbations due to atmospheric drag and the non-spherical shape of the planet. Any non-Keplerian force can be dealt with in this way as illustrated by Fig. 6.5 that shows the effect of the ionic propulsion system (that was active over a major part of the spacecraft's journey) of the Smart 1 probe.

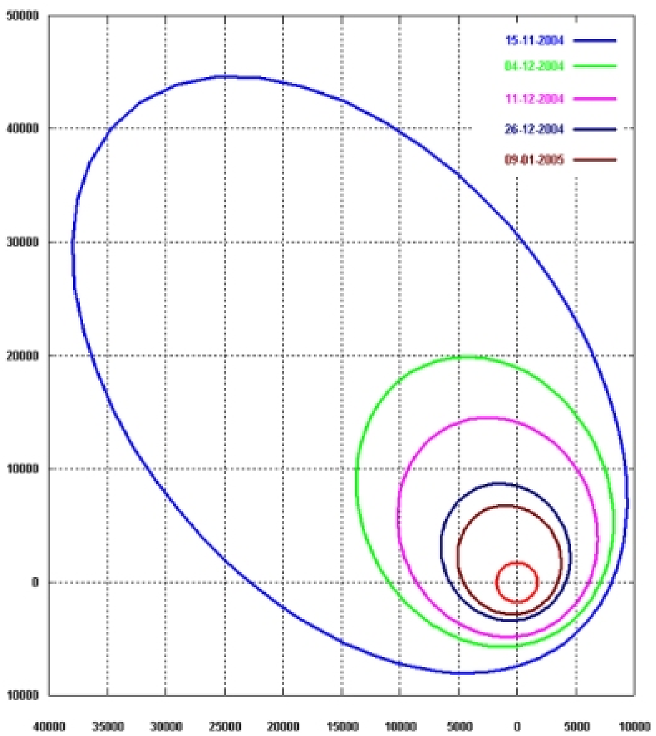


Figure 6.5: Illustration of the osculating orbit of the Smart 1 probe as it approached the Moon. In this specific case, the force $\vec{P}$ was due to the ionic propulsion system.

6.5.3 The interior structure of celestial bodies

By using the theory of perturbations of a Keplerian orbit developed in this chapter, it is possible to constrain the values of the moments of inertia of a celestial body. Indeed, measuring the trajectory of a spacecraft orbiting a planet, moon, asteroid or comet and comparing this orbit to the theoretical predictions allows to determine *a posteriori* the various terms of the gravitational potential (J_2, J_4,...).
Why is this of interest? The reason is that the knowledge of the moments of inertia provides some characterization of the internal distribution of mass that goes beyond the mere information of the mean density.

Consider a uniform sphere of radius R and mass M. The moment of inertia (see Sect. 5.5) about an axis of symmetry is $0.4\,M\,R^2$. Planets and, even more so, the minor bodies of the Solar System are not perfectly spherical. Most planets are indeed flattened by rotation and display a roughly ellipsoidal shape. This means that their moment of inertia about the rotation axis ($\mathcal{I}_{zz}$) is larger than the moments ($\mathcal{I}_{xx}$ and $\mathcal{I}_{yy}$) about the other two axes. The difference $2\,\mathcal{I}_{zz} - \mathcal{I}_{xx} - \mathcal{I}_{yy} = 2\,J_2\,M\,R_e^2$ (see eq. 5.27) is an indication of how much excess mass is concentrated towards the equator.

From equation 5.20 and restricting ourselves to the J_2 term, we obtain:

$$U = -\frac{G\,M}{r} + \frac{G\,M\,R_e^2}{r^3}\,J_2\,(\frac{3}{2}\sin^2\theta - \frac{1}{2}) \tag{6.107}$$

This expression of the potential is valid in an inertial frame of reference. If instead, we treat the potential in a frame of reference rotating with the planet (e.g. by treating a point at the surface of the planet), we need to correct the above expression by subtracting the potential associated with the centrifugal force: $\frac{1}{2}\,\omega^2\,r^2\,\cos^2\theta$.
We thus obtain

$$U = -\frac{G\,M}{r} + \frac{G\,M\,R_e^2}{r^3}\,J_2\,(\frac{3}{2}\sin^2\theta - \frac{1}{2}) - \frac{1}{2}\,\omega^2\,r^2\,\cos^2\theta \tag{6.108}$$

For a fluid planet, the potential has the same value all over the planet's surface. This property then allows us to predict the actual shape of the planet. If we call R_e and R_p, the radii at the equator and at the pole, we can express the flattening of the planet as

$$f = \frac{R_e - R_p}{R_e} \simeq \epsilon$$

Hence the flattening f is essentially equivalent to the ellipticity of the spheroid ϵ (see also equation 5.37). As a next step, we then express the condition that the potentials at the pole and near the equator have the same value. This leads to an approximate expression for f:

$$f \simeq \frac{\frac{3}{2}\,J_2 + \frac{1}{2}\,\frac{\omega^2\,R_e^3}{G\,M}}{1 - 3\,J_2} \tag{6.109}$$

which, considering the numerical values of the terms in the denominator, can usually be further simplified into

$$f \simeq \frac{3}{2}\,J_2 + \frac{1}{2}\,\frac{\omega^2\,R_e^3}{G\,M} \tag{6.110}$$

which is equivalent to equation 5.43. Assuming a fluid Earth, we can use the numbers in Table 5.1 and $\omega = 7.272 \times 10^{-5}\,\mathrm{s}^{-1}$ to compute the Earth's flattening as $f = 3.346 \times 10^{-3}$. This number is in excellent agreement with the observed value $f = 3.353 \times 10^{-3}$ (World Geodetic System 84). Therefore, the fluid approximation seems acceptable to describe the Earth's interior.
By means of the theory of hydrostatic planets or from the knowledge of the rate of precession (see chapter 8), we can eventually express the ratio between $\mathcal{I}_{zz}$ and $M\,R_e^2$. For the Earth, this ratio amounts to 0.3308, which is less

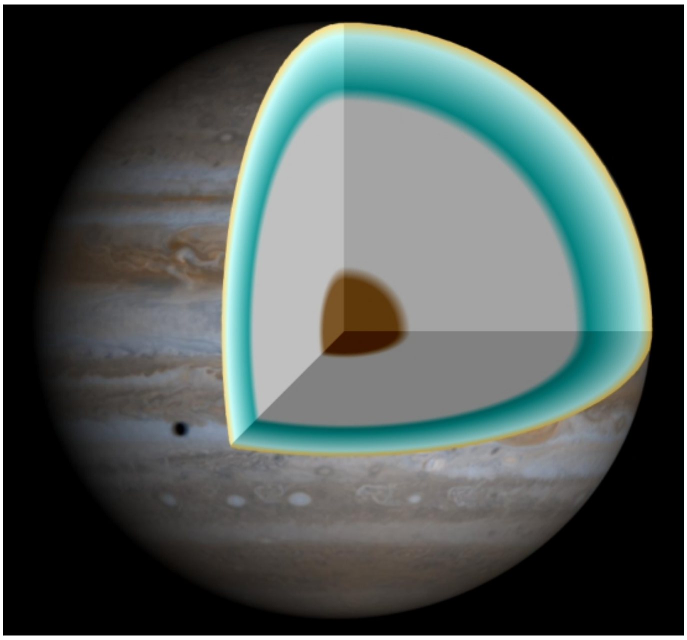


Figure 6.6: Illustration of the current theoretical view of the interior of Jupiter: the planet is thought to have a rocky core surrounded by a deep layer of metallic hydrogen and an outer layer of molecular hydrogen.

than the value of a uniform sphere (0.4), indicating a concentration of mass towards the center of our planet. This so-called differentiation is observed for most planets and moons in the Solar System.
The measurement of the gravitational potential of a celestial body hence allows us to get important constraints about its internal structure. Simple two-layer body models consisting of a mantle and a core can be constrained by this method. There are several unknows to determine: the core and mantle densities as well as the core radius. In general, one needs to adopt one of these parameters and derive the other two. More complex theoretical models of planetary formation predict the existence of several layers in the planetary interior. These models can also be tested against the observational determinations of the moments of inertia.

6.6 Exercises

6.1 Express the perturbating potential $U = \eta/r^3$ in terms of the elements of the osculating orbit. Limit the periodic terms to e^3, but keep the full independent term of the expansion to show that:

$$U = \frac{\eta}{a^3}\left[(1-e^2)^{-3/2} + \left(3\,e + \frac{27\,e^3}{8}\right)\cos M + \frac{9\,e^2}{2}\cos(2\,M) + \frac{53\,e^3}{8}\cos(3\,M) + \mathcal{O}(e^4)\right]$$

Insert this expression into the Lagrange equations 6.47 to establish the equations of the perturbations of the elements of the osculating orbit, keeping for each Lagrange equation the three most significant orders in e^n for the periodic terms.
Suggestion: use (but do not demonstrate) expression 6.89.

6.2 Express the Lagrange equations for the independent term in J_3 of the Earth's potential:

$$U_3 = \frac{3\,\mu}{2\,a^4}\,R^3\,J_3\,\sin\omega\,\sin i\left(\frac{5}{4}\sin^2 i - 1\right)\frac{e}{(1-e^2)^{5/2}}$$

Under which circumstances do the secular perturbations of Ω and e associated with the J_3 term cancel out?

6.3 Considering the perturbations due to the J_2 term, compare the values of the critical inclinations for Molniya-type orbits around the planets Earth and Mars. Compare also the inclination required for a circular heliosynchronous orbit at an altitude of 1000 km above the surface of either planet Mars or planet Earth.
Suggestion: use the parameters quoted in Tables 5.1 and 2.1 (pay attention to the units!). Use equations 6.101 and 6.102, but do not re-demonstrate these equations.

6.4 Show that the independent term of the geopotential in J_3

$$U_3 = \frac{3\,\mu}{2\,a^4}\,R_e^3\,J_3\,\sin\omega\,\sin i\left(\frac{5}{4}\,\sin^2 i - 1\right)\frac{e}{(1-e^2)^{5/2}}$$

gives rise to a long-period perturbation of the eccentricity of the orbit given by

$$\delta e = -\frac{J_3}{2\,J_2}\,\frac{R_e}{a}\,\sin i\,\sin\omega$$

where $\omega = \omega_0 + \overline{\omega_1}\,J_2\,t$.
Express the period of this perturbation. What are the values of this period and of the amplitude of the perturbation for a satellite orbiting the Earth with $a_0 = 8900\,\mathrm{km}$, $e_0 = 0.2$, and $i = 35°$?
Suggestion: use the values from Table 5.1 along with the Lagrange equations 6.47 and use also equation 6.102, but do not re-demonstrate it.

6.5 Consider an artificial satellite of mass m orbiting the Earth. The Earth is assimilated to a sphere of radius R and mass M, surrounded by an atmosphere of density $\rho(r) = \rho_0\,\exp\left[-k\,(r - r_0)\right]$ with k, ρ_0, r_0 being positive constants. The drag force experienced by the satellite is $\vec{F} = -b\,\rho(r)\,|\dot{\vec{r}}|\,\dot{\vec{r}}$.
Use relations 6.24 and 6.25, to express the time derivatives of the distance at perigee r_p and apogee r_a. Using the fact that $\frac{dE}{dt} = \frac{a}{r}\,n$ with E the eccentric anomaly, and $n = \sqrt{\frac{GM}{a^3}}$, express $\frac{dr_p}{dE}$ and $\frac{dr_a}{dE}$ as a function of the elements of the osculating orbit and of E.

6.6 Consider the J_4 term of the geopotential. Using the relation $\sin\theta = \sin i\,\sin(\omega + \phi)$, show that the potential per unit mass associated with J_4 for a satellite in orbit around the Earth can be expressed:

$$\begin{aligned} U' &= GM\,J_4\,\frac{R_e^4}{a^5}\left(\frac{a}{r}\right)^5\left[\frac{105}{64}\,\sin^4 i - \frac{35}{16}\,\sin^4 i\,\cos(2\,\omega + 2\,\phi) + \frac{35}{64}\,\sin^4 i\,\cos(4\,\omega + 4\,\phi) - \frac{15}{8}\,\sin^2 i \right.\\ &\left. + \frac{15}{8}\,\sin^2 i\,\cos(2\,\omega + 2\,\phi) + \frac{3}{8}\right] \end{aligned}$$

Here θ, i, ω and ϕ are the instantaneous latitude of the satellite, the inclination of the orbit with respect to the equator, the longitude of perigee and the true anomaly, respectively.

Using the fact that

$$\frac{1}{\pi}\int_0^{2\,\pi}\left(\frac{a}{r}\right)^5\cos p\,\phi\,dM = \frac{1}{(1-e^2)^{7/2}}\left[2\,\delta_{p,0}\,(1 + \frac{3\,e^2}{2}) + (3\,e + \frac{3\,e^3}{4})\,\delta_{p,1} + \frac{3\,e^2}{2}\,\delta_{p,2} + \frac{e^3}{4}\,\delta_{p,3}\right]$$

compute the independent term of U'.
Finally using the Lagrange equations 6.47, establish the secular term of $\dot{\Omega}$ for the potential associated with J_4.

6.7 Consider a satellite orbiting an idealized planet with the shape of a spheroid of ellipticity ε. The orbit is located in the planet's equatorial plane and is almost circular ($e << 1$) with a semi-major axis a. The potential experienced by the satellite can be expressed as

$$V(r) = -\frac{GM}{r}\left(1 + \frac{\varepsilon}{5}\,\frac{R^2}{r^2}\right)$$

Show that the major axis of the orbit precesses with time at a velocity

$$\frac{d\,\omega}{dt} = \frac{3\,\varepsilon}{5}\,\frac{R^2}{a^2}\,n$$

where n is the mean orbital angular velocity of the satellite.

Chapter 7

The N-body problem

In this chapter, we consider the problem of the motion of N spherical or point-like masses that form an isolated system (i.e. there are no external forces acting on the system, the only forces are internal and arise from the mutual gravitational effects). We shall label these masses as m_k with $k = 0, ..., n$ (where $n = N - 1$) and their positions are given as $\vec{OP_k}$ with respect to the origin O of an inertial frame of reference. Newton's equations for this system can thus be expressed as

$$m_k \frac{d^2 \vec{OP_k}}{dt^2} = \sum_{i=0}^{k-1} \frac{G\, m_k\, m_i}{|\vec{P_k P_i}|^3} \vec{P_k P_i} + \sum_{i=k+1}^{n} \frac{G\, m_k\, m_i}{|\vec{P_k P_i}|^3} \vec{P_k P_i} \tag{7.1}$$

This problem does not have a general solution. In fact, the resolution of these equations requires the knowledge of $6 \times N$ constants of integration. However, as we will show below, there are only 10 classical integrals that can be defined for the general problem formulated by equation 7.1.
In this chapter, we will therefore focus on some particular solutions as well as on situations that can be reduced to a two-body problem with some perturbations reflecting the action of the other masses.

7.1 Integrals of the equations of motion

The very first integrals to consider concern the uniform velocity straight-line motion of the center of mass C of the system. Indeed, since there are no external forces acting on the system, its center of mass follows a straight-line motion at a constant speed.

$$\sum_{k=0}^{n} m_k \frac{d^2 \vec{OP_k}}{dt^2} = \vec{0}$$

$$\Rightarrow \left(\sum_{k=0}^{n} m_k\right) \frac{d^2 \vec{OC}}{dt^2} = \vec{0}$$

$$\vec{OC}(t) = \vec{OC}(t_0) + \vec{V_C}\, t \tag{7.2}$$

The latter equation involves 6 scalar constants (the 3 components of two vectors), hence reducing the number of independent variables from $6 \times N$ to $6 \times (N - 1)$. This is somewhat equivalent to a $N - 1$ body problem. In practice one could use the center of mass as the origin of the inertial axes and solve the equations of $N - 1$ bodies (m_1 to m_{N-1}) and eventually derive the position of the last body from $m_0\, \vec{CP_0} = -\sum_{i=1}^{n} m_i\, \vec{CP_i}$

The next quantity that is constant as a function of time is the total angular momentum. Indeed, since there are no external forces acting on the system, and since the moments of the different gravitational forces compensate each other by pairs, we can write

$$\begin{aligned}
\sum_{k=0}^{n} m_k\, \vec{OP_k} \wedge \frac{d^2\, \vec{OP_k}}{dt^2} &= \sum_{k=0}^{n} \sum_{i\neq k} \vec{OP_k} \wedge \frac{G\, m_i\, m_k}{|\vec{P_kP_i}|^3}\, \vec{P_kP_i} \\
&= \sum_{k=0}^{n} \sum_{i\neq k} (\vec{OP_k} + \vec{P_kP_i}) \wedge \frac{G\, m_i\, m_k}{|\vec{P_kP_i}|^3}\, \vec{P_kP_i} \\
&= -\sum_{k=0}^{n} \sum_{i\neq k} \vec{OP_i} \wedge \frac{G\, m_i\, m_k}{|\vec{P_kP_i}|^3}\, \vec{P_iP_k} = \vec{0}
\end{aligned}$$

$$\Rightarrow \frac{d}{dt}\left(\sum_{k=0}^{n} \vec{OP_k} \wedge m_k\, \frac{d\, \vec{OP_k}}{dt}\right) = \vec{0}$$

$$\sum_{k=0}^{n} \vec{OP_k} \wedge m_k\, \frac{d\, \vec{OP_k}}{dt} = \vec{Cst} \tag{7.3}$$

Since the center of mass is the orgin of an inertial frame, we can also express the conservation of the angular momentum if the positions are measured from the center of mass:

$$\sum_{k=0}^{n} \vec{CP_k} \wedge m_k\, \frac{d\, \vec{CP_k}}{dt} = \vec{h} \tag{7.4}$$

Finally, we can express the conservation of the total energy of the system (again this stems from the fact that no external forces act on the N-body system). Let T be the total kinetic energy. We thus obtain:

$$\begin{aligned}
\frac{dT}{dt} &= \sum_{k=0}^{n} m_k\, \frac{d\, \vec{OP_k}}{dt} \cdot \frac{d^2\, \vec{OP_k}}{dt^2} \\
&= \sum_{k=0}^{n} \sum_{i\neq k} \frac{d\, \vec{OP_k}}{dt} \cdot \frac{G\, m_i\, m_k}{|\vec{P_kP_i}|^3}\, \vec{P_kP_i} \\
&= \sum_{k=0}^{n-1} \sum_{i=k+1}^{n} \left(\frac{d\, \vec{OP_k}}{dt} - \frac{d\, \vec{OP_i}}{dt}\right) \cdot \frac{G\, m_i\, m_k}{|\vec{P_kP_i}|^3}\, \vec{P_kP_i} \\
&= -\sum_{k=0}^{n-1} \sum_{i=k+1}^{n} \frac{d\, \vec{P_kP_i}}{dt} \cdot \frac{G\, m_i\, m_k}{|\vec{P_kP_i}|^3}\, \vec{P_kP_i} \\
&= \frac{d}{dt}\left(\sum_{k=0}^{n-1} \sum_{i=k+1}^{n} \frac{G\, m_i\, m_k}{|\vec{P_kP_i}|}\right)
\end{aligned}$$

$$\frac{1}{2}\sum_{k=0}^{n} m_k \left|\frac{d\, \vec{OP_k}}{dt}\right|^2 - \left(\sum_{k=0}^{n-1} \sum_{i=k+1}^{n} \frac{G\, m_i\, m_k}{|\vec{P_kP_i}|}\right) = \mathcal{E} \tag{7.5}$$

In the latter relation, the potential U can be expressed as either of the two forms:

$$U = -\left(\sum_{k=0}^{n-1} \sum_{i=k+1}^{n} \frac{G\, m_i\, m_k}{|\vec{P_kP_i}|}\right) = -\frac{1}{2}\sum_{k=0}^{n-1} \sum_{i\neq k} \frac{G\, m_i\, m_k}{|\vec{P_kP_i}|}$$

Again, we stress that the N-body problem does not have a general analytical solution. Usually, the equations (7.1) are solved numerically. Nevertheless, the above integrals of the motion are important to check the impact of the error propagation on the results of the numerical integration. Indeed, we know that the solution must satisfy to equations 7.2, 7.4 and 7.5 and the comparison between the results of the numerical resolution of Newton's equations (7.1) and the value of the integrals of the motion is therefore a fundamental test of the validity of the solution.

7.2 The relative motion

As we have seen above, the N-body problem can be somewhat simplified by accounting for the integral of the motion related to the center of mass C and by expressing the equations of an $N-1$ body problem. Let us thus express the positions of the various masses with respect to C:

$$\vec{u_k} = \vec{CP_k}$$
$$\Rightarrow \vec{u_0} = -\sum_{k=1}^{n} \frac{m_k}{m_0}\,\vec{u_k}$$

Newton's equations can now be expressed as

$$\frac{d^2\,\vec{u_k}}{dt^2} = G\,m_0\,\frac{\vec{u_0}-\vec{u_k}}{|\vec{u_0}-\vec{u_k}|^3} + G\sum_{i=1,i\neq k}^{n} m_i\,\frac{\vec{u_i}-\vec{u_k}}{|\vec{u_i}-\vec{u_k}|^3}$$

or

$$m_k\,\frac{d^2\,\vec{u_k}}{dt^2} = -\vec{\nabla}_k\,U \tag{7.6}$$

where $\vec{\nabla}_k\,U = \left(\frac{\partial U}{\partial x_k}, \frac{\partial U}{\partial y_k}, \frac{\partial U}{\partial z_k}\right)$ and $U = -\sum_{j=0}^{n-1}\sum_{i=j+1}^{n} \frac{G\,m_i\,m_j}{[(x_i-x_j)^2+(y_i-y_j)^2+(z_i-z_j)^2]^{1/2}}$.
It is often more advantageous to express the equations not with respect to the center of mass, but with respect to one of the masses m_0 (which we usually choose to be the most massive one).

$$\vec{r_k} = \vec{P_0P_k} = \vec{u_k} - \vec{u_0}$$
$$\Rightarrow \vec{r_0} = \vec{0}$$

Newton's equations can now be expressed as

$$\begin{aligned}\frac{d^2\,\vec{r_k}}{dt^2} &= \sum_{i=0,i\neq k}^{n} G\,m_i\,\frac{\vec{r_i}-\vec{r_k}}{|\vec{r_i}-\vec{r_k}|^3} - \sum_{i=1}^{n} G\,m_i\,\frac{\vec{r_i}}{|\vec{r_i}|^3} \\ &= -G\,(m_0+m_k)\,\frac{\vec{r_k}}{|\vec{r_k}|^3} + \sum_{i=1,i\neq k}^{n} G\,m_i\left(\frac{\vec{r_i}-\vec{r_k}}{|\vec{r_i}-\vec{r_k}|^3} - \frac{\vec{r_i}}{|\vec{r_i}|^3}\right)\end{aligned} \tag{7.7}$$

The first term on the right-hand side of equation 7.7 reflects the pure Keplerian acceleration experienced by mass m_k from m_0, whilst the sum reflects a differential acceleration due to the other masses of the system. As we shall see hereafter, this formulation of the equations of Newton opens up the possibility to treat the effect of the other bodies on m_k as a perturbation with respect to the effect of m_0. This result can also be expressed as

$$\frac{d^2\,\vec{r_k}}{dt^2} = -\frac{G\,(m_0+m_k)\,\vec{r_k}}{|\vec{r_k}|^3} + \vec{\nabla}_k\,V_k \tag{7.8}$$

where $V_k = \sum_{i=1,i\neq k}^{n} G\,m_i\left(\frac{1}{|\vec{r_i}-\vec{r_k}|} - \frac{\vec{r_i}\cdot\vec{r_k}}{|\vec{r_i}|^3}\right)$.
Note that the V_k potential is a different function for each mass m_k and hence it does not allow to construct a Hamiltonian for the global problem.

7.3 The 3-body problem

As an illustration of the N-body problem, let us consider here the case where $N = 3$. We can express Newton's equations as

$$\frac{d^2\,\vec{r_1}}{dt^2} = -G\,(m_0 + m_1)\,\frac{\vec{r_1}}{|\vec{r_1}|^3} + G\,m_2\left(\frac{\vec{r_2} - \vec{r_1}}{|\vec{r_2} - \vec{r_1}|^3} - \frac{\vec{r_2}}{|\vec{r_2}|^3}\right) \tag{7.9}$$

$$\frac{d^2\,\vec{r_2}}{dt^2} = -G\,(m_0 + m_2)\,\frac{\vec{r_2}}{|\vec{r_2}|^3} + G\,m_1\left(\frac{\vec{r_1} - \vec{r_2}}{|\vec{r_1} - \vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3}\right) \tag{7.10}$$

As an example, we consider the case of the Earth's motion around the Sun accounting for the influence of Jupiter, the most massive planet of the Solar System. In this case, the indices 0, 1 and 2 refer to the Sun ($\odot$), the Earth ($\oplus$) and Jupiter ($\jupiter$) respectively.

$$\frac{d^2\,\vec{r_\oplus}}{dt^2} + G\,(m_\odot + m_\oplus)\,\frac{\vec{r_\oplus}}{|\vec{r_\oplus}|^3} = G\,m_\jupiter\left(\frac{\vec{r_\jupiter} - \vec{r_\oplus}}{|\vec{r_\jupiter} - \vec{r_\oplus}|^3} - \frac{\vec{r_\jupiter}}{|\vec{r_\jupiter}|^3}\right) = \vec{\mathcal{P}}$$

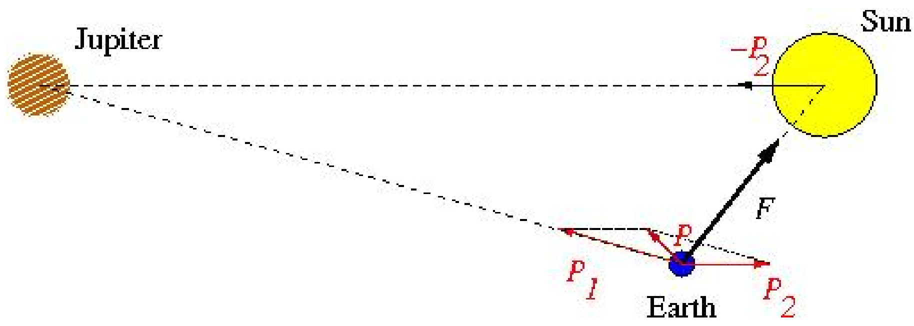


Figure 7.1: Schematic illustration of the 3-body interaction between the Sun, the Earth and Jupiter. Note that the vectors, the distances and the dimensions of the different bodies are not to scale.

Here, $\vec{\mathcal{P}}$ is the perturbation of the Keplerian acceleration $\vec{\mathcal{F}} = -G\,(m_\odot + m_\oplus)\,\frac{\vec{r_\oplus}}{|\vec{r_\oplus}|^3}$. $\vec{\mathcal{P}}$ consists of two components, $\vec{\mathcal{P}}_1$ which is the acceleration of the Earth due to the attraction of Jupiter and $\vec{\mathcal{P}}_2$ which is the opposite of the acceleration of the Sun due to the attraction of Jupiter. The sum of these two vectors yields $\vec{\mathcal{P}}$ which must then be compared to $\vec{\mathcal{F}}$ (see Fig. 7.1).
We can now consider two extreme situations to evaluate the order of magnitude of $\vec{\mathcal{P}}$. If we assume that the configuration of the three bodies is such that the distance between the Sun and Jupiter is equal to the distance between Jupiter and the Earth (the three bodies form an isosceles triangle), we obtain that $|\vec{\mathcal{P}}| = \frac{r_\oplus}{r_\jupiter}|\vec{\mathcal{P}}_2| = \frac{G\,m_\jupiter\,r_\oplus}{r_\jupiter^3}$. Hence, we find that in this case

$$\frac{|\vec{\mathcal{P}}|}{|\vec{\mathcal{F}}|} = \frac{m_\jupiter}{m_\odot + m_\oplus}\left(\frac{r_\oplus}{r_\jupiter}\right)^3$$

The Sun's mass is equal to $1.989\,10^{33}$ g, whilst the Earth's mass is $5.976\,10^{27}$ g and Jupiter is 317.8 times more massive than the Earth. The mean distance between Jupiter and the Sun amounts to 5.203 AU. Hence, the above ratio becomes $\frac{|\vec{\mathcal{P}}|}{|\vec{\mathcal{F}}|} = 6.8\,10^{-6}$. If we repeat the same reasoning for a configuration where the three bodies are aligned (the Earth being between the Sun and Jupiter), we obtain that $\frac{|\vec{\mathcal{P}}|}{|\vec{\mathcal{F}}|} = 1.88\,10^{-5}$. We conclude that on average

$$\frac{|\vec{\mathcal{P}}|}{|\vec{\mathcal{F}}|} \simeq 10^{-5}$$

We can perform a similar calculation to evaluate the impact of the Moon's differential attraction on a satellite in a geostationary orbit (radius of 42160 km). Noting that $m_{\leftmoon}\,/m_{\earth} = 0.0123$ and the Moon's distance from Earth is 380 000 km, we find that the perturbation is of the same order of magnitude as the one due to Jupiter on the Earth's motion.

7.3.1 The Lagrange solutions of the 3-body problem

Unlike the general N-body problem, the 3-body problem has some analytical solutions. These are known as the Lagrange solutions and lead to the so-called Lagrangian points in the case of a circular restricted three body problem (see Sect. 7.3.2). For now, we consider that the three masses m_0, m_1 and m_2 have arbitrary values (this assumption will be changed in the next subsection). In this case, Lagrange considered two particular configurations of the three masses where the equations of the problem can be solved.

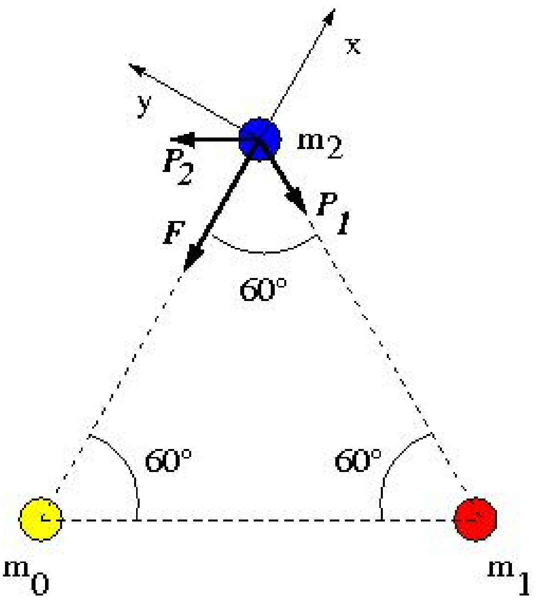


Figure 7.2: Schematic illustration of the axes used for the Lagrange solution of the three-body problem.

Consider a situation where m_0, m_1 and m_2 occupy the summits of an equilateral triangle, $|\vec{r_1}| = |\vec{r_2}| = |\vec{r_2} - \vec{r_1}| = r_2$, inside the orbital plane of m_1 around m_0. In this case, the projection of equation 7.10 onto the axes of the system (see Fig. 7.2) yields:

$$\begin{aligned}
\ddot{x}_2 &= -G\,(m_0 + m_2)\,\frac{1}{r_2^2} - \frac{1}{2}\,G\,m_1\left(\frac{1}{r_2^2} + \frac{1}{r_2^2}\right) = -\frac{G\,(m_0 + m_1 + m_2)}{r_2^2} \\
\ddot{y}_2 &= -\frac{G\,m_1}{r_2^2}\,\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = 0
\end{aligned}$$

Therefore, the equations of the problem reduce to those of two decoupled equivalent two-body problems:

$$\frac{d^2\,\vec{r_1}}{dt^2} = -G\,(m_0 + m_1 + m_2)\,\frac{\vec{r_1}}{|\vec{r_1}|^3} \tag{7.11}$$

$$\frac{d^2\,\vec{r_2}}{dt^2} = -G\,(m_0 + m_1 + m_2)\,\frac{\vec{r_2}}{|\vec{r_2}|^3} \tag{7.12}$$

From these considerations, it becomes clear that, provided that we have $|\dot{\vec{r_1}}| = |\dot{\vec{r_2}}|$ and $\vec{r_1} \wedge \dot{\vec{r_1}} = \vec{r_2} \wedge \dot{\vec{r_2}}$ at time $t = 0$, m_1 and m_2 actually describe conical trajectories of the same eccentricity, the same semi-major axis and the same focus (m_0), but rotated by 60° with respect to each other (see Fig. 7.3 for the case of an elliptical orbit).

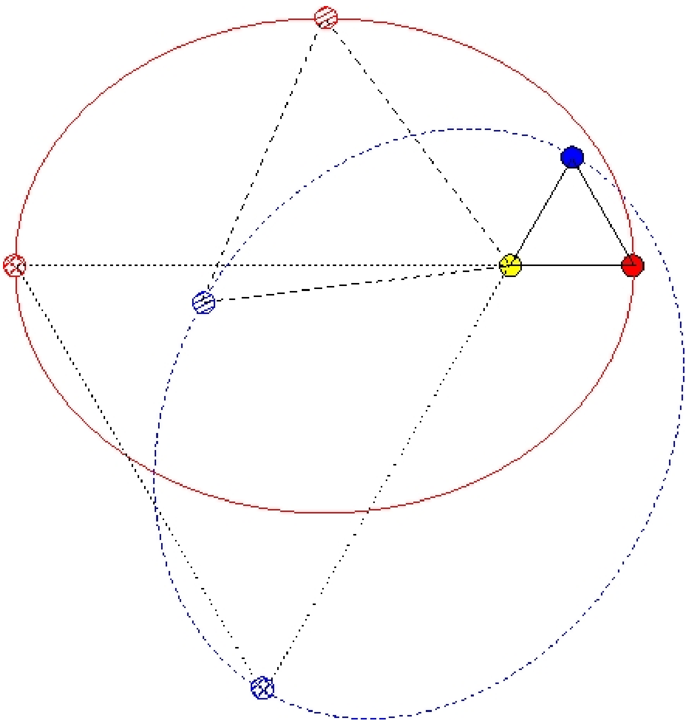


Figure 7.3: The Lagrange solution of the 3-body problem in case of an equilateral triangular configuration (shown here at three different phases of the motion).

Another set of solutions is obtained if we consider that the three bodies are aligned. This can be expressed as $\vec{r_2} = \xi\,\vec{r_1}$ where ξ is a constant. In this case, the equations become

$$\frac{d^2\,\vec{r_1}}{dt^2} = -G\,(m_0 + m_1)\,\frac{\vec{r_1}}{r_1^3} + G\,m_2\left(\frac{\xi - 1}{|\xi - 1|^3}\,\frac{\vec{r_1}}{r_1^3} - \frac{\xi}{|\xi|^3}\,\frac{\vec{r_1}}{r_1^3}\right) \tag{7.13}$$

$$\frac{d^2\,\vec{r_2}}{dt^2} = -G\,(m_0 + m_2)\,\frac{\xi}{|\xi|^3}\,\frac{\vec{r_1}}{r_1^3} + G\,m_1\left(-\frac{\xi - 1}{|\xi - 1|^3}\,\frac{\vec{r_1}}{r_1^3} - \frac{\vec{r_1}}{r_1^3}\right) \tag{7.14}$$

$$= \xi\left[-G\,(m_0 + m_1)\,\frac{\vec{r_1}}{r_1^3} + G\,m_2\left(\frac{\xi - 1}{|\xi - 1|^3}\,\frac{\vec{r_1}}{r_1^3} - \frac{\xi}{|\xi|^3}\,\frac{\vec{r_1}}{r_1^3}\right)\right] \tag{7.15}$$

where the last equation (7.15) stems from the fact that $\vec{r_2} = \xi\,\vec{r_1}$. This relation can then be translated into a condition on the parameter ξ:

$$f(\xi) = m_0 + m_1 + m_2\left(\frac{\xi}{|\xi|^3} + \frac{1 - \xi}{|1 - \xi|^3}\right) - \frac{1}{|\xi|^3}\left(m_0 + m_2 + m_1\,\frac{|\xi|^3}{\xi}\left(1 - \frac{1 - \xi}{|1 - \xi|^3}\right)\right) = 0 \tag{7.16}$$

The function $f(\xi)$ is illustrated in Fig. 7.4. As we can see from this figure, $f(\xi)$ has three zeros, one in each of the domains $\xi < 0.0$, $\xi \in]0.0, 1.0[$ and $\xi > 1.0$ respectively. Therefore, provided that we have $\xi^2\,\vec{r_1} \wedge \dot{\vec{r_1}} = \vec{r_2} \wedge \dot{\vec{r_2}}$ at time $t = 0$, these Lagrange solutions are such that m_1 and m_2 actually describe coplanar and homothetic (with the factor ξ) conical trajectories around the same focus (m_0, see Fig. 7.5 for the case of an elliptical orbit with $0 < \xi < 1$).

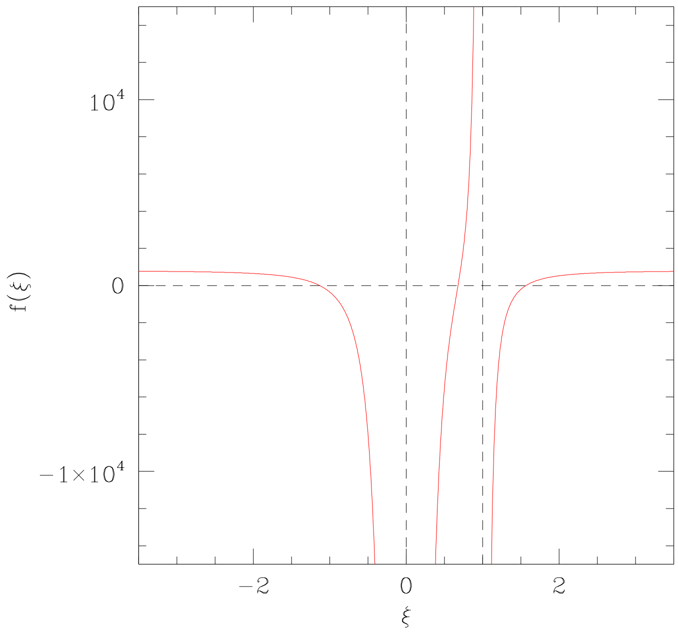


Figure 7.4: The function $f(\xi)$ in the case where $m_0 = 800$, $m_1 = 200$ and $m_2 = 1$.

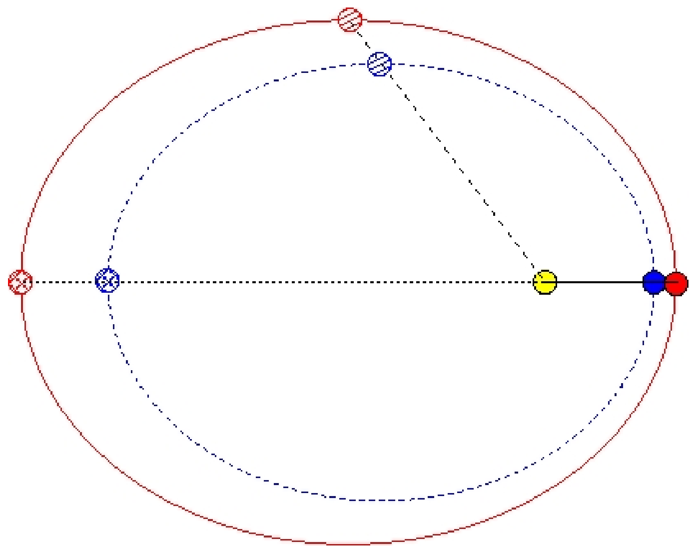


Figure 7.5: The Lagrange solution of the 3-body problem in case of an aligned configuration.

7.3.2 The circular restricted 3-body problem

A particular situation arises if we consider the 3-body problem (equations 7.9 and 7.10) in the case where $m_2 << \min{(m_0, m_1)}$. Under this assumption, we are dealing with the *restricted three-body problem*, where we can now assume that the orbit of m_1 around m_0 is not affected by m_2. Indeed, equation 7.9 reduces to the pure Keplerian formulation of a two body problem.

In this subsection, we consider the particular situation where m_1 revolves around m_0 on a circular orbit of radius a_1 (this is the so-called *circular restricted three-body problem*). The angular velocity of this circular motion is then given by $n_1 = \sqrt{\frac{G\,(m_0+m_1)}{a_1^3}}$. Let $\vec{e_z}$ be the unity vector perpendicular to the plane of the motion. The angular velocity vector can thus be written $\vec{\Omega} = n_1\,\vec{e_z}$. Since $m_2 << \min{(m_0, m_1)}$, Newton's equation for the motion of

m_2 is given by

$$\begin{aligned}\frac{d^2\,\vec{r_2}}{dt^2} &= -G\,m_0\,\frac{\vec{r_2}}{|\vec{r_2}|^3} + G\,m_1\left(\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3}\right)\\ &= \vec{\nabla}\left(\frac{G\,m_0}{r_2} + \frac{G\,m_1}{|\vec{r_1}-\vec{r_2}|} - m_1\,G\,\frac{\vec{r_1}\cdot\vec{r_2}}{a_1^3}\right)\end{aligned}$$

We can express this equation in a non-inertial frame of reference ($\vec{e_x}$, $\vec{e_y}$, $\vec{e_z}$) centred on P_0 and rotating with the motion of m_1. Here $\vec{e_x}$ is chosen to be the unity vector along the direction of $\vec{r_1}$. The absolute acceleration (expressed in an inertial frame of reference) is then related to the relative acceleration (in the rotating frame of reference) by:

$$\frac{d^2\,\vec{r_2}}{dt^2} = \frac{\delta^2\,\vec{r_2}}{\delta t^2} + 2\,\vec{\Omega}\wedge\frac{\delta\,\vec{r_2}}{\delta t} + \vec{\Omega}\wedge(\vec{\Omega}\wedge\vec{r_2})$$

We thus obtain:

$$\begin{aligned}\frac{\delta^2\,\vec{r_2}}{\delta t^2} + 2\,n_1\,\vec{e_z}\wedge\frac{\delta\,\vec{r_2}}{\delta t} + n_1^2\,\vec{e_z}\wedge(\vec{e_z}\wedge\vec{r_2}) &= -G\,m_0\,\frac{\vec{r_2}}{|\vec{r_2}|^3} + G\,m_1\left(\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3}\right)\\ &= \vec{\nabla}\left(\frac{G\,m_0}{r_2} + \frac{G\,m_1}{|\vec{r_1}-\vec{r_2}|} - m_1\,G\,\frac{\vec{r_1}\cdot\vec{r_2}}{a_1^3}\right)\end{aligned}$$

Now, if we project this equation on the x, y and z axes, we find:

$$\ddot{x} - 2\,n_1\,\dot{y} = \frac{\partial}{\partial x}\left(\frac{G\,m_0}{r_2} + \frac{G\,m_1}{|\vec{r_1}-\vec{r_2}|} - m_1\,G\,\frac{\vec{r_1}\cdot\vec{r_2}}{a_1^3}\right) + n_1^2\,x \tag{7.17}$$

$$\ddot{y} + 2\,n_1\,\dot{x} = \frac{\partial}{\partial y}\left(\frac{G\,m_0}{r_2} + \frac{G\,m_1}{|\vec{r_1}-\vec{r_2}|} - m_1\,G\,\frac{\vec{r_1}\cdot\vec{r_2}}{a_1^3}\right) + n_1^2\,y \tag{7.18}$$

$$\ddot{z} = \frac{\partial}{\partial z}\left(\frac{G\,m_0}{r_2} + \frac{G\,m_1}{|\vec{r_1}-\vec{r_2}|} - m_1\,G\,\frac{\vec{r_1}\cdot\vec{r_2}}{a_1^3}\right) \tag{7.19}$$

where $\frac{\delta^2\,\vec{r_2}}{\delta t^2} = (\ddot{x},\ddot{y},\ddot{z})$ and $\frac{\delta\,\vec{r_2}}{\delta t} = (\dot{x},\dot{y},\dot{z})$.
The right-hand members of equations 7.17 – 7.19 can be expressed as the gradient of the so-called Roche potential

$$\frac{G\,m_0}{\sqrt{x^2+y^2+z^2}} + \frac{G\,m_1}{\sqrt{(a_1-x)^2+y^2+z^2}} - m_1\,G\,\frac{x}{a_1^2} + \frac{n_1^2}{2}\,(x^2+y^2)$$

Since $\frac{G\,m_1}{a_1^3} = \frac{m_1\,n_1^2}{m_0+m_1}$ the Roche potential can finally be expressed as

$$\Phi = \frac{G\,m_0}{\sqrt{x^2+y^2+z^2}} + \frac{G\,m_1}{\sqrt{(a_1-x)^2+y^2+z^2}} + \frac{n_1^2}{2}\left[\left(x - \frac{m_1}{m_0+m_1}\,a_1\right)^2 + y^2\right] \tag{7.20}$$

From equations 7.17 – 7.19, we thus infer the following important result:

$$\frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) = \Phi - C_J \tag{7.21}$$

C_J is called Jacobi's integral and is a constant as a function of time. Equation 7.21 implies that Φ must be larger than C_J for a motion to be possible. We finally note that the Roche potential can be written in a more symmetrical form by adopting the notations $\rho_0 = \sqrt{x^2+y^2+z^2}$ and $\rho_1 = \sqrt{(a_1-x)^2+y^2+z^2}$:

$$\Phi = n_1^2\,a_1^3\left\{\left[\frac{m_0}{m_0+m_1}\left(\frac{1}{\rho_0} + \frac{\rho_0^2}{2\,a_1^3}\right) + \frac{m_1}{m_0+m_1}\left(\frac{1}{\rho_1} + \frac{\rho_1^2}{2\,a_1^3}\right)\right] - \frac{1}{2\,a_1^3}\left(z^2 + \frac{m_0\,m_1}{(m_0+m_1)^2}\,a_1^2\right)\right\} \tag{7.22}$$

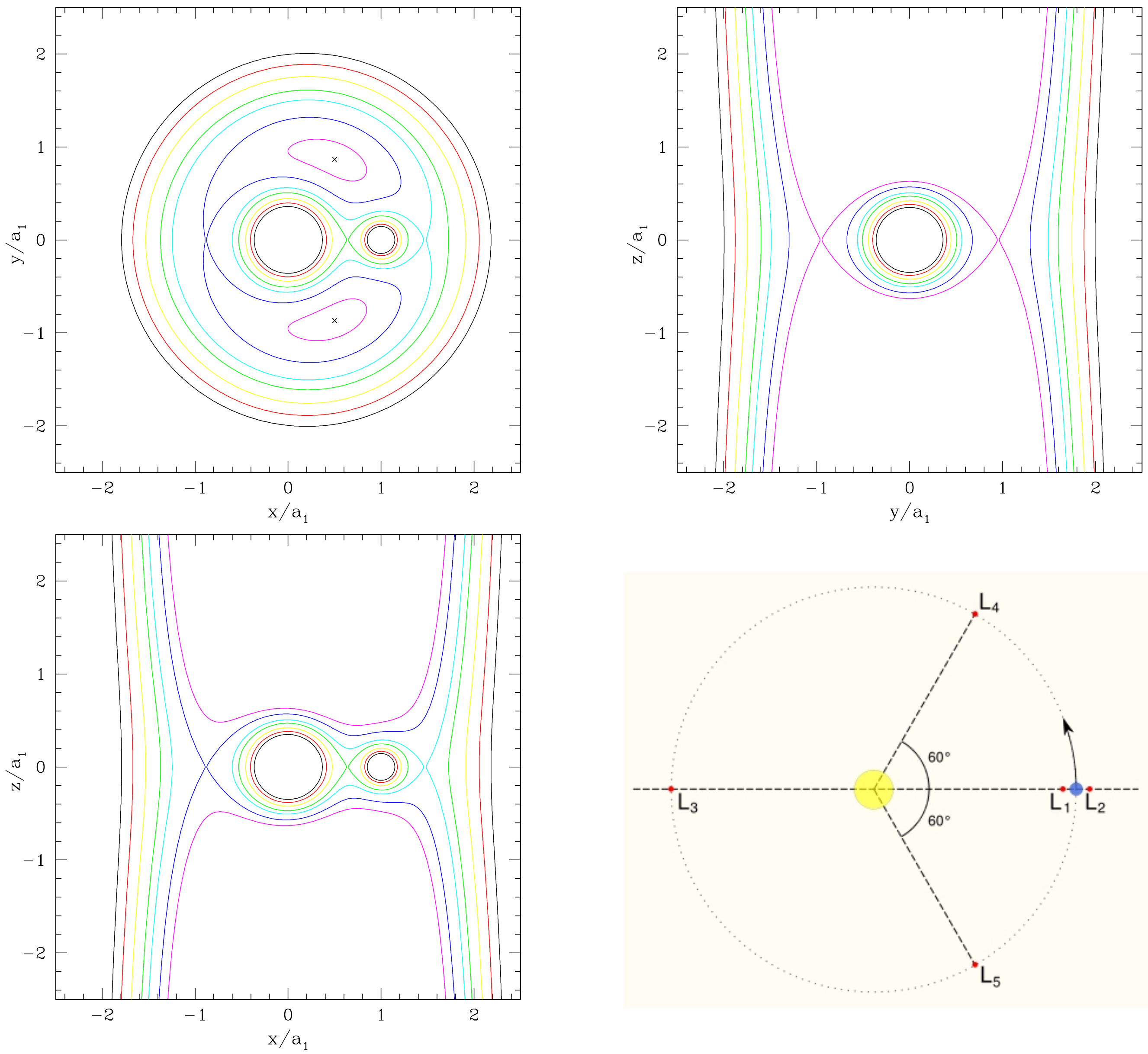


Figure 7.6: Top left: section of the equipotential surfaces of the Roche potential (expressed as $\frac{\Phi}{n_1^2\,a_1^2}$) for $m_0 = 4\,m_1$ by the plane $z = 0$. The various contours correspond to values of $\frac{C_J}{n_1^2\,a_1^2}$ equal to 1.46, 1.60, 1.78, 1.90, 2.10, 2.30 and 2.50. Top right: section of the same surfaces but in the plane $x = 0$. Bottom left: section of the same surfaces but in the plane $y = 0$. Bottom right: schematic view of the location of the Lagrangian points of the Sun - Earth system.

For a given value C_J of the Jacobi integral, the motion is only possible over those parts of space where $\Phi \geq C_J$. The sections of the equipotential surfaces of the Roche potential by several planes are shown in Figure 7.6. First of all, we note that the surfaces $\Phi = C_J$ are symmetrical with respect to the xy and xz planes. Near m_0 and m_1, the contours are essentially spherical or ellipsoidal and they deform into a drop-like shape as one moves away from m_0 or m_1. The equipotentials are contained within the cylinder of equation $n_1^2\,\left[\left(x - \frac{m_1}{m_0+m_1}\,a_1\right)^2 + y^2\right] = 2\,C_J$

and they reach this cylinder asymptotically for $z \rightarrow \infty$. Therefore, for large values of C_J and at large distance from the masses, the equipotentials tend towards a cylindrical shape with a symmetry axis[1] parallel to $\vec{e_z}$.
One important feature is the existence of five so-called Lagrangian points or libration points that are points of relative equilibrium given by $\vec{\nabla}\,\Phi = \vec{0}$ (see below). These points are seen as intersections of contours (L_1, L_2 and L_3) or minima of Φ (L_4 and L_5). The Lagrangian points correspond of course to the particular Lagrangian solutions of the restricted 3-body problem that we have discussed in the previous section. In the circular case, the conicals found in Sect. 7.3.1 reduce to circular trajectories and the Lagrangian points represent these circular orbits in the frame of reference rotating with $\vec{P_0 P_1}$.
The coordinates of the Lagrangian points are solution of

$$\vec{\nabla}\,\Phi = \vec{0} \Rightarrow \left\{ \begin{array}{lllll} \frac{\partial\,\Phi}{\partial x} & = & -\frac{G\,m_0\,x}{[x^2+y^2+z^2]^{3/2}} - \frac{G\,m_1\,(x-a_1)}{[(a_1-x)^2+y^2+z^2]^{3/2}} + n_1^2\left(x - \frac{m_1}{m_0+m_1}\,a_1\right) & = 0 \\ \frac{\partial\,\Phi}{\partial y} & = & -\frac{G\,m_0\,y}{[x^2+y^2+z^2]^{3/2}} - \frac{G\,m_1\,y}{[(a_1-x)^2+y^2+z^2]^{3/2}} + n_1^2\,y & = 0 \\ \frac{\partial\,\Phi}{\partial z} & = & -\frac{G\,m_0\,z}{[x^2+y^2+z^2]^{3/2}} - \frac{G\,m_1\,z}{[(a_1-x)^2+y^2+z^2]^{3/2}} & = 0 \end{array} \right. \tag{7.23}$$

From the third equation hereabove, it follows immediately that the Lagrangian points are all located in the orbital plane ($z = 0$). The second equation then leads to two possibilities: $y = 0$ or $y \neq 0$. In the first case, we are left with the following equation

$$\frac{-G\,m_0\,x}{|x|^3} - \frac{G\,m_1\,(x-a_1)}{|x-a_1|^3} + n_1^2\left(x - \frac{m_1}{m_0+m_1}\,a_1\right) = 0 \tag{7.24}$$

From the definition of n_1 and taking $x = \xi\,a_1$, we thus obtain

$$f(\xi) = \frac{m_0}{m_0+m_1}\,\frac{\xi}{|\xi|^3} + \frac{m_1}{m_0+m_1}\,\frac{\xi-1}{|\xi-1|^3} - \left(\xi - \frac{m_1}{m_0+m_1}\right) = 0 \tag{7.25}$$

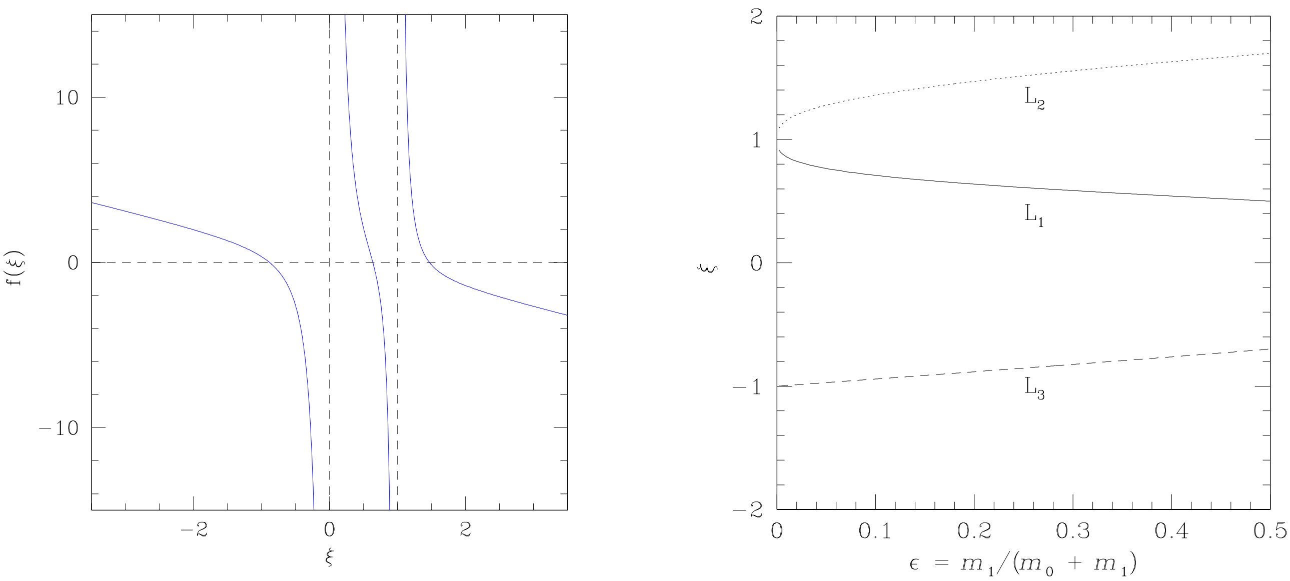


Figure 7.7: Left: the function $f(\xi)$ in the case where $m_0 = 4\,m_1$. Right: zeros of $f(\xi)$ as a function of the parameter $\varepsilon = \frac{m_1}{m_0+m_1}$.

[1]This result arises from the centrifugal force that introduces a term proportional to $x^2 + y^2$ in the Roche potential.

This function is shown in Fig. 7.7. As we can see this function has three zeroes. The corresponding Lagrangian points L_1, L_2 and L_3 are located along the axis of the system (since $y = 0$ and $z = 0$).
For $y \neq 0$, we obtain

$$-\frac{G\,m_0}{[x^2+y^2]^{3/2}} - \frac{G\,m_1}{[(a_1-x)^2+y^2]^{3/2}} + n_1^2 = 0$$

which then leads to

$$\begin{aligned}(x-a_1)^2+y^2 &= a_1^2\\ x^2+y^2 &= a_1^2\end{aligned}$$

The last two Lagrangian points, L_4 and L_5, are thus located at $x = a_1/2$ and $y = \pm\frac{\sqrt{3}}{2}\,a_1$.

Stability of the Lagrangian points

Let us now consider the stability of these Lagrangian points. For this purpose, we call $(x_0, y_0, 0)$ the coordinates of any of the five points L_i and we assume a small perturbation such that

$$\begin{aligned}x &= x_0 + \delta x\\ y &= y_0 + \delta y\\ z &= \delta z\end{aligned}$$

From equations 7.17 – 7.19 we infer that

$$\delta\ddot{x} = \frac{\partial^2\Phi}{\partial x^2}\,\delta x + \frac{\partial^2\Phi}{\partial x\,\partial y}\,\delta y + \frac{\partial^2\Phi}{\partial x\,\partial z}\,\delta z + 2\,n_1\,\delta\dot{y} \tag{7.26}$$

$$\delta\ddot{y} = \frac{\partial^2\Phi}{\partial x\,\partial y}\,\delta x + \frac{\partial^2\Phi}{\partial y^2}\,\delta y + \frac{\partial^2\Phi}{\partial y\,\partial z}\,\delta z - 2\,n_1\,\delta\dot{x} \tag{7.27}$$

$$\delta\ddot{z} = \frac{\partial^2\Phi}{\partial x\,\partial z}\,\delta x + \frac{\partial^2\Phi}{\partial y\,\partial z}\,\delta y + \frac{\partial^2\Phi}{\partial z^2}\,\delta z \tag{7.28}$$

All the partial derivatives in these equations being evaluated at L_i. Since $z = 0$ at the Lagrangian points, we find that $\frac{\partial^2\Phi}{\partial y\,\partial z} = \frac{\partial^2\Phi}{\partial x\,\partial z} = 0$. On the other hand, at L_i we also have

$$\frac{\partial^2\Phi}{\partial z^2} = -\frac{G\,m_0}{[x^2+y^2]^{3/2}} - \frac{G\,m_1}{[(a_1-x)^2+y^2]^{3/2}} = -\lambda^2 < 0$$

Equation 7.28 thus reduces to

$$\delta\ddot{z} + \lambda^2\,\delta z = 0$$

which leads to

$$\delta z = Z_1\,\cos(\lambda\,t) + Z_2\,\sin(\lambda\,t) \tag{7.29}$$

Hence the Lagrangian points are stable against a perturbation along the z axis.
Now, introducing the notations $u = \frac{\partial^2\Phi}{\partial x^2}$, $v = \frac{\partial^2\Phi}{\partial x\,\partial y}$ and $w = \frac{\partial^2\Phi}{\partial y^2}$, and testing solutions of the type $\delta x = X\,\exp(\alpha\,t)$ and $\delta y = Y\,\exp(\alpha\,t)$, we then obtain

$$\begin{pmatrix}\alpha^2-u & -v-2\,n_1\,\alpha\\ -v+2\,n_1\,\alpha & \alpha^2-w\end{pmatrix}\begin{pmatrix}X\\ Y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix} \tag{7.30}$$

This system of linear equations admits a solution different from $X = Y = 0$ only if the determinant of the matrix of the coefficients is zero:

$$\Rightarrow (\alpha^2 - u)\,(\alpha^2 - w) - (v + 2\,n_1\,\alpha)\,(v - 2\,n_1\,\alpha) = 0$$

This condition leads to

$$\alpha^4 + (4\,n_1^2 - u - w)\,\alpha^2 + (u\,w - v^2) = 0 \tag{7.31}$$

For the first three Lagrangian points one finds that $u = \frac{2\,G\,m_0}{|x|^3} + \frac{2\,G\,m_1}{|x-a_1|^3} + n_1^2 = 2\,\lambda^2 + n_1^2 > 0$, $v = 0$ and $w = -\frac{G\,m_0}{|x|^3} - \frac{G\,m_1}{|x-a_1|^3} + n_1^2 = -\lambda^2 + n_1^2$. It can be shown that $w < 0$ in each of the three L_i $i = 1, 2, 3$ (see Fig. 7.8). Hence $u\,w - v^2 < 0$ and equation 7.31 admits two real solutions for α^2 that have opposite signs. We therefore conclude that there exists a positive value of α and thus the equilibrium is unstable.

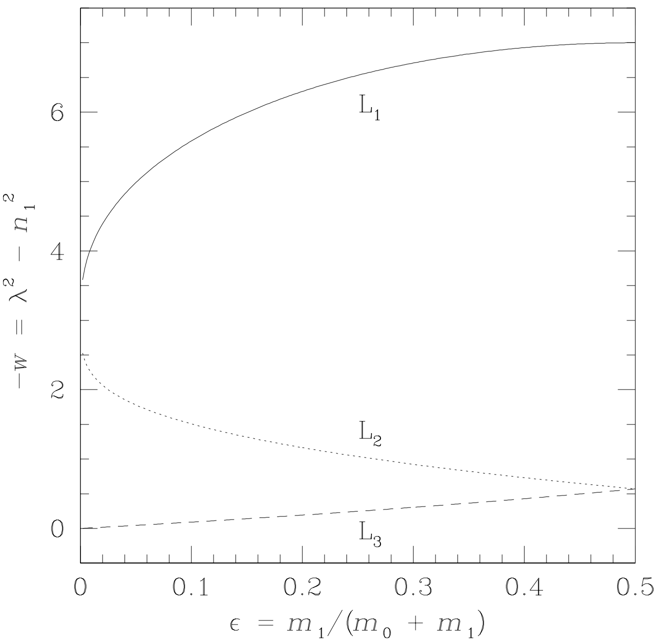


Figure 7.8: The quantity $-w$ is shown as function of $\varepsilon = \frac{m_1}{m_0+m_1} = \frac{1}{1+q}$ at the three linear Lagrangian points L$_1$, L$_2$ and L$_3$. As one can clearly see, $w < 0$ for every possible value of $q = \frac{m_0}{m_1}$.

For the L_4 and L_5 points, we obtain that $u = \frac{3}{4}\,n_1^2$, $v = \frac{3\sqrt{3}}{4}\,(m_0 - m_1)\,\frac{G}{a_1^3}$ and $w = \frac{9}{4}\,n_1^2$. Inserting these results into equation 7.31, we obtain

$$\alpha^4 + n_1^2\,\alpha^2 + \frac{27}{4}\,n_1^4\,\frac{m_1\,m_0}{(m_1 + m_0)^2} = 0 \tag{7.32}$$

which leads to

$$\Delta = n_1^4 - 27\,n_1^4\,\frac{m_1\,m_0}{(m_1 + m_0)^2} \tag{7.33}$$

$$\alpha^2 = \frac{-n_1^2 \pm \sqrt{\Delta}}{2} \tag{7.34}$$

The latter equation yields purely imaginary solutions for α provided that

$$0 \leq \Delta < n_1^4$$

If we define $q = m_0/m_1 > 1$, this condition translates into a constraint on q. Indeed, the criterion on Δ is met for $q > (25 + \sqrt{621})/2 = 24.96$. In this case, the triangular Lagrangian points are stable equilibrium positions.

It has to be stressed that while L_4 and L_5 correspond to local extrema of the potential Φ, their stability is actually the result of the action of the Coriolis force.
There are a number of situations inside the Solar System where this stability condition is fulfilled: e.g. the Trojan asteroids in the Sun - Jupiter system or the satellites Telesto and Calypso that are located at the triangular Lagrangian points of the Saturn - Thetys system (the same is true for the moons Helene and Polydeuces that occupy the L_4 and L_5 points of the Saturn - Dione system).
For values of q smaller than 24.96, L_4 and L_5 are unstable.

Orbits around stable L_4 or L_5 points

Let us assume that q largely exceeds the critical value 24.96, so that

$$\varepsilon = \frac{1}{1+q} << (25.96)^{-1} = 0.0385$$

Equation 7.32 can be reformulated as

$$\alpha^4 + n_1^2\,\alpha^2 + \frac{27}{4}\,n_1^4\,\varepsilon\,(1-\varepsilon) = 0 \tag{7.35}$$

Since, we are in a situation where the L_4 and L_5 points are stable, equation 7.35 admits the two negative roots

$$\alpha^2 \simeq \frac{-n_1^2 \pm n_1^2\,\sqrt{1-27\,\varepsilon}}{2} \simeq \frac{n_1^2}{2}\,\left[-1 \pm \left(1 - \frac{27\,\varepsilon}{2}\right)\right] \tag{7.36}$$

which in turn lead to the four purely imaginary solutions for α:

$$\alpha \simeq \pm\frac{3\,\sqrt{3\,\varepsilon}}{2}\,n_1\,j \tag{7.37}$$

$$\alpha \simeq \pm\sqrt{1 - \frac{27\,\varepsilon}{4}}\,n_1\,j \simeq \left(1 - \frac{27\,\varepsilon}{8}\right)\,n_1\,j \tag{7.38}$$

where j is the imaginary unit ($j^2 = -1$). The former two solutions correspond to a long-period oscillation, whilst the latter two oscillations occur on a period that is only slightly longer than the orbital period of m_1 around m_0. The amplitudes along the x and y axes, associated with these solutions must obey equation 7.30. Let us now consider a system of axes x' and y', centred on L_4 and rotated by $-\frac{\pi}{6}$ ($-30°$) with respect to the x, y axes (see Fig. 7.9)[2]. We thus have that

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} \tag{7.39}$$

which leads to

$$\begin{pmatrix} \alpha^2 - \frac{3\,n_1^2}{4} & -\frac{3\,\sqrt{3}}{4}\,n_1^2\,(1-2\,\varepsilon) - 2\,n_1\,\alpha \\ -\frac{3\,\sqrt{3}}{4}\,n_1^2\,(1-2\,\varepsilon) + 2\,n_1\,\alpha & \alpha^2 - \frac{9\,n_1^2}{4} \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{7.40}$$

We thus derive the relation between the X' and Y' parameters:

$$\left(\frac{\sqrt{3}}{2}\,\alpha^2 - \frac{3\,\sqrt{3}}{4}\,\varepsilon\,n_1^2 + n_1\,\alpha\right)\,X' + \left(\frac{\alpha^2}{2} - \frac{3\,n_1^2}{2} + \frac{9\,n_1^2}{4}\,\varepsilon - \sqrt{3}\,n_1\,\alpha\right)Y' = 0 \tag{7.41}$$

[2] Of course the same development can be performed in L_5 rotating the axes by $\frac{\pi}{6}$.

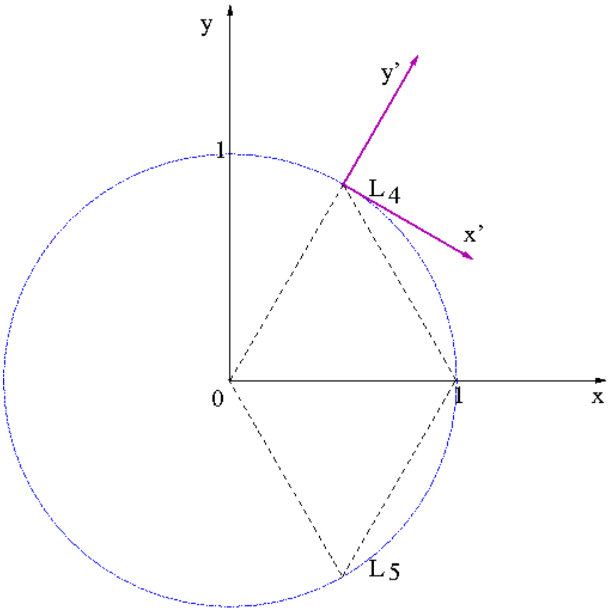


Figure 7.9: The system of axes x' and y' rotated by an angle $-\frac{\pi}{6}$ with respect to the conventional x and y axes. The x' axis is tangent to an imaginary circle of radius unity centred on m_0, whilst the y' axis is along the direction of the radius of the circle.

For the low frequency terms, $\alpha \simeq \pm\frac{3\sqrt{3\varepsilon}}{2}\, n_1\, j$ and restricting ourselves to the dominant terms (remembering that $\varepsilon << 1$), we thus obtain

$$Y' \simeq \pm\sqrt{3\,\varepsilon}\, j\, X'$$

The two long-period oscillation modes hence combine into

$$\delta x' = X'_{lp}\, \cos\left[\frac{3\sqrt{3\,\varepsilon}}{2}\, n_1\, t - \Phi_{lp}\right]$$

and

$$\delta y' = -\sqrt{3\,\varepsilon}\, X'_{lp}\, \sin\left[\frac{3\sqrt{3\,\varepsilon}}{2}\, n_1\, t - \Phi_{lp}\right]$$

where X'_{lp} and Φ_{lp} are set by the initial conditions. This so-called libration motion has a much larger amplitude along the tangential direction than along the radial direction.

For the high frequency terms, $\alpha \simeq \left(1 - \frac{27\,\varepsilon}{8}\right)\, n_1\, j$, we find that (again restricting ourselves to the dominant terms)

$$Y' \simeq \pm\frac{j\, X'}{2}$$

These oscillation modes hence combine into

$$\delta x' = X'_{sp}\, \cos\left[\left(1 - \frac{27\,\varepsilon}{8}\right)\, n_1\, t - \Phi_{sp}\right]$$

and

$$\delta y' = -\frac{X'_{sp}}{2}\, \sin\left[\left(1 - \frac{27\,\varepsilon}{8}\right)\, n_1\, t - \Phi_{sp}\right]$$

where X'_{sp} and Φ_{sp} are again set by the initial conditions. When expressed in the non-rotating inertial frame of reference, the short-period oscillations yield a roughly elliptical motion of m_2 about m_0, with the major axis slowly precessing at the rate $\frac{27}{8}\,\varepsilon\, n_1$.

Orbits around the linear libration points

The fact that the libration points L_1, L_2 and L_3 are intrinsically unstable does not necessarily imply that there exist no solutions for stable orbits about these points. Indeed, if the initial conditions are appropriate, the amplitude of the unstable exponential component of the solution vanishes and one is left with a stable solution. The trouble is that any small perturbation due e.g. to a slight eccentricity of the orbit of m_1 around m_0 or to the influence of other masses in the Solar System will eventually produce a non-zero amplitude for the real and positive root of α. Therefore, without any external force that could compensate this instability, the orbit of m_2 will eventually become unstable even if it was originally stable.
Let us first re-express equations 7.26 - 7.28 in the case there is no external force acting on m_2:

$$\begin{aligned} \delta\ddot{x} - 2\,n_1\,\delta\dot{y} - (2\,\lambda^2 + n_1^2)\,\delta x &= 0 \qquad (7.42)\\ \delta\ddot{y} + 2\,n_1\,\delta\dot{x} + (\lambda^2 - n_1^2)\,\delta y &= 0 \qquad (7.43)\\ \delta\ddot{z} + \lambda^2\,\delta z &= 0 \qquad (7.44) \end{aligned}$$

As stated above, the last equation yields a stable solution:

$$\delta z = Z\,\cos(\lambda\,t + \phi_z)$$

For the first two equations, we consider that the initial conditions of the problem are such that the only remaining solution is the pure imaginary root for α. Therefore, we now consider a solution of the kind $\delta x = \Re\{X_c\,\exp\left[j\,(\omega\,t + \phi_0)\right]\}$ and $\delta y = \Re\{Y_c\,\exp\left[j\,(\omega\,t + \phi_0)\right]\}$, which yields

$$\begin{pmatrix} -(\omega^2 + 2\,\lambda^2 + n_1^2) & -2\,n_1\,\omega\,j \\ 2\,n_1\,\omega\,j & -(\omega^2 + n_1^2 - \lambda^2) \end{pmatrix} \begin{pmatrix} X_c \\ Y_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (7.45)$$

The latter system implies

$$\omega^4 + (\lambda^2 - 2\,n_1^2)\,\omega^2 + (n_1^4 + n_1^2\,\lambda^2 - 2\,\lambda^4) = 0$$

A positive solution for ω^2 is

$$\omega^2 = \frac{\lambda\,\sqrt{9\,\lambda^2 - 8\,n_1^2} - \lambda^2 + 2\,n_1^2}{2} \qquad (7.46)$$

There further exists a relation between the amplitudes of the motion along x and y:

$$X_c = \frac{-(\omega^2 + n_1^2 - \lambda^2)\,j\,Y_c}{2\,n_1\,\omega}$$

Thus we see that $\delta x = X\,\sin(\omega\,t + \phi_0)$ and $\delta y = Y\,\cos(\omega\,t + \phi_0)$ with $X = \frac{\omega^2 + n_1^2 - \lambda^2}{2\,n_1\,\omega}\,Y$.

Hence, in this solution, the test mass moves on an ellipse in the (x, y) plane centred on the libration point with a frequency ω and oscillates with a different frequency λ along the z direction perpendicular to the plane. The combination of these two motions yields a so-called **Lissajous** orbit. A priori, the values of Z and Y are arbitrary, although they must be sufficiently small for the linearization of the equations of motion to remain valid.
As stated above, any small perturbation will render this solution unstable. For spacecraft moving around the L_1 or L_2 points (see lectures on *Space Exploration*), one can actively compensate this instability by applying some modest corrections using the propulsion system. Let us re-express equations 7.26 - 7.28 with the presence of an external force $\vec{F}$:

$$\begin{aligned} \delta\ddot{x} - 2\,n_1\,\delta\dot{y} - (2\,\lambda^2 + n_1^2)\,\delta x &= F_x \qquad (7.47)\\ \delta\ddot{y} + 2\,n_1\,\delta\dot{x} + (\lambda^2 - n_1^2)\,\delta y &= F_y \qquad (7.48)\\ \delta\ddot{z} + \lambda^2\,\delta z &= F_z \qquad (7.49) \end{aligned}$$

Table 7.1: Properties of the Lissajous orbits around the L_1 and L_2 points of the Sun - Earth and Earth - Moon systems.

	n_1 (rad/day)	λ/n_1	ω/n_1	X/Y
L_1 (Sun - Earth)	0.01720	2.015	2.086	0.310
L_2 (Sun - Earth)	0.01720	1.985	2.057	0.314
L_1 (Earth - Moon)	0.22997	2.269	2.334	0.279
L_2 (Earth - Moon)	0.22997	1.786	1.863	0.349

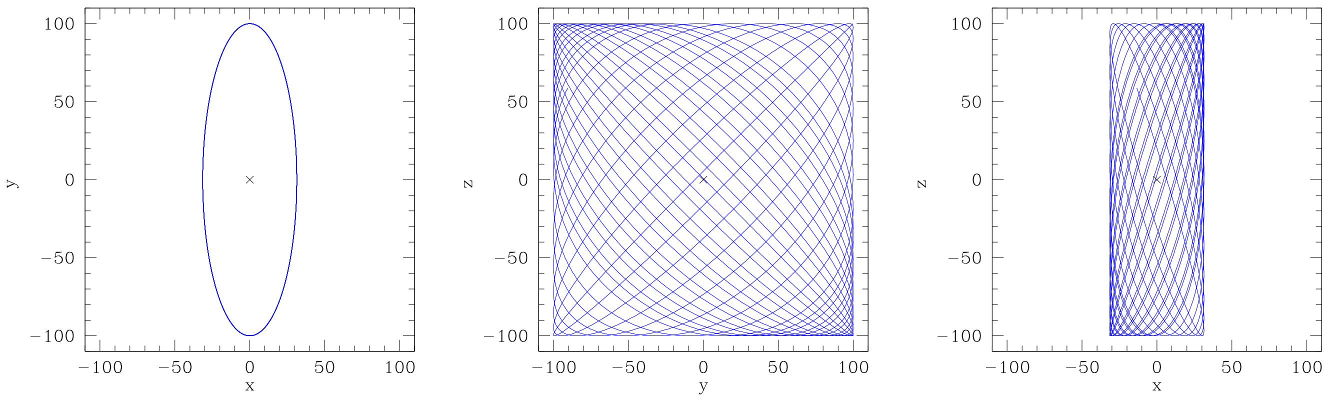


Figure 7.10: Projection of the Lissajous type orbit around the Sun - Earth L_2 point into various planes. The cross at the center of each image yields the position of L_2. The orbit is shown for $Y = Z = 100$ (in arbitrary distance units) and for a time interval of 10 years.

It can be shown, that the orbit can be stabilized by choosing $F_x = -c_1\,\delta\dot{x} - (2\,\lambda^2 + 1 + c_2)\,\delta x$, $F_y = 0$ and $F_z = 0$, with $c_1 > 0$ and $c_2 > 2$.
The view of the Lissajous orbit in the (y, z) plane (see Fig. 7.10) shows the situation as seen from Earth. In some cases, e.g. orbits around L_1 of the Sun-Earth system, it is important to avoid that the spacecraft's projection on the sky gets to close to the projection of the Lagrangian point (e.g. for telecommunication purposes). In these situations, one implements an avoidance zone around the projection of the libration point and an impulsive control of the z coordinate prevents the spacecraft from entering this region. In practice, the impulsive control of z aims at forcing the oscillation along the z axis to have the same period as the oscillations in the orbital plane. This concept leads to so-called **halo** orbits.
We emphasize that the above treatment is correct as long as the linearization of the equations 7.26 - 7.28 remains valid and the problem can be described in the framework of the restricted three-body problem (i.e. circular orbit for m_0 and m_1, no other forces...). For large amplitudes (X, Y and Z), the linear treatment is no longer valid and higher order terms need to be considered. These higher order terms can actually lead to a halo solution by bringing the periods of the in-orbit and out-of-orbit motions to the same value.

7.3.3 The sphere of influence

In celestial mechanics and space exploration, one often has to deal with problems of interplanetary spaceflight where the mass of the spacecraft is negligible compared to the mass of a celestial body (be it a planet or the Sun). In these cases, a good first approximation of the motion can be obtained by assuming that in the vicinity of a planet, only the attraction of the planet needs to be accounted for, whilst at large distances from the planet only the attraction of the Sun matters. These approximations constitute the concepts of the **sphere of influence** and the **patched conics**. Within the sphere of influence, one considers that the spacecraft is submitted mainly to the action of the gravitational force of a single body and the effect of any other body is treated as a perturbation of the conical (i.e. the Keplerian) motion.

Let us start by investigating under what circumstances we can reduce the 3-body problem of the motion of a low-mass object P_2 under the influence of the Sun (P_0) and a planet (P_1) to a perturbed 2-body problem.
First, we consider that P_2 undergoes mainly the attraction by P_0 (heliocentric description of the motion). For this purpose, we recall Newton's equation for the motion of P_2 (see equation 7.10), where we express the Keplerian part of the motion of P_2 around P_0:

$$\frac{d^2\,\vec{r_2}}{dt^2} = -G\,(m_0+m_2)\,\frac{\vec{r_2}}{|\vec{r_2}|^3} + G\,m_1\left(\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3}\right)$$

In order to consider the effect of P_1 as a small perturbation, the following condition must be satisfied, with η being a small quantity:

$$R = \frac{m_1}{m_0+m_2}\,\frac{\left|\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3}\right|}{\frac{1}{|\vec{r_2}|^2}} < \eta \tag{7.50}$$

Let ϕ be the angle between $\vec{r_2}-\vec{r_1}$ and $-\vec{r_1}$. Defining $\alpha = \frac{|\vec{r_2}-\vec{r_1}|}{|\vec{r_1}|}$, we obtain

$$\left|\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3}\right| = \frac{1}{|\vec{r_1}-\vec{r_2}|^2}\,(1+\alpha^4-2\,\alpha^2\,\cos\phi)^{1/2}$$

and

$$|\vec{r_2}|^2 = |\vec{r_1}|^2\,(1+\alpha^2-2\,\alpha\,\cos\phi)$$

Using these results, we can reformulate the ratio R as

$$R = \frac{m_1}{m_0+m_2}\,\frac{1}{\alpha^2}\,(1+\alpha^2-2\,\alpha\,\cos\phi)\,(1+\alpha^4-2\,\alpha^2\,\cos\phi)^{1/2}$$

Since α decreases when P_2 approaches P_1, we can write the condition on R by taking the dominant term of R:

$$\frac{m_1}{m_0+m_2}\,\frac{1}{\alpha^2} \simeq \frac{m_1}{m_0}\,\frac{1}{\alpha^2} < \eta \tag{7.51}$$

In a similar manner, we consider now the situation where P_2 undergoes mainly the attraction by P_1 (planetocentric description of the motion). Newton's equation for the motion of P_2 can now be expressed taking the Keplerian part of the motion of P_2 around P_1:

$$\frac{d^2\,(\vec{r_2}-\vec{r_1})}{dt^2} = -G\,(m_1+m_2)\,\frac{\vec{r_2}-\vec{r_1}}{|\vec{r_2}-\vec{r_1}|^3} + G\,m_0\left(\frac{\vec{r_1}}{|\vec{r_1}|^3} - \frac{\vec{r_2}}{|\vec{r_2}|^3}\right)$$

In order to consider the effect of P_0 as a small perturbation, one must have:

$$R' = \frac{m_0}{m_1+m_2}\,\frac{\left|\frac{\vec{r_1}}{|\vec{r_1}|^3} - \frac{\vec{r_2}}{|\vec{r_2}|^3}\right|}{\frac{1}{|\vec{r_2}-\vec{r_1}|^2}} < \eta \tag{7.52}$$

With the same definitions of ϕ and α as above, we can now express the value of

$$\left| \frac{\vec{r_1}}{|\vec{r_1}|^3} - \frac{\vec{r_2}}{|\vec{r_2}|^3} \right| = \frac{\alpha}{|\vec{r_1}|^2} \left(1 + 3\,\cos^2\phi + \mathcal{O}(\alpha)\right)^{1/2}$$

This then leads to a new expression of R':

$$R' \simeq \frac{m_0}{m_1 + m_2}\,\alpha^3\,\sqrt{1 + 3\,\cos^2\phi} \geq \frac{m_0\,\alpha^3}{m_1 + m_2}$$

As a result, we can again take the dominant term of R' to reformulate the condition on R':

$$\frac{m_0}{m_1 + m_2}\,\alpha^3 < \eta \tag{7.53}$$

Comparing equations 7.51 and 7.53, we see that both the planetocentric and the heliocentric approximations are of the same level of accuracy if

$$\frac{m_1}{m_0}\frac{1}{\alpha^2} = \frac{m_0}{m_1}\,\alpha^3$$

The two approximations to R and R' are illustrated in Fig. 7.11 and their equality implies

$$\alpha_0 = \left(\frac{m_1}{m_0}\right)^{2/5}$$

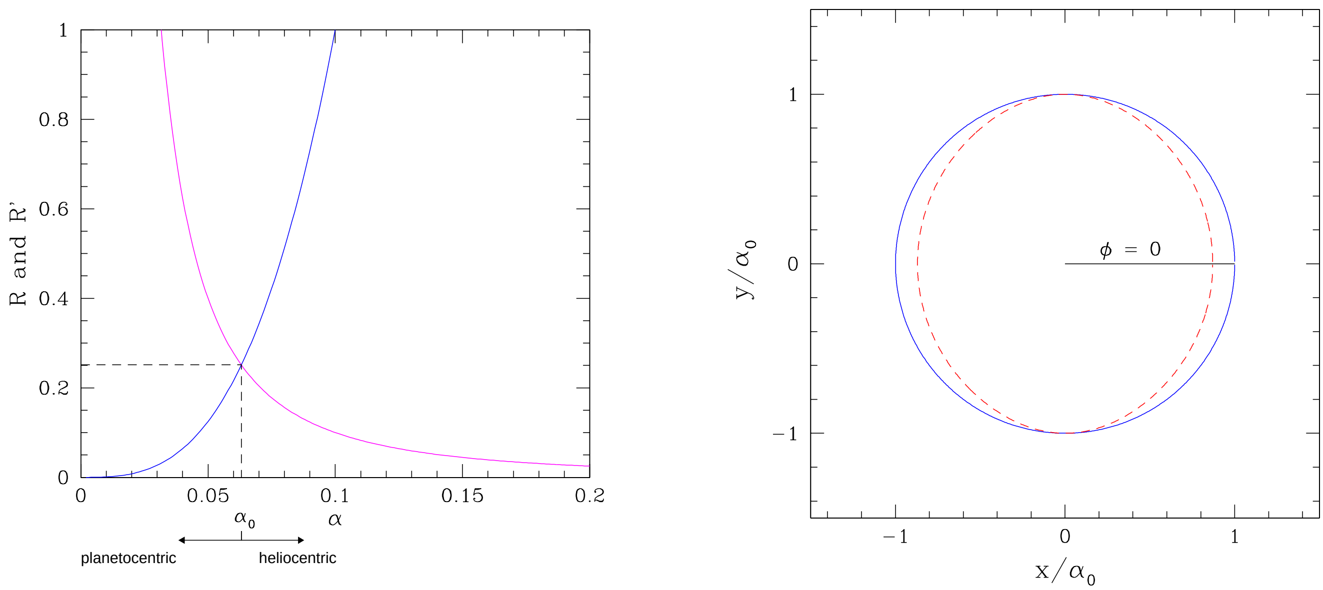


Figure 7.11: Left: the approximations to R and R' are shown as a function of α for $m_0/m_1 = 1000$. For values below α_0, the planetocentric approximation is the most accurate one, whilst for larger values of α, the heliocentric approximation should be used. Right: the more accurate representation of the 'sphere' of influence accounting for the first order dependence on ϕ (dashed line) is shown in comparison with the sphere of radius α_0 (solid line).

The sphere of influence of P_1 is the region of radius $\alpha_0\,|\vec{r_1}|$ and centred on P_1 where the perturbation due to P_0 on the planetocentric motion is less than the perturbation due to P_1 on the heliocentric motion (see Fig. 7.11). It has to be stressed that this does not imply that the condition $R' < \eta$ is actually satisfied. In fact, one usually considers

Table 7.2: The spheres of influence for the planets of the Solar System.

Planet	m_1/m_0	α_0	R_0	a_1 (AU)	$\alpha_0\,a_1$ (10^6 km)
Mercury	$1.66\,10^{-7}$	0.0019	0.044	0.387	0.112
Venus	$2.45\,10^{-6}$	0.0057	0.075	0.723	0.616
Earth	$3.00\,10^{-6}$	0.0062	0.079	1.00	0.925
Mars	$3.23\,10^{-7}$	0.0025	0.050	1.52	0.577
Jupiter	$9.55\,10^{-4}$	0.0620	0.249	5.20	48.2
Saturn	$2.86\,10^{-4}$	0.0382	0.196	9.54	54.6
Uranus	$4.37\,10^{-5}$	0.0180	0.134	19.2	51.8
Neptune	$5.18\,10^{-5}$	0.0193	0.138	30.1	86.8

that the action of a third body can be treated as a perturbation only if $\eta \leq 0.01$. However, as we can see from Table 7.2, R_0, the value of R and R' evaluated in α_0 is larger than 0.01 for all planets of the Solar System.
Actually, the sphere of influence is not perfectly spherical. A somewhat more accurate representation can be obtained if we account for the factor $(1 + 3\,\cos^2\phi)^{1/2}$. This then leads to the solution for the radius of the sphere of influence:

$$\alpha_0 = (1 + 3\,\cos^2\phi)^{-1/10}\,\left(\frac{m_1}{m_0}\right)^{2/5}$$

The latter correction is however of minor importance, since the correcting factor varies only over a narrow range between 0.8706 and 1.0 (see Fig. 7.11).

7.3.4 The Tisserand invariant

Let us consider a circular restricted 3-body problem where $m_0 >> m_1 >> m_2$. Such a situation holds for instance if m_0, m_1 and m_2 correspond to the masses of the Sun, a planet and a spacecraft or a minor body, respectively. Let us call $\vec{R_2}$ the position vector of m_2 in an (almost) inertial frame of reference centred on the center of mass of m_0 and m_1.
We can write

$$\vec{R_2} = \vec{r_2} - \frac{m_1}{m_0 + m_1}\,a_1\,\vec{e_x}$$

where $\vec{r_2} = (x, y, z)$ is the position vector of m_2 measured from m_0, a_1 is the constant distance between m_0 and m_1, and $\vec{e_x}$ is the unit vector pointing from m_0 to m_1. The latter rotates at a constant angular velocity $\vec{\Omega} = \sqrt{\frac{G\,(m_0+m_1)}{a_1^3}}\,\vec{e_z}$ (see Sect. 7.3.2).
The velocity of m_2 thus becomes:

$$\dot{\vec{R_2}} = \frac{\delta\,\vec{r_2}}{\delta t} + \vec{\Omega} \wedge \vec{R_2}$$

where $\frac{\delta\,\vec{r_2}}{\delta t}$ is the relative velocity in the rotating frame of reference tied to m_0 and m_1 (see Sect. 7.3.2). From there, we find that

$$\frac{\delta\,\vec{r_2}}{\delta t} = \dot{\vec{R_2}} - \vec{\Omega} \wedge \vec{R_2}$$

which leads to

$$|\frac{\delta\,\vec{r_2}}{\delta t}|^2 = |\dot{\vec{R_2}}|^2 + |\vec{\Omega} \wedge \vec{R_2}|^2 - 2\,\vec{\Omega}\cdot\vec{h} \tag{7.54}$$

where, $\vec{h} = \vec{R_2} \wedge \dot{\vec{R_2}}$ is the angular momentum of m_2.
The Roche potential (equation 7.20) introduced in Sect. 7.3.2 can be expressed as

$$\Phi = \frac{G\,m_0}{\rho_0} + \frac{G\,m_1}{\rho_1} + \frac{1}{2}|\vec{\Omega} \wedge \vec{R_2}|^2 \tag{7.55}$$

with $\rho_0 = \sqrt{x^2 + y^2 + z^2}$ and $\rho_1 = \sqrt{(a_1 - x)^2 + y^2 + z^2}$.
The Jacobi integral (equation 7.21) hence becomes

$$\begin{aligned} C_J &= \Phi - \frac{1}{2}|\frac{\delta\,\vec{r_2}}{\delta t}|^2 \\ &= \frac{G\,m_0}{\rho_0} + \frac{G\,m_1}{\rho_1} - \frac{1}{2}|\dot{\vec{R_2}}|^2 + \vec{\Omega} \cdot \vec{h} \\ &\simeq \vec{\Omega} \cdot \vec{h} - \varepsilon \end{aligned} \tag{7.56}$$

where the latter relation involves the orbital energy of m_2 if it were on a Keplerian orbit about m_0 not undergoing any influence from m_1 (see below):

$$\varepsilon = \frac{1}{2}|\dot{\vec{R_2}}|^2 - \frac{G\,m_0}{\rho_0}$$

Indeed, if m_2 is far away from m_1, and since $m_1 << m_0$, we can neglect $\frac{G\,m_1}{\rho_1}$ compared to $\frac{G\,m_0}{\rho_0}$.
In the restricted 3-body problem, C_J is a constant, and we thus obtain the expression of the Tisserand invariant[3] in the most general case:

$$\vec{\Omega} \cdot \vec{h} - \varepsilon \simeq Cst \tag{7.57}$$

Let us assume that m_2 is well outside the sphere of influence of m_1. According to what we have seen in the previous section, the motion of m_2 can thus essentially be described as a Keplerian orbit about m_0. From time to time, this motion can bring m_2 sufficiently close to m_1 to make it enter the sphere of influence of m_1. The interaction between m_1 and m_2 changes the orbit of the latter about m_0. The Tisserand invariant tells us that the elements of the orbit of m_2 cannot change in an arbitrary way: they must be such that equation 7.57 is fulfilled.

As an illustration, we consider a spacecraft on an elliptical orbit about the Sun with initial semi-major axis a, initial orbital inclination i (with respect to the orbital plane of planet Jupiter) and initial eccentricity e. If this spacecraft flies-by Jupiter (in a so-called **gravity assist** manoeuvre), its orbital parameters change. Let us assume that the new orbit in the heliocentric frame of reference is still an ellipse, but now of semi-major axis a', inclination i' and eccentricity e'. Under these circumstances, we can write

$$\vec{\Omega} \cdot \vec{h} - \varepsilon = \frac{G\,M_\odot}{2}\left[\frac{1}{a} + 2\sqrt{\frac{a\,(1-e^2)}{a_♃^3}}\,\cos i\right]$$

Hence the Tisserand invariant for elliptical orbits in the Sun - Jupiter system (often called the Jupiter Tisserand invariant) becomes

$$\frac{a_♃}{a} + 2\sqrt{\frac{a\,(1-e^2)}{a_♃}}\,\cos i \simeq \frac{a_♃}{a'} + 2\sqrt{\frac{a'\,(1-e'^2)}{a_♃}}\,\cos i' \simeq Cst \tag{7.58}$$

Of course, the general form of the Tisserand invariant (equation 7.57) not only holds for elliptical orbits, but for any Keplerian trajectory. For instance, a comet entering the Solar System on a parabolic or hyperbolic orbit ($e \geq 1$)

[3] Named after Félix Tisserand (1845 - 1896) who first introduced this quantity as a criterion to identify comets having undergone a close encounter with Jupiter.

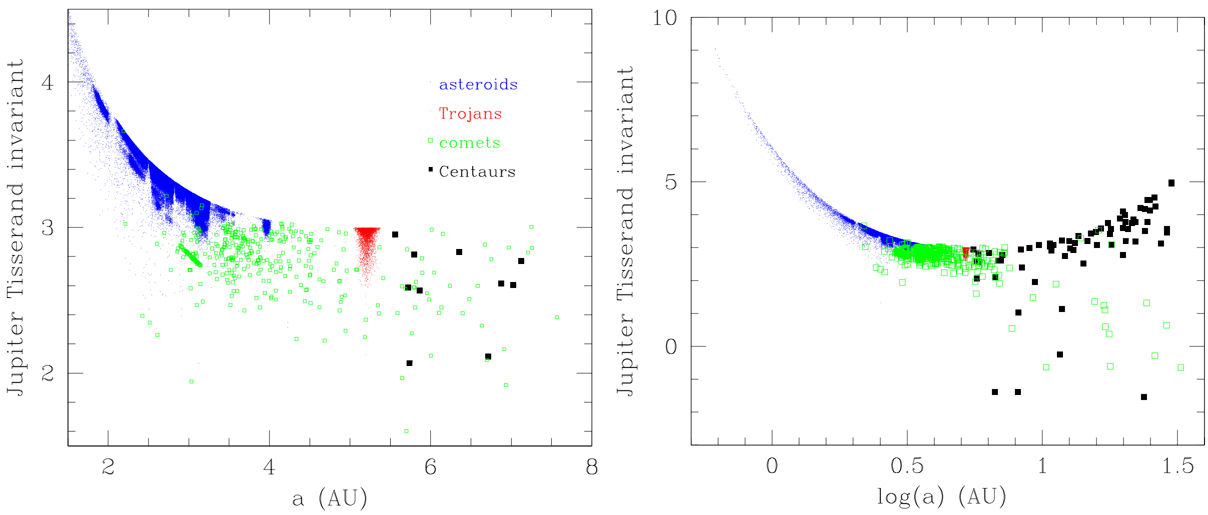


Figure 7.12: The Jupiter Tisserand invariant is shown as a function of the semi-major axis for different categories of minor bodies in the Solar System. On the left, we focus on the vicinity of Jupiter, whilst the right panel provides a more global view of the Solar System. Data were taken from NASA's Jet Propulsion Laboratory Small Body Database available at `http://www.ssd.jpl.nasa.gov/sbdb_query.cgi`.

can be influenced by the attraction of Jupiter (or another big planet) in such a way that its new heliocentric orbit becomes an ellipse ($e' < 1$). The reverse is also possible: a spacecraft on an elliptical orbit about the Sun can gain sufficient velocity by an encounter with a major planet to make it leave the Solar System on a parabolic or hyperbolic trajectory (see lectures on *Space Exploration*).
The invariance of the Jupiter Tisserand parameter allows to distinguish comets from asteroids. Indeed, as shown in Fig. 7.12, the comets have usually lower values than asteroids. This indicates that the comets, even though they may now orbit the Sun within the Jovian orbit, actually originated beyond the Jovian orbit. In the right panel of Fig. 7.12, we note some objects which have a negative value of their Tisserand parameter. These are actually objects that orbit the Sun in the opposite sense to Jupiter (hence $i > 90°$).

7.4 The motion of N bodies in a planetary system

We consider here the situation where m_0, the mass of the central body (the Sun in the case of the Solar System) is much larger than m_k for every $k > 0$. We aim at expressing the motion of each of the masses m_k as a Keplerian motion around m_0 subject to perturbations by all the other masses. According to equations 7.7 and 7.8, Newton's equation of the motion of m_k can be written as

$$\frac{d^2\,\vec{r_k}}{dt^2} = -\frac{G\,(m_0 + m_k)\,\vec{r_k}}{|\vec{r_k}|^3} + \sum_{n=1,n\neq k}^{N} \vec{\nabla}_k\,U_{k,n} \tag{7.59}$$

where $U_{k,n} = G\,m_n\,\left(\frac{1}{|\vec{r_n}-\vec{r_k}|} - \frac{\vec{r_n}\cdot\vec{r_k}}{|\vec{r_n}|^3}\right)$.

To illustrate the principle, we consider the mutual interaction between two planets $k = 1$ and $N-1 = 2$. We choose here the indices in such a way that $\rho = \frac{r_1}{r_2} < 1$ without any loss of generality. We also introduce $\alpha = \frac{a_1}{a_2} < 1$ such that $\rho = \alpha\,\frac{r_1}{a_1}\,\frac{a_2}{r_2}$. The distance between the two planets is $\Delta_{1,2} = |\vec{r_1} - \vec{r_2}|$. We further define two unit vectors $\vec{u_1}$ and $\vec{u_2}$ such that $\vec{r_1} = r_1\,\vec{u_1}$ and $\vec{r_2} = r_2\,\vec{u_2}$ as well as the angle γ between the two vectors $\vec{u_1}$ and $\vec{u_2}$.

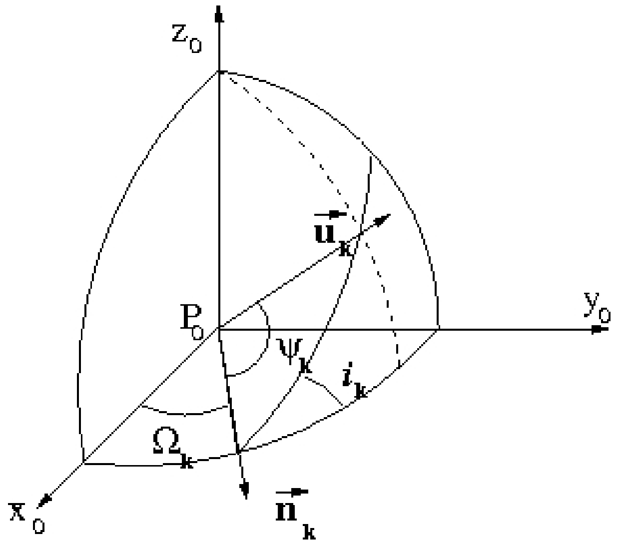


Figure 7.13: Illustration of the Ω_k, i_k and ψ_k angles of the osculating orbit of P_k in the frame of reference P_0, x_0, y_0, z_0. The vectors $\vec{n_k}$ and $\vec{u_k}$ are unit vectors pointing towards the ascending node and towards the instantaneous position of P_k respectively. The same set of angles and vectors are defined for each of the planets considered in the problem.

Using these definitions, we can write

$$U_{1,2} = G\,m_2\,\left(\frac{1}{\Delta_{1,2}} - \frac{r_1\,\cos\gamma}{r_2^2}\right)$$

$$U_{2,1} = G\,m_1\,\left(\frac{1}{\Delta_{1,2}} - \frac{r_2\,\cos\gamma}{r_1^2}\right)$$

We can express the inverse of the separation between the two planets as

$$\begin{aligned}\frac{1}{\Delta_{1,2}} &= (r_1^2 + r_2^2 - 2\,r_1\,r_2\,\cos\gamma)^{-1/2} \\ &= \frac{1}{r_2}\,(1 - 2\,\rho\,\cos\gamma + \rho^2)^{-1/2} \\ &= \frac{1}{r_2}\sum_{n=0}^{\infty}\rho^n\,P_n(\cos\gamma) \qquad (7.60)\end{aligned}$$

In the latter relation, $P_n(\cos\gamma)$ are the Legendre polynomials (see Sect. 5.4). To second order in α, the separation between the two planets can thus be written as

$$\frac{1}{\Delta_{1,2}} = \frac{1}{a_2}\left(\frac{a_2}{r_2} + \alpha\,\frac{r_1}{a_1}\left(\frac{a_2}{r_2}\right)^2\cos\gamma + \alpha^2\left(\frac{r_1}{a_1}\right)^2\left(\frac{a_2}{r_2}\right)^3\left(\frac{3}{2}\,\cos^2\gamma - \frac{1}{2}\right) + \mathcal{O}(\alpha^3)\right) \qquad (7.61)$$

The next steps are then to express the fractions $\frac{\rho^n}{r_2} = \frac{r_1^n}{r_2^{n+1}}$ as functions of the osculating elements of the orbits of the two planets.

In this context, let us remind that, with the exception of Mercury, most planets of our Solar System have orbits with small eccentricities and with a rather low inclination with respect to the plane of the ecliptic (see Table 2.1). As a result, some of the classical osculating elements, such as ω and Ω are somewhat ill defined and we use instead the following elements that are more regular for almost circular, low-inclination orbits[4]:

$$z = e\,\exp\left(j\,\varpi\right)$$

[4]Note that an overlined complex number (such as $\overline{X}$) stands for its conjugate. We recall that to avoid confusion, we use the notation i for the inclination angle, whilst j as part of the argument of an exponential function stands for the imaginary unit ($j^2 = -1$).

$$X = e\,\exp(j\,M) = e\,\exp[j\,(L-\varpi)]$$
$$\zeta = \sin(i/2)\,\exp(j\,\Omega)$$
$$Y = \sin(i/2)\,\exp[j\,(L-\Omega)]$$

where $\varpi = \Omega + \omega$ and $L = M + \varpi$.
For instance, from the relations in Sect. 4.3, we find that

$$\begin{aligned}
\frac{a}{r} &= 1 + \frac{1}{2}(X+\overline{X}) + \frac{1}{2}(X^2+\overline{X}^2) + \mathcal{O}(e^3)\\
\frac{r}{a} &= 1 - \frac{1}{2}(X+\overline{X}) - \frac{1}{4}(X^2 - 2\,X\,\overline{X} + \overline{X}^2) + \mathcal{O}(e^3)\\
\theta = \exp[j\,(\phi - M)] &= 1 + (X-\overline{X}) + \frac{1}{8}(9\,X^2 - 8\,X\,\overline{X} - \overline{X}^2) + \mathcal{O}(e^3)
\end{aligned}$$

From these relations, we see that the development up to the dth power of e of any term of the kind $\left(\frac{r}{a}\right)^n \exp[j\,m\,(\phi - M)]$ can be expressed as a product of series of increasing powers of X:

$$\left(\frac{r}{a}\right)^n \exp[j\,m\,(\phi - M)] = \sum_{0\le p+p'\le d} X^p \overline{X}^{p'} C^{n,m}_{p,p'} + \mathcal{O}(e^{d+1})$$

where the summation is actually a double summation over the (positive) indices p and p', under the condition that the sum of the indices $p + p'$ does not exceed d.
Since

$$\frac{r_1^n}{r_2^{n+1}} = \frac{\alpha^n}{a_2}\left(\frac{r_1}{a_1}\right)^n \left(\frac{a_2}{r_2}\right)^{n+1}$$

we see that $\Delta_{1,2}^{-1}$ can be expressed as a quadruple sum of terms of the kind

$$\begin{aligned}
\frac{r_1^n}{r_2^{n+1}} &= \frac{\alpha^n}{a_2} \sum_{0\le p_1+p_1'+p_2+p_2'\le d} C^{n,0}_{p_1,p_1'}\, C^{-n-1,0}_{p_2,p_2'} X_1^{p_1}\,\overline{X_1}^{p_1'}\, X_2^{p_2}\,\overline{X_2}^{p_2'}\\
&= \frac{\alpha^n}{a_2} \sum_{0\le p_1+p_1'+p_2+p_2'\le d} C^{n,0}_{p_1,p_1'}\, C^{-n-1,0}_{p_2,p_2'} e_1^{(p_1+p_1')}\, e_2^{(p_2+p_2')} \times\\
&\quad \exp\{j\,[(p_1-p_1')\,(L_1-\varpi_1) + (p_2-p_2')\,(L_2-\varpi_2)]\}
\end{aligned} \tag{7.62}$$

To compute the distance $\Delta_{1,2}$ between the two planets and to evaluate the scalar product in $U_{1,2}$, the cosine of the angle γ between the vectors $\vec{r_1}$ and $\vec{r_2}$ still needs to be expressed as a function of the osculating elements. This can be done by noting that the components of the vectors $\vec{u_1}$ and $\vec{u_2}$ are given by (see Figure 7.13):

$$\vec{u_1} = (\cos\psi_1\,\cos\Omega_1 - \sin\psi_1\,\sin\Omega_1\,\cos i_1, \cos\psi_1\,\sin\Omega_1 + \sin\psi_1\,\cos\Omega_1\,\cos i_1, \sin\psi_1\,\sin i_1)$$

and the same expression holds for $\vec{u_2}$ by replacing the index "1" by "2". Here $\psi_1 = l_1 - \Omega_1$, where l_1 is the so-called true longitude along the osculating orbit.
One can then compute $\cos\gamma = \vec{u_1}\cdot\vec{u_2}$, and accounting for the fact that $\psi_1 = l_1 - \Omega_1$, we obtain

$$\begin{aligned}
\cos\gamma = \Re\Big\{&\cos^2\frac{i_1}{2}\,\cos^2\frac{i_2}{2}\,\exp[j\,(l_1-l_2)] + \sin^2\frac{i_1}{2}\,\sin^2\frac{i_2}{2}\,\exp[j\,(l_1-l_2-2\,\Omega_1+2\,\Omega_2)]\\
&+\sin^2\frac{i_1}{2}\,\cos^2\frac{i_2}{2}\,\exp[j\,(l_1+l_2-2\,\Omega_1)] + \cos^2\frac{i_1}{2}\,\sin^2\frac{i_2}{2}\,\exp[j\,(l_1+l_2-2\,\Omega_2)]\\
&+\frac{1}{2}\sin i_1\,\sin i_2\,(\exp[j\,(l_1-l_2-\Omega_1+\Omega_2)] - \exp[j\,(l_1+l_2-\Omega_1-\Omega_2)])\Big\}
\end{aligned}$$

Finally, by introducing the complex variables ζ, z, X and Y along with their conjugates into the above relation and by noting that $l = L + (\phi - M)$, we eventually obtain an expression of $\cos\gamma$ as a series of terms that are products of various powers of ζ_1, ζ_2, z_1 and z_2 as well as of the trigonometric functions $\exp\left[j\,(p\,L_1 + (p+q)\,L_2)\right]$. These expressions can then be inserted into the development of the potentials.

Once the potentials[5] $U_{k,n}$ have been expressed in terms of the osculating elements, they are added for the different values of n and the resulting expressions of U_k are then inserted into the Lagrange equations[6] and this yields the set of differential equations for the elements of the osculating orbit that need then to be integrated. As for the general theory of perturbations, one again distinguishes between secular (in this case terms that are independent of L_1 or L_2) and periodic terms.
This kind of development is not a simple task and requires substantial computing power. For instance, for the eight planets of our Solar System, a development up to second order in m_k/m_0 and up to fifth order in eccentricity e and inclination i contains more than 150 000 terms.

The variations of a_k are periodic and do not include a secular term. Actually, the so-called secular variations of the z_k and ζ_k parameters are not genuine secular trends but rather very long-period periodic modulations with a frequency of the order $\frac{m_k}{m_0}\,n_{0,k}$, i.e. about three orders of magnitude slower than the mean motion $n_{0,k}$. Since the periods of planetary revolutions are of order of years up to decades and even centuries, it becomes clear that these variations are indeed extremely slow.

Another feature that appears in the motion of a planetary system is the **resonance** between different planets. Indeed, from the above relations of the U_k terms, it appears that the integration of the Lagrange equations produces denominators of the kind $(p\,n_{0,k} - (p+q)\,n_{0,n})$ with p and q being small integers. For some combinations of p and q, these denominators can actually be rather small in comparison to $n_{0,k}$ and hence produce a rather large amplitude in the corresponding term of the perturbation. Each of these terms multiplies a power e^q at least. The most prominent case in our Solar System (see Table 2.1) is the resonance between Jupiter and Saturn, where $2\,n_{♃} - 5\,n_{♄} = -4.02"\,\mathrm{day}^{-1}$ (which corresponds to a period of 883 years). Since in this case, $q = 3$, the Jupiter - Saturn resonance corresponds to an e^3 term. A very similar situation occurs between Uranus and Neptune where $n_{♅} - 2\,n_{♆}$ corresponds to a period of 4239 years and multiplies a term in e^1.

It is actually the fact that the motion of Uranus is perturbed by the attraction of Neptune that led to the discovery of the latter (see Chapter 1). When the motion of Neptune was also found to apparently deviate from the predictions, the hypothesis of another planet (planet X) revolving beyond the orbit of Neptune was formulated. To explain the apparent effect on Neptune, the mass of planet X was estimated to be about $12\,\mathrm{M}_{♁}$ in 1848. Percival Lowell (1855 - 1916) spent part of his life searching for this mysterious planet. In 1930, Clyde Tombaugh (1906 - 1997) discovered the dwarf planet Pluto, then believed to be planet X. However, Pluto was about a factor ten dimmer (and hence less massive) than predicted. For a long time, it was assumed that Pluto might indeed be large and dark, but with more powerful telescopes, its mass was continuously reduced. In 1978, Charon the largest of Pluto's moons was discovered and accurate mass determinations became available. The mass of Pluto is now determined to be $0.0021\,\mathrm{M}_{♁}$. Such a low mass is by no means sufficient to explain the apparent deviations of Neptune. However, with the data from modern spacecraft, the masses of the outer planets are now much better known, the transneptunian minor planets are better known and modern calculations can explain the motion of the planets without the need for a perturbing planet beyond the orbit of Neptune.
In this context, let us stress that the orbit of Pluto is stable, although it crosses Neptune's orbit, because the orbital

[5]Remember that these potentials are different for each planet.
[6]A set of Lagrange equations equivalent to 6.47 but for the case where e and i are small can be derived from the definition of the osculating elements used here.

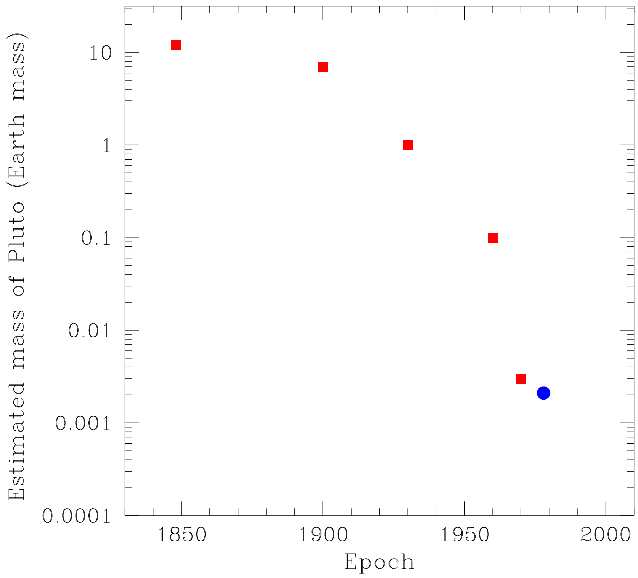


Figure 7.14: The estimated (filled squares) or dynamically determined (filled dot) mass of planet X (before 1930) and Pluto (after 1930) as a function of time. The mass is given in units of the Earth's mass ($M_{\oplus}$).

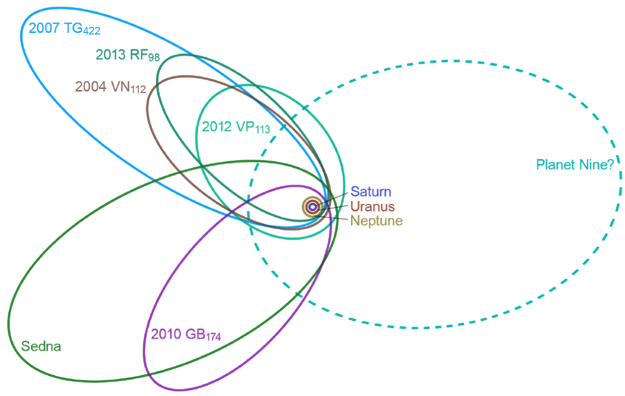


Figure 7.15: Orbital clustering of distant Kuiper Belt objects. This situation could reveal the existence of a yet undetected massive planet on an eccentric orbit ($e \sim 0.6$) having its periastron anti-aligned with the Kuiper Belt objects (Batygin & Brown 2016).

periods are in a $2\,n_{♆} : 3\,n_{♇}$ resonance. This resonance ensures that Neptune and Pluto remain at a large distance ($>$ 17 AU) from each other, at the times their orbits cross. Without this resonance, the dwarf planet would be ejected by the strong perturbations due to Neptune.

Another interesting example of resonance in the Solar System is provided by the Kirkwood gaps. The latter are gaps in the observed distribution of semi-major axes of the asteroids of the main asteroid belt (located between about 2.1 and 3.5 AU) due to orbital resonances with Jupiter. The most important gaps are found for semi-major axes of 2.50, 2.82, 2.95 and 3.27 AU and correspond respectively to the 3:1, 5:2, 7:3 and 2:1 resonances. Let us consider the 3:1 resonance: asteroids with $a = 2.50$ AU have an orbital period of 3.95 years and will thus accomplish 3 orbits whilst Jupiter revolves once around the Sun. Although, there is still no complete theory for the effect of these resonances on the distribution of asteroids, it is known that Jupiter's attraction perturbs the orbits of these asteroids generating orbital chaos and planet crossing (the eccentricity of the asteroid's orbit becomes large enough for the asteroid to cross the orbit of Mars). From numerical simulations of the problem, it has been shown that the eccentricity of Jupiter's orbit is of prime importance in this context (Saha 1992).

Finally, another intriguing example of a possible resonance concerns orbital grouping in longitude of perihelion (ω) of very distant Kuiper Belt objects (see Fig. 7.15). This feature is surprising as the gravitational perturbation exerted by the giant planets of the Solar System should lead to apsidal precession that would randomize the values of ω. Yet, Batygin & Brown (2016) drew attention to a group of objects with $a > 250$ AU that share the same value of ω to within 8°. Batygin & Brown (2016) showed that such a grouping could be maintained by the action of a hypothetical Planet Nine with a mass of at least 10 $M_{\oplus}$ and a highly eccentric orbit whose perihelion would be anti-aligned with that of the Kuiper Belt objects.

7.4.1 The Laplace resonance

A fascinating case of resonance exists among the Galilean moons Io, Europa and Ganymede. This is a so-called Laplace resonance which occurs when three or more orbiting bodies have a simple integer ratio between their orbital periods. In the present case, we have an orbital resonance of 1:2:4. This situation is illustrated in Fig. 7.16.

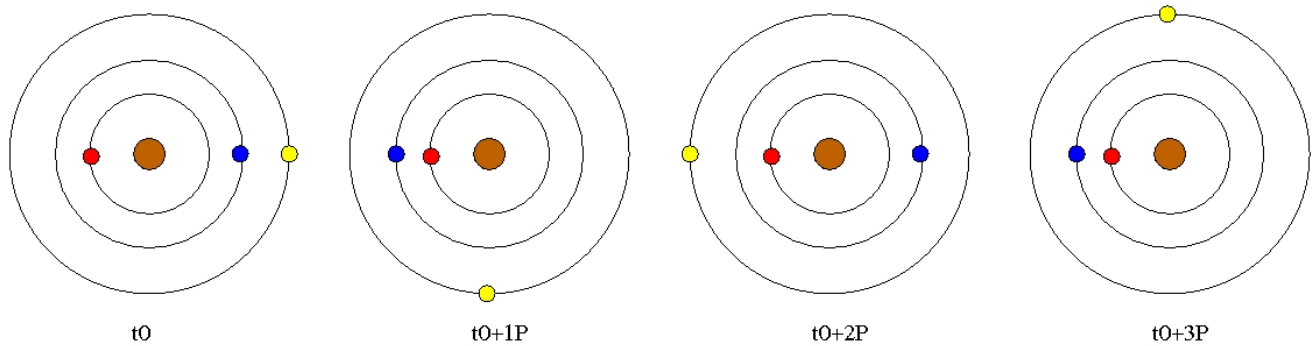


Figure 7.16: Illustration of the Laplace resonance among the three inner Galilean moons: Io, Europa and Ganymede. The configuration of the three satellites with respect to Jupiter is illustrated at four different times, separated by one orbital period of Io each.

Let us consider this situation in more details (Showman 1997). Actually, the ratio of mean motions of Io and Europa is not exactly 2:1. Hence, their conjunction drifts at a mean angular velocity $2\,n_{\rm Europa} - n_{\rm Io} = -0.74^\circ\,{\rm day}^{-1}$. However, the Io - Europa conjunction is actually locked to Io's perijove (i.e. its closest approach to Jupiter) and Europa's apojove:

$$\omega_{\rm Io} = (2\,n_{\rm Europa} - n_{\rm Io})\,t - \theta_1$$

$$\omega_{\rm Europa} = (2\,n_{\rm Europa} - n_{\rm Io})\,t - \theta_2$$

where θ_1 and θ_2 librate about 0° and 180° respectively.
The same holds for the Europa - Ganymede conjunction which occurs when Europa is near perijove.

$$\omega_{\rm Europa} = (2\,n_{\rm Ganymede} - n_{\rm Europa})\,t - \theta_3$$

where, this time, θ_3 librates about 0°. The Laplace resonance is then defined as a 1:1 commensurability between the rates of motion of the Io - Europa and Europa - Ganymede conjunctions. In other words,

$$\dot{\omega}_{\rm Io} = \dot{\omega}_{\rm Europa}$$

and $(2\,n_{\rm Ganymede} - 3\,n_{\rm Europa} + n_{\rm Io})\,t$ librates about 180°. This latter relation implies that a triple conjunction of the three moons is impossible. The fact that the three moons periodically line up in the same configuration actually prevents the orbit of Io and Europa from being circularized by Jupiter's tidal effect. Indeed, the gravitational interaction between these moons stretches their orbits into elliptical shapes.
These orbital resonances have wide-ranging consequences. Indeed, Io's strong volcanism is most probably triggered by Jupiter's tidal action which is modulated by the eccentricity of Io's orbit. The eccentricity itself is forced by the resonance with the other two jovian satellites.
Another spectacular example of nearly resonant orbits was found among the seven exo-planets of the Trappist-1 system. These planets have orbital periods about their host star of 1.51, 2.42, 4.04, 6.10, 9.21, 12.35 and 18.77 days. The ratios of the orbital periods of the five innermost planets are very close to the ratios of integer numbers: 5/8, 3/8, 1/4, 1/6 where we have taken the ratio between the orbital period of the inner planet and each of the following

four. If we rather consider the ratio between the orbital periods of a planet and its nearest neighbour, we obtain ratios of 5/8, 5/3, 3/2, 3/2, 4/3 and 3/2. With the latter formulation, we see that even the two outer planets are found to be in near-resonance. This resonant configuration plays a key role in maintaining these exo-planets in a stable configuration.

7.4.2 Perihelion precession in the Solar System

Let us consider that all the planets of the Solar System move in the same plane (the ecliptic) around the Sun. If we are interested in long-term trends, we can average the mutual interaction of the planets over their orbital period. Carl Friedrich Gauß (1777 - 1855) thus proposed to model each planet as a ring of radius a_k centred on the Sun. We can then evaluate the potential due to such a uniform ring of mass m_k at some point $\vec{r_j}$ inside the ecliptic.

$$U_{j,k} = -\frac{G\,m_k}{2\,\pi}\int_0^{2\,\pi}\frac{1}{|\vec{r_j}-\vec{r_k}|}d\phi$$

For $a_j > a_k$, we have

$$\frac{1}{|\vec{r_j}-\vec{r_k}|} = \frac{1}{a_j}\sum_{n=0}^{\infty}\left(\frac{a_k}{a_j}\right)^n P_n(\cos\phi)$$

whilst for $a_j < a_k$, we have

$$\frac{1}{|\vec{r_j}-\vec{r_k}|} = \frac{1}{a_k}\sum_{n=0}^{\infty}\left(\frac{a_j}{a_k}\right)^n P_n(\cos\phi)$$

Therefore, we see that the total potential at the position of the planet number j can be written

$$U_j = -\frac{G\,M_\odot}{r} - \sum_{n=0}^{\infty} s_n\left[\sum_{k<j}\frac{G\,m_k}{r}\left(\frac{a_k}{r}\right)^n + \sum_{j<k}\frac{G\,m_k}{a_k}\left(\frac{r}{a_k}\right)^n\right]$$

Here we identify the planets by their number expressing increasing distances from the Sun outwards (e.g. $k = 1 \equiv$ Mercury, $k = 2 \equiv$ Venus,...). The coefficients $s_n = \frac{1}{2\,\pi}\int_0^{2\,\pi} P_n(\cos\phi)\,d\phi$ are zero for odd values of n and amount to $P_n(0)^2$ for even values of n. Thus $s_0 = 1$, $s_2 = \frac{1}{4}$, $s_4 = \frac{9}{64}$, $s_6 = \frac{25}{256}$,... We will use this potential hereafter to investigate the precession of the longitude of perihelion ($\dot{\omega}$) of the planets of the Solar System.

Before we do so, let us first consider a mass moving on a circular orbit under the effect of the central force per unit mass $\vec{f}(r) = f(r)\,\vec{e_r}$. In polar coordinates, the equations of the motion along the $\vec{e_r}$ and $\vec{e_\theta}$ axes become, respectively:

$$\ddot{r} - r\,\dot{\theta}^2 = f(r) \quad (7.63)$$

$$2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta} = 0 \quad (7.64)$$

The latter of these equations yields the conservation of angular momentum $r^2\,\dot{\theta} = h = Cst$. Inserting this result into the former equation, we obtain

$$\ddot{r} - \frac{h^2}{r^3} = f(r) \quad (7.65)$$

If the mass is moving on a strictly circular orbit of radius a, we have $\ddot{r} = 0$ and thus $-h^2 = a^3\,f(a)$. Let us now assume that the orbit is not exactly circular, and let us define $\Delta r = r - a$ with $\Delta r << a$. Expanding to first order in $\frac{\Delta r}{a}$, we obtain that

$$\ddot{\Delta r} - \frac{h^2}{a^3}\left(1 - 3\,\frac{\Delta r}{a}\right) = f(a) + \left.\frac{d\,f}{dr}\right)_a\Delta r$$

which leads to

$$\ddot{\Delta r} + \left[-\frac{3\,f(a)}{a} - \frac{d\,f}{dr}\Big)_a\right]\Delta r = 0$$

If the term between brackets is positive, then Δr oscillates periodically between a minimum and a maximum value with a period

$$P = \frac{2\,\pi}{\left[-\frac{3\,f(a)}{a} - \frac{d\,f}{dr}\Big)_a\right]^{1/2}} \tag{7.66}$$

For the conventional two-body problem, $f(r) = -\mu\,r^{-2}$, and the above relation simply yields Kepler's third law. The angle that separates the two extreme values of Δr then becomes

$$\frac{P\,\dot{\theta}}{2} = \frac{\pi}{\left[3 + \frac{d\,f}{dr}\Big)_a \frac{a}{f(a)}\right]^{1/2}} \tag{7.67}$$

Let us now apply these results to the Gauss approximation of the planetary perturbations. From the potential established above, we infer

$$f(r) = -\frac{G\,M_\odot}{r^2} - \sum_{n=0}^{\infty} s_n \left[\sum_{k<j} \frac{G\,m_k}{r^2}\left(\frac{a_k}{r}\right)^n (n+1) - \sum_{j<k} \frac{G\,m_k}{a_k^2}\left(\frac{r}{a_k}\right)^{n-1} n\right] \tag{7.68}$$

$$\frac{d\,f}{dr} = \frac{2\,G\,M_\odot}{r^3} + \sum_{n=0}^{\infty} s_n \left[\sum_{k<j} \frac{G\,m_k}{r^3}\left(\frac{a_k}{r}\right)^n (n+1)(n+2) + \sum_{j<k} \frac{G\,m_k}{a_k^3}\left(\frac{r}{a_k}\right)^{n-2} n(n-1)\right] \tag{7.69}$$

To first order in $\frac{m_k}{M_\odot}$, we then obtain:

$$\left[3 + \frac{d\,f}{dr}\Big)_{a_j} \frac{a_j}{f(a_j)}\right]^{-1/2} = 1 + \frac{1}{2}\sum_{n=0}^{\infty} s_n\,n\,(n+1)\left[\sum_{k<j}\frac{m_k}{M_\odot}\left(\frac{a_k}{a_j}\right)^n + \sum_{j<k}\frac{m_k}{M_\odot}\left(\frac{a_j}{a_k}\right)^{n+1}\right] \tag{7.70}$$

The rate of perihelion precession hence becomes

$$\dot{\omega} = \frac{n_j}{2}\sum_{n=0}^{\infty} s_n\,n\,(n+1)\left[\sum_{k<j}\frac{m_k}{M_\odot}\left(\frac{a_k}{a_j}\right)^n + \sum_{j<k}\frac{m_k}{M_\odot}\left(\frac{a_j}{a_k}\right)^{n+1}\right] \tag{7.71}$$

where n_j is the mean orbital motion of planet j. The results of this rather simple approach are shown in Fig. 7.17. Overall, we find that, despite the crude approximations, the agreement between our very simple treatment and the actual measurements is not too bad. The worst discrepancy is observed for Venus, where this approach overpredicts the actual rate of precession. Venus is the planet that undergoes the strongest influence by planet Mercury. Mercury has an orbit that is quite eccentric and quite inclined with respect to the plane of the ecliptic. In the approach adopted here neither the eccentricity of the perturber's orbit nor its inclination are accounted for. A better agreement can be achieved using a more sophisticated approach where the inclinations of the different orbital planes as well as the eccentricities of the orbits are accounted for, or alternatively by means of the so-called Laplace-Lagrange secular evolution theory (see Fitzpatrick 2012)[7].

[7] See also the work of Lo et al. (2013) who present an approach that accounts for the eccentricity of the perturber's orbit whilst still using the Gauss approximation.

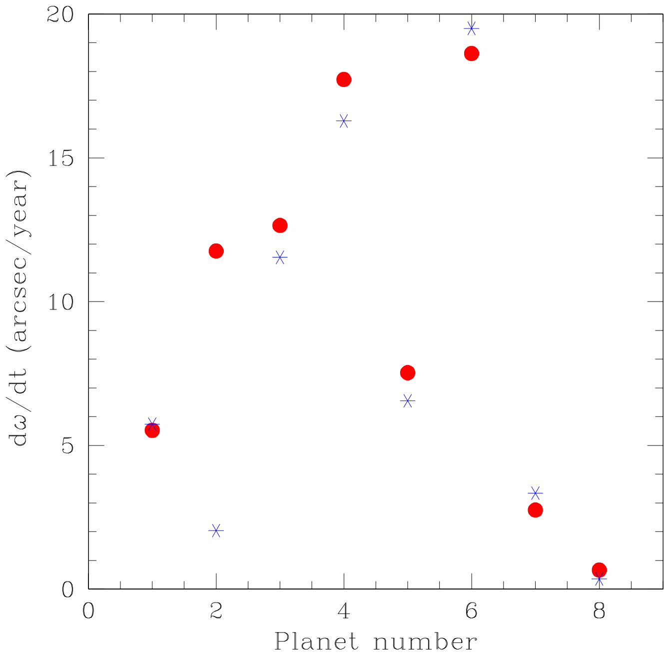


Figure 7.17: Perihelion precession rates for the planets of the Solar System. The stars yield the observed rates (adopted from Fitzpatrick 2012), whilst the filled symbols indicate the results of our estimates using equation 7.71 up to $n = 16$.

7.5 Exercises

7.1 General relativity leads to a correction of the gravitational potential of the Sun as experienced by planet Mercury.

$$U = -\frac{G\,M_\odot}{r} - \frac{G\,M_\odot\,h^2}{c^2\,r^3}$$

where c is the speed of light in the vacuum and h is the angular momentum (which is constant). Demonstrate that the constant term of the Fourier expansion of the relativistic correction amounts to

$$\overline{U'} = \frac{-G\,M_\odot\,h^2}{c^2\,a^3\,(1-e^2)^{3/2}}$$

Use the Lagrange equations to infer the general relativistic correction for the precession of Mercury's perihelion

$$\dot{\omega}_{GR} = \frac{3\,(G\,M_\odot)^{3/2}}{c^2\,a^{5/2}\,(1-e^2)}$$

What is the value of this precession rate and how does it compare to the classical (Newtonian) term which amounts to 5.32 arcsec yr^{-1}?

7.2 A comet approaching the Sun on a parabolic orbit of perihelion distance r_p and inclination i with respect to Jupiter's orbital plane is perturbed by a close encounter with Jupiter such that its new orbit in the heliocentric frame of reference becomes an ellipse of semi-major axis a', eccentricity e' and inclination i'. Demonstrate that

$$\sqrt{\frac{2\,r_p}{a_{♃}}}\,\cos i \simeq \frac{a_{♃}}{2\,a'} + \sqrt{\frac{a'}{a_{♃}}\,(1-e'^2)}\,\cos i'$$

Same question for a comet initially on a hyperbolic orbit with a perihelion distance r_p, inclination i and asymptotes with an acute angle α with respect to each other. Demonstrate that

$$\frac{a_{♃}}{2\,r_p}\,(1-e)+\sqrt{\frac{r_p}{a_{♃}}\,(1+e)}\,\cos i\simeq\frac{a_{♃}}{2\,a'}+\sqrt{\frac{a'}{a_{♃}}\,(1-e'^2)}\,\cos i'$$

with $e=\frac{1}{\cos\frac{\alpha}{2}}$.

Chapter 8

The rotation of a rigid celestial body

In addition to their orbital motion, most objects in the Universe are observed to be in rotation. Whilst this is obviously true for stars, planets and minor bodies, we shall only consider the simplest situation, i.e. the rotation of rigid bodies where the angular velocity vector $\vec{\omega}$ is constant everywhere inside the body. However, even in this simplified case, the description of the phenomenon becomes rather complex when several objects act upon each other. In this chapter, we restrict ourselves to a first introduction to the subject.

8.1 Fundamental concepts

Let CM be the center of mass of a rigid (at least to first approximation) body such as a planet, an asteroid,... The angular momentum around the center of mass can be written as

$$\begin{aligned}
\vec{L}_{CM} &= \int (\vec{r} \wedge \dot{\vec{r}})\, \rho(\vec{r})\, dV \\
&= \int [\vec{r} \wedge (\dot{\vec{r}}_{CM} + \vec{\omega} \wedge \vec{r})]\, \rho(\vec{r})\, dV \\
&= \int [\vec{r} \wedge (\vec{\omega} \wedge \vec{r})]\, \rho(\vec{r})\, dV \\
&= \mathcal{I}\, \vec{\omega}
\end{aligned} \tag{8.1}$$

where $\mathcal{I}$ is the moment of inertia tensor defined by $\mathcal{I}_{ij} = -\int (x_i\, x_j)\, \rho\, dV$ for $i, j = 1, 2, 3$ and $i \neq j$ and $\mathcal{I}_{ii} = \sum_{j\neq i, j=1}^{3} \int x_j^2\, \rho\, dV$.
It is always possible to define so-called principal axes of inertia $(\vec{e_x}', \vec{e_y}', \vec{e_z}')$ where the tensor can be expressed as

$$\mathcal{I} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \tag{8.2}$$

The rate of change of the angular momentum is given by the moment of the external forces:

$$\begin{aligned}
\frac{d\,\vec{L}_{CM}}{dt} &= \frac{\delta\,\vec{L}_{CM}}{\delta t} + \vec{\omega} \wedge \vec{L}_{CM} \\
&= \int (\dot{\vec{r}} \wedge \dot{\vec{r}})\, \rho(\vec{r})\, dV + \int (\vec{r} \wedge \ddot{\vec{r}})\, \rho(\vec{r})\, dV \\
&= \int (\vec{r} \wedge \vec{F})\, dV = \vec{\mathcal{M}}
\end{aligned} \tag{8.3}$$

where $\vec{F}$ is the external force per unit volume. This equation can also be expressed by three scalar differential equations, sometimes called Euler's equations of rotational motion:

$$A\,\dot{\omega}_1 + (C - B)\,\omega_2\,\omega_3 = \mathcal{M}_1 \tag{8.4}$$
$$B\,\dot{\omega}_2 + (A - C)\,\omega_3\,\omega_1 = \mathcal{M}_2 \tag{8.5}$$
$$C\,\dot{\omega}_3 + (B - A)\,\omega_1\,\omega_2 = \mathcal{M}_3 \tag{8.6}$$

On the other hand, the kinetic energy can be expressed as

$$\begin{aligned} T &= \frac{1}{2}\int \dot{\vec{r}}\cdot\dot{\vec{r}}\,\rho(\vec{r})\,dV \\ &= \frac{1}{2}\,M\,\dot{r}_{CM}^2 + \frac{1}{2}\,\vec{\omega}\cdot\mathcal{I}\vec{\omega} = T_{CM} + T_{rot} \end{aligned} \tag{8.7}$$

In the principal axes of inertia, the kinetic energy of the rotation is given by

$$T_{\rm rot} = \frac{1}{2}\,[A\,\omega_1^2 + B\,\omega_2^2 + C\,\omega_3^2]$$

The rotation of a rigid body is usually described by means of Euler's angles (see Fig. 8.1).

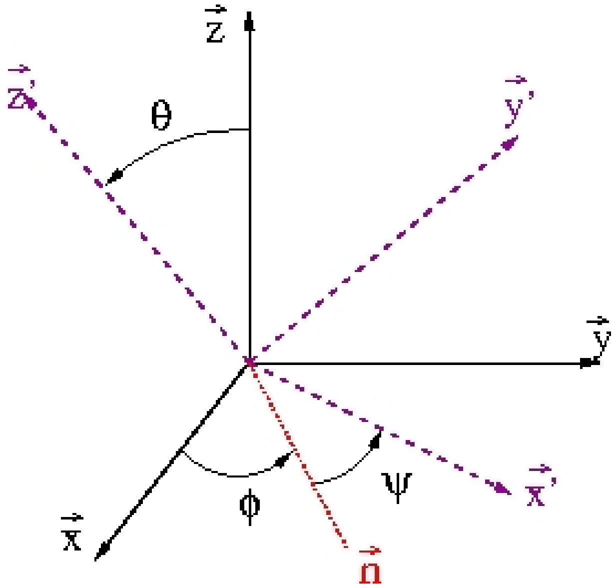


Figure 8.1: Illustration of Euler's angles. The (x, y, z) axes refer to an inertial frame of reference, whilst (x', y', z') designate the coordinates corresponding to the principal axes.

Euler's angles express the position of the moving system of coordinates (x', y', z') with respect to the absolute frame of reference (x, y, z). $\vec{n}$ is the direction of the line of nodes, which is defined here as the intersection between the (x', y') and (x, y) planes. ϕ is the precession angle (rotation around $\vec{e_z}$), θ is the nutation angle (rotation around $\vec{n}$) and ψ is the spin angle (rotation around $\vec{e_z}'$).
The instantaneous angular velocity can thus be written as

$$\vec{\omega} = \dot{\phi}\,\vec{e_z} + \dot{\theta}\,\vec{n} + \dot{\psi}\,\vec{e_z}'$$

Since $\vec{e_z} = \cos\theta\,\vec{e_z}' + \sin\theta\,\sin\psi\,\vec{e_x}' + \sin\theta\,\cos\psi\,\vec{e_y}'$ and $\vec{n} = \cos\psi\,\vec{e_x}' - \sin\psi\,\vec{e_y}'$, we can express the angular velocity in the principal axes as

$$\vec{\omega} = (\dot{\phi}\,\sin\theta\,\sin\psi + \dot{\theta}\,\cos\psi)\,\vec{e_x}' + (\dot{\phi}\,\sin\theta\,\cos\psi - \dot{\theta}\,\sin\psi)\,\vec{e_y}' + (\dot{\phi}\,\cos\theta + \dot{\psi})\,\vec{e_z}' \tag{8.8}$$

The components of $\vec{\omega}$ can thus be written as

$$\omega_1 = \dot{\phi}\,\sin\theta\,\sin\psi + \dot{\theta}\,\cos\psi \tag{8.9}$$
$$\omega_2 = \dot{\phi}\,\sin\theta\,\cos\psi - \dot{\theta}\,\sin\psi \tag{8.10}$$
$$\omega_3 = \dot{\phi}\,\cos\theta + \dot{\psi} \tag{8.11}$$

For a freely rotating rigid body[1], $\mathcal{L} = T$ and

$$p_\phi = \frac{\partial T}{\partial\dot{\phi}} = (A\,\omega_1\,\sin\psi + B\,\omega_2\,\cos\psi)\,\sin\theta + C\,\omega_3\,\cos\theta \tag{8.12}$$
$$p_\theta = \frac{\partial T}{\partial\dot{\theta}} = A\,\omega_1\,\cos\psi - B\,\omega_2\,\sin\psi \tag{8.13}$$
$$p_\psi = \frac{\partial T}{\partial\dot{\psi}} = C\,\omega_3 \tag{8.14}$$

These expressions are rather complex and it is advantageous to use an alternative set of angles that are defined in Fig. 8.2.

8.2 The Andoyer canonical variables

In the same way as for the two-body problem, we can introduce a set of canonical elements that describe the problem of a rotating rigid body. These are the so-called Andoyer angles (l, g, h) illustrated in Fig. 8.2 and their conjugated momenta (L, G, H):

$$L = |\vec{L}_{CM}|\,\cos J \tag{8.15}$$
$$G = |\vec{L}_{CM}| \tag{8.16}$$
$$H = |\vec{L}_{CM}|\,\cos K \tag{8.17}$$

It can indeed be shown that the Andoyer elements form a set of canonical coordinates.
In the principal axes $(\vec{e_x}', \vec{e_y}', \vec{e_z}')$, we can write

$$\vec{L}_{CM} = (G\,\sin J\,\sin l, G\,\sin J\,\cos l, G\,\cos J)$$

and

$$\vec{\omega} = \mathcal{I}^{-1}\,\vec{L}_{CM} = (A^{-1}\,G\,\sin J\,\sin l, B^{-1}\,G\,\sin J\,\cos l, C^{-1}\,G\,\cos J)$$

This yields the following expression for the rotational kinetic energy:

$$T = \frac{1}{2}\,\vec{\omega}\cdot\vec{L}_{CM} = \frac{1}{2}\,(G^2 - L^2)\,(A^{-1}\,\sin^2 l + B^{-1}\,\cos^2 l) + \frac{L^2}{2\,C} \tag{8.18}$$

[1] We recall that the relation $p_i = \frac{\partial T}{\partial\dot{q}_i}$ remains valid also as long as the potential U of an external force does not depend on the generalized velocities $\dot{q}_i$.

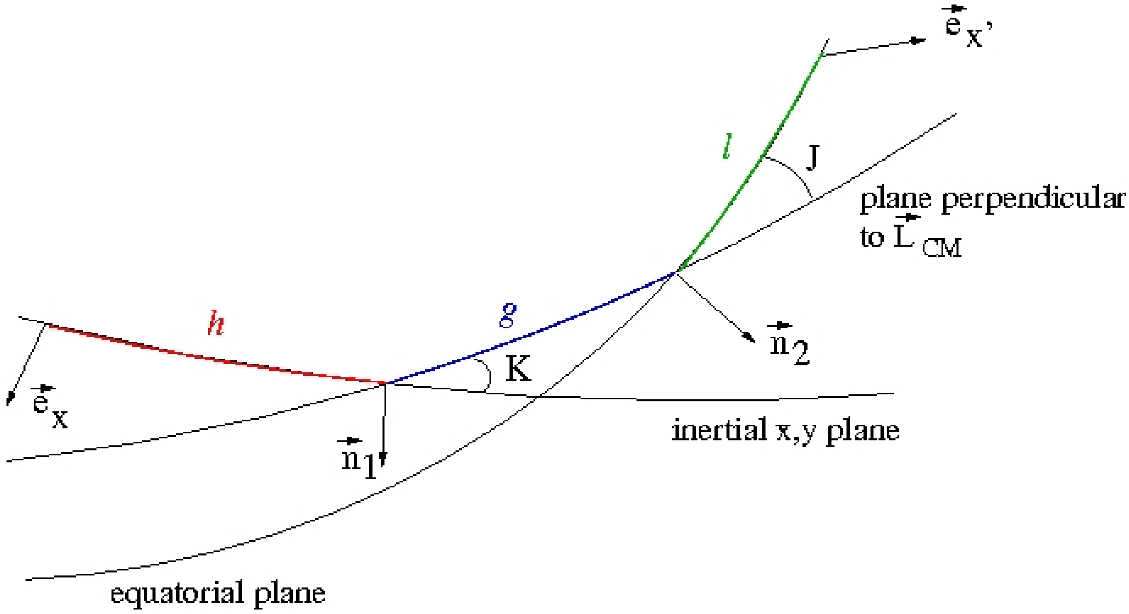


Figure 8.2: Definition of the Andoyer angular elements (l, g, h) and the angles K and J. The equatorial plane is defined as the principal plane perpendicular to $\vec{e_z}'$.

In the case where no external forces apply on the rigid body, $\mathcal{H} = T$ and the Hamiltonian is thus independent of g, h and H. Using the canonical equations of Hamilton, we obtain that

$$\dot{L} = -\frac{\partial \mathcal{H}}{\partial l} = (G^2 - L^2)\left[\frac{1}{B} - \frac{1}{A}\right] \sin l \, \cos l \tag{8.19}$$

$$\dot{G} = -\frac{\partial \mathcal{H}}{\partial g} = 0 \tag{8.20}$$

$$\dot{H} = -\frac{\partial \mathcal{H}}{\partial h} = 0 \tag{8.21}$$

$$\dot{l} = \frac{\partial \mathcal{H}}{\partial L} = L\left[\frac{1}{C} - \frac{\sin^2 l}{A} - \frac{\cos^2 l}{B}\right] \tag{8.22}$$

$$\dot{g} = \frac{\partial \mathcal{H}}{\partial G} = G\left[\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right] \tag{8.23}$$

$$\dot{h} = \frac{\partial \mathcal{H}}{\partial H} = 0 \tag{8.24}$$

Hence, h, G and H are constants if no external forces act upon the body.

If, in addition $A = B$, $\mathcal{H}$ does not depend explicitly on l and L is also constant. In the latter case, $\dot{l} = L\,(C^{-1} - A^{-1})$ and $\dot{g} = \frac{G}{A}$. Hence l and g grow at constant rates with time. Finally, we note that if $A = B = C$ (e.g. for a perfectly spherical body), the only remaining motion is a rotation about the direction of the $\vec{L}_{CM}$ vector at a constant rate.

8.2.1 The modified Andoyer elements

We note that if K or J are equal to zero, either h and g or g and l are no longer defined and it is advantageous to use the modified Andoyer elements: $p = l + g + h$, $q = g + h$, $s = -h$, $P = L$, $Q = G - L$ and $S = G - H$. It can be shown that these modified elements preserve the canonical form of the equations of Hamilton.

8.3 Perturbations due to an orbiting secondary mass

Let us consider a point-like mass m that orbits around the body of mass M that we wish to study. We assume that the orbit of m is circular and is located in the inertial plane. The gravitational potential at a position $\vec{r'}$ for the point-like mass m can then be expressed as

$$U(\vec{r'}) = -G\,m \int\int\int_V \frac{\rho(\vec{r})}{|\vec{r} - \vec{r'}|}\,dV$$

The $\frac{1}{|\vec{r}-\vec{r'}|}$ term can be expressed as

$$\begin{aligned}
\frac{1}{|\vec{r} - \vec{r'}|} &= (r^2 + r'^2 - 2\,r\,r'\,\cos\gamma)^{-1/2} \\
&= \frac{1}{r'}\left(1 - 2\,\frac{r}{r'}\,\cos\gamma + \frac{r^2}{r'^2}\right)^{-1/2} \\
&= \frac{1}{r'}\sum_{n=0}^{\infty}\left(\frac{r}{r'}\right)^n\,P_n(\cos\gamma)
\end{aligned} \tag{8.25}$$

In the latter relation, $P_n(\cos\gamma)$ are the Legendre polynomials (see Sect. 5.4) and γ is the angle between $\vec{r}$ and $\vec{r'}$. In this way, the potential can be expressed as a sum of potentials corresponding to the different (integer) values of n.

$$U = \sum_{n=0}^{\infty} U_n$$

Here, we restrict ourselves to $n \leq 2$. One obviously finds that $U_0 = \frac{-G\,M\,m}{r'}$ and $U_1 = 0$ (for a system of coordinates centred on the center of mass of M). After some calculations, one can further establish that

$$U_2 = \frac{4}{3}\,\alpha\,n^2\,C - 2\,\alpha\,n^2\,C\,\frac{x'^2 + y'^2}{r'^2} - 4\,\beta\,n^2\,C\,\frac{x'^2 - y'^2}{r'^2}$$

where $\alpha = \frac{3}{4}\,\frac{m}{M+m}\,\frac{2\,C-A-B}{2\,C}$, $\beta = \frac{3}{4}\,\frac{m}{M+m}\,\frac{B-A}{4\,C}$ and $n^2 = \frac{G\,(M+m)}{r'^3}$, whilst x' and y' refer to the components of the $\vec{r'}$ vector that indicates the position of the perturbing mass m in the frame of the principal axes of inertia.
As a further simplification, we shall assume that $J = 0$ (i.e. the vector $\vec{L}_{CM}$ is perpendicular to the equatorial plane). This then implies that g and l are no longer determined independently and we have to use the modified Andoyer elements. Let $\lambda = \lambda_0 + n\,t$ be the right ascension of the mass m at time t. Hence

$$\frac{x'}{r'} = \cos(h-\lambda)\,\cos(l+g) - \sin(h-\lambda)\,\sin(l+g)\,\cos K \tag{8.26}$$

$$\frac{y'}{r'} = -\cos(h-\lambda)\,\sin(l+g) - \sin(h-\lambda)\,\cos(l+g)\,\cos K \tag{8.27}$$

The Hamiltonian can then be written as

$$\begin{aligned}\mathcal{H} &= \frac{L^2}{2\,C} + n\,\Lambda + \alpha\,n^2\,C\,\sin^2 K\,[1 - \cos(2\,\lambda - 2\,h)] \\ &\quad -\beta\,n^2\,C\,[\cos(2\,\lambda - 2\,h + 2\,g + 2\,l)\,(1-\cos K)^2 \\ &\quad + \cos(2\,\lambda - 2\,h - 2\,g - 2\,l)\,(1+\cos K)^2 + 2\,\cos(2\,g + 2\,l)\,(1-\cos^2 K)] \qquad (8.28)\end{aligned}$$

or actually using the modified Andoyer elements

$$\begin{aligned}\mathcal{H} &= \frac{P^2}{2\,C} + n\,\Lambda + \alpha\,n^2\,C\,\sin^2 K\,[1 - \cos(2\,\lambda + 2\,s)] \\ &\quad -\beta\,n^2\,C\,[\cos(2\,\lambda + 2\,p + 4\,s)\,(1-\cos K)^2 \\ &\quad + \cos(2\,\lambda - 2\,p)\,(1+\cos K)^2 + 2\,\cos(2\,p + 2\,s)\,(1-\cos^2 K)] \qquad (8.29)\end{aligned}$$

There are several comments to be made about this Hamiltonian.

- The Hamiltonian is designed for the study of the rotation of the mass M. Therefore, we have excluded those terms that are not directly related to rotation (such as U_0 and the kinetic energy of the center of mass CM).
- Since we have made the assumption that $J = 0$, the kinetic energy of the rotation can simply be expressed as $\frac{L^2}{2\,C}$.
- We have introduced the generalized momentum Λ that allows us to include λ, the right ascension of the mass m, as one of the variables of the problem, whilst preserving the simple relation $\dot{\lambda} = n = \frac{\partial\mathcal{H}}{\partial\Lambda}$ valid for a circular orbital motion.
- As for any Hamiltonian, the terms that are important are those that are variable. We have therefore a priori excluded those terms that are simple additive constants.

To illustrate the applications of this Hamiltonian, we consider the problem of the precession of the Earth's rotation axes under the influence of the Moon and the Sun. Let us first note that since $S = P\,(1 - \cos K)$, we have $\frac{\partial\,\cos K}{\partial S} = \frac{-1}{P}$ and the rate of precession can be obtained from

$$\dot{s} = \frac{\partial\mathcal{H}}{\partial S} = \frac{-1}{P}\frac{\partial\mathcal{H}}{\partial\cos K} \qquad (8.30)$$

$$\begin{aligned}&= 2\,\alpha\,n^2\,\frac{C}{P}\,\cos K\,[1 - \cos(2\,\lambda + 2\,s)] - 2\,\beta\,n^2\,\frac{C}{P}\,[\cos(2\,\lambda + 2\,p + 4\,s)\,(1-\cos K) \\ &\quad - \cos(2\,\lambda - 2\,p)\,(1+\cos K) + 2\,\cos(2\,p + 2\,s)\,\cos K] \qquad (8.31)\end{aligned}$$

In the latter expression, there are a number of terms that are periodic functions of time. These are the trigonometric functions that depend on $2\,\lambda$ or $2\,(p{+}s)$. Whilst the former are periodic with half the orbital period of the perturbing body m, the latter vary with half the rotational period of mass M. Since we are interested in what happens over a long period of time, we can average the above derivative and we then obtain a time-averaged rate of precession:

$$< \dot{s} >= 2\,\alpha\,n^2\,\frac{C}{P}\,\cos K \qquad (8.32)$$

Whilst one has to keep in mind that we have made a number of simplifying assumptions, it is nevertheless interesting to see that this relation allows to make a rather accurate prediction of the precession rate of the Earth.
Indeed, let us first consider the action of the Moon. We have to stress here that the Moon does not revolve the Earth inside the inertial plane of the ecliptic. However, the inclination of the Moon's orbit ($i = 5°$) with respect

to the ecliptic is sufficiently small so that we can consider it negligible to first approximation. The term $\frac{2C-A-B}{2C}$ amounts to $1/306$ in the case of the Earth, whilst the equatorial plane is inclined by $K = 23.5^\circ$ with respect to the plane of the ecliptic. $\frac{C}{P}$ can be approximated as the inverse of the frequency of the sidereal day (13713 s rad^{-1}). $m/(M+m)$ amounts to 0.01215 and $n = 2.66170\times10^{-6}$ rad s^{-1}. We thus obtain $< \dot{s} >_{☾} = 5.306\times10^{-12}$ rad s^{-1} which corresponds to a period of 37521 yr.
As a next step, we consider the action of the Sun. In this case, $m/(M + m) = 0.999997$ and $n = 1.9966 \times 10^{-7}$ rad s^{-1} yielding $< \dot{s} >_{\odot} = 2.457 \times 10^{-12}$ rad s^{-1} which corresponds to a period of 81023 yr. The sum of the two effects then yields a total rate of precession of $< \dot{s} > = 7.7639 \times 10^{-12}$ rad s^{-1} which corresponds to a period of 25645 yr in excellent agreement with the actual period of 25765 yr.
If some of the simplifying assumptions are dropped, one can build a more rigorous theory of the perturbations of the rotation of a solid body. For instance, it is possible to study the impact of the Sun and the major planets of the Solar System on the rotation axis of Mars which displays almost chaotic long-term variations due to various resonance effects.

8.4 The Cassini states

Let us consider the impact of the Earth's attraction on the Moon. The plane of the Moon's orbit around the Earth (equivalent to the plane of the apparent orbit of the Earth around the Moon) slowly precesses and can thus not be considered an inertial frame of reference. Let us thus assume that the frame of reference $(\vec{e_x}, \vec{e_y}, \vec{e_z})$ tied to the orbital motion of the perturbing mass m (the Earth in this case) is rotating at a constant rate $\vec{\Omega}$ with respect to the inertial frame of reference $(\vec{g_x}, \vec{g_y}, \vec{g_z})$. If $\vec{\omega}$ is the rotational velocity of the principal axes $(\vec{e_x}', \vec{e_y}', \vec{e_z}')$ with respect to $(\vec{e_x}, \vec{e_y}, \vec{e_z})$, then the total rotation vector becomes

$$\vec{\omega}' = \vec{\omega} + \vec{\Omega}$$

The kinetic energy hence becomes

$$T = \frac{1}{2}\,\vec{\omega}' \cdot \vec{L}_{CM} = \frac{1}{2}\,\vec{\omega} \cdot \mathcal{I}\,\vec{\omega} + \vec{\Omega} \cdot \mathcal{I}\,\vec{\omega} + \frac{1}{2}\,\vec{\Omega} \cdot \mathcal{I}\,\vec{\Omega}$$

Since $\mathcal{H} = \sum_{i=1}^{3} p_i\,\dot{q}_i - \mathcal{L}$ and $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i}$, we obtain that

$$\begin{aligned} \mathcal{H} &= \vec{\omega} \cdot \mathcal{I}\,\vec{\omega} + \vec{\Omega} \cdot \mathcal{I}\,\vec{\omega} - T + U \\ &= T - \vec{\Omega} \cdot \vec{L}_{CM} + U \end{aligned} \tag{8.33}$$

Hence, the corrective term of the Hamiltonian is equal to $-\vec{\Omega}\cdot\vec{L}_{CM} = P\,[\sin K\,(\Omega_x\,\sin s + \Omega_y\,\cos s) - \Omega_z\,\cos K]$, where we have expressed the components of the vectors in the $(\vec{e_x}, \vec{e_y}, \vec{e_z})$ frame.
If we assume that the orbital plane of mass m precesses at a uniform rate Ω around an axis inclined by an angle i with respect to $\vec{e}_z$ in the plane $(\vec{e}_y, \vec{e}_z)$, we have that $\Omega_x = 0$, $\Omega_y = \Omega\,\sin i$ and $\Omega_z = \Omega\,\cos i$ (see Fig. 8.3). Hence

$$-\vec{\Omega} \cdot \vec{L}_{CM} = P\,\Omega\,(\sin i\,\sin K\,\cos s - \cos i\,\cos K)$$

With these results, the Hamiltonian can eventually be written

$$\begin{aligned} \mathcal{H} &= \frac{P^2}{2\,C} + n\,\Lambda + P\,\Omega\,(\sin i\,\sin K\,\cos s - \cos i\,\cos K) \\ &\quad +\alpha\,n^2\,C\,\sin^2 K\,[1 - \cos{(2\,\lambda + 2\,s)}] - \beta\,n^2\,C\,[\cos{(2\,\lambda + 2\,p + 4\,s)}\,(1 - \cos K)^2 \\ &\quad + \cos{(2\,\lambda - 2\,p)}\,(1 + \cos K)^2 + 2\,\cos{(2\,p + 2\,s)}\,(1 - \cos^2 K)] \end{aligned} \tag{8.34}$$

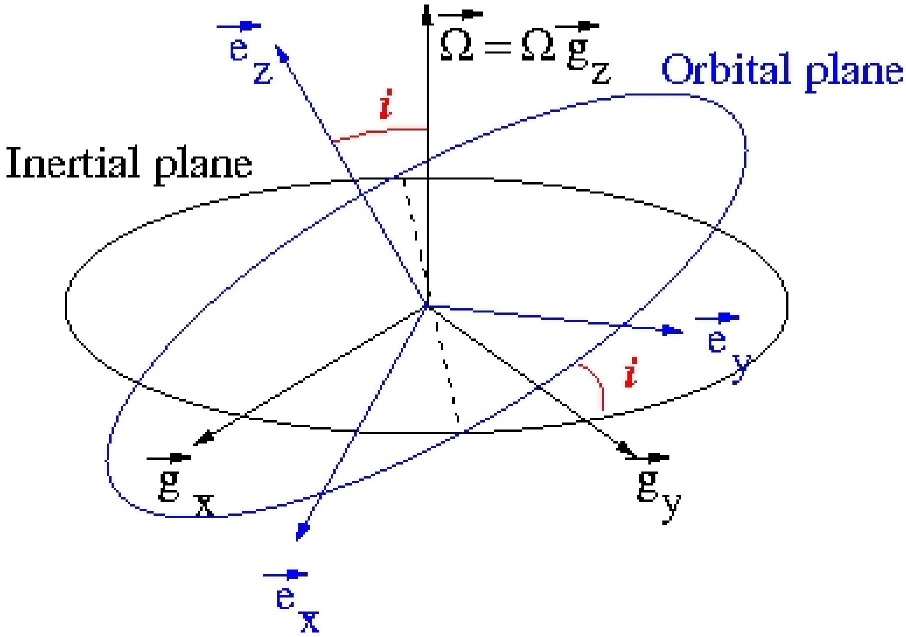


Figure 8.3: Illustration of the system of axes used in this section. The orbital plane precesses around $\vec{\Omega}$.

Again, we wish to quantify the long-term effects associated with this Hamiltonian. To do so, we again average out the periodic terms. However, in the case of the Moon, the rotational period and the period of revolution around the Earth are the same, so that we have a resonance between λ and p.
If we introduce $\sigma = p - \lambda$, it can be shown that the transformation $(p, \lambda, P, \Lambda) \rightarrow (\sigma, \lambda', P, \Lambda')$ is canonical provided that $\Lambda' = \Lambda + P$. We hence obtain the time-averaged Hamiltonian

$$\begin{aligned} \mathcal{H} &= \frac{P^2}{2\,C} + n\,\Lambda' - n\,P + P\,\Omega\,(\sin i\,\sin K\,\cos s - \cos i\,\cos K) \\ &\quad + \alpha\,n^2\,C\,\sin^2 K - \beta\,n^2\,C\,\cos(2\,\sigma)\,(1+\cos K)^2 \end{aligned} \tag{8.35}$$

The important variables in this problem are σ and s as well as their conjugated momenta P and $S = P\,(1 - \cos K)$. By noting that $\frac{\partial \cos K}{\partial P} = \frac{1-\cos K}{P}$ and $\frac{\partial \cos K}{\partial S} = \frac{-1}{P}$, we can then use the canonical equations of Hamilton to obtain the following set of differential equations:

$$\dot{\sigma} = \frac{P}{C} - n + \Omega\,[\sin i\,\sin K\,\cos s - \cos i\,\cos K] + \frac{1-\cos K}{P}\,\frac{\partial \mathcal{H}}{\partial \cos K} \tag{8.36}$$

$$\dot{s} = \frac{-1}{P}\,\frac{\partial \mathcal{H}}{\partial \cos K} \tag{8.37}$$

$$\dot{P} = -2\,\beta\,n^2\,C\,(1+\cos K)^2\,\sin(2\,\sigma) \tag{8.38}$$

$$\dot{S} = \Omega\,P\,\sin i\,\sin K\,\sin s \tag{8.39}$$

where the derivative of the Hamiltonian with respect to $\cos K$ in equations 8.36 and 8.37 is given by

$$\frac{\partial \mathcal{H}}{\partial \cos K} = \frac{-P\,\Omega}{\sin K}\,[\sin i\,\cos K\,\cos s + \cos i\,\sin K] - 2\,n^2\,C\,[\alpha\,\cos K + \beta\,(1+\cos K)\,\cos(2\,\sigma)] \tag{8.40}$$

Equations 8.36 to 8.39 posses an equilibrium solution ($\dot{\sigma} = \dot{s} = \dot{P} = \dot{S} = 0$), for any value of i and K (see however below), that is given by

$$\sigma = k\,\frac{\pi}{2} \tag{8.41}$$
$$s = k'\,\pi \tag{8.42}$$
$$\frac{\partial\,\mathcal{H}}{\partial\cos K} = 0 \tag{8.43}$$
$$P = n\,C + \Omega\,C\,\cos\,(K - i) \tag{8.44}$$

where k and k' are integer numbers. These conditions define the so-called Cassini states. For the Moon, $k = 0$ and $k' = 1$. The condition on σ implies that the Moon's axis with the largest inertia points towards the Earth. The meaning of $s = \pi$ is illustrated in Fig. 8.4.

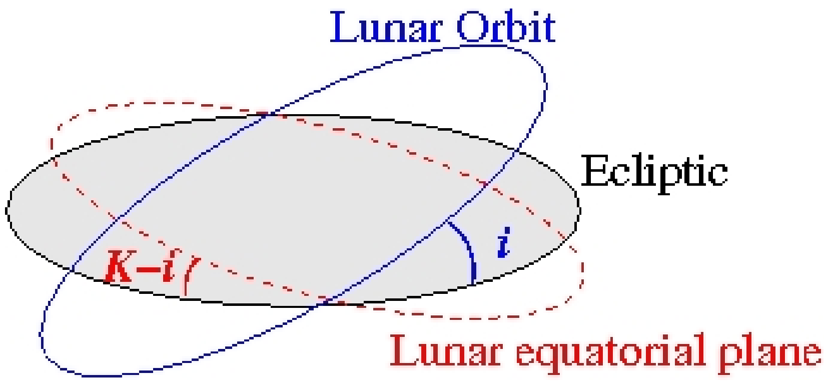


Figure 8.4: In the case of the Moon, the line of nodes of the Moon's orbit with respect to the ecliptic is aligned with the intersection between the Moon's equator and the ecliptic. This is indicated by the Cassini condition $s = \pi$.

The condition on $\frac{\partial\,\mathcal{H}}{\partial\cos K}$ actually yields a relation between i and K:

$$\frac{P\,\Omega}{\sin K}\,[\sin\,(i - K)] = 2\,n^2\,C\,[\alpha\,\cos K + \beta\,(1 + \cos K)]$$

Hence, in the Cassini states, K and i are not independent. For low values of i and K, this relation becomes

$$i = [1 + \frac{2\,n^2\,C}{P\,\Omega}(\alpha + 2\,\beta)]\,K$$

When some of the simplifying assumptions made in this chapter (circular orbit for m, no other body affecting the motion,...) are dropped, the Moon is strictly speaking no longer in a Cassini state. However, it remains actually close to such a state and presents small amplitude librations around the Cassini state.

8.5 Tides

As we have seen throughout this course, the gravitational interaction of two massive celestial bodies is usually more complex than the Keplerian problem. One of the consequences of the finite extension of these masses is their tidal interaction. Tides are a universal phenomenon. They concern of course the well known phenomenon of lunar tides on Earth, but they are also acting upon other objects in the Solar System, such as the Galilean satellites of Jupiter. Tides are also present in binary stellar systems, exoplanetary systems, and even when stars orbit supermassive black holes. In this section, we provide a basic description of the phenomenon allowing to understand some of its fundamental features.

Let us consider two masses m and m' that orbit their common center of mass C on circular orbits. We are interested in the tidal deformation of mass m and we assume that m' can be approximated as a point-like mass. For the time being, we assume that m does not rotate. The latter assumption is illustrated in the left panel of Fig. 8.5, in the frame of reference of the center of mass C.

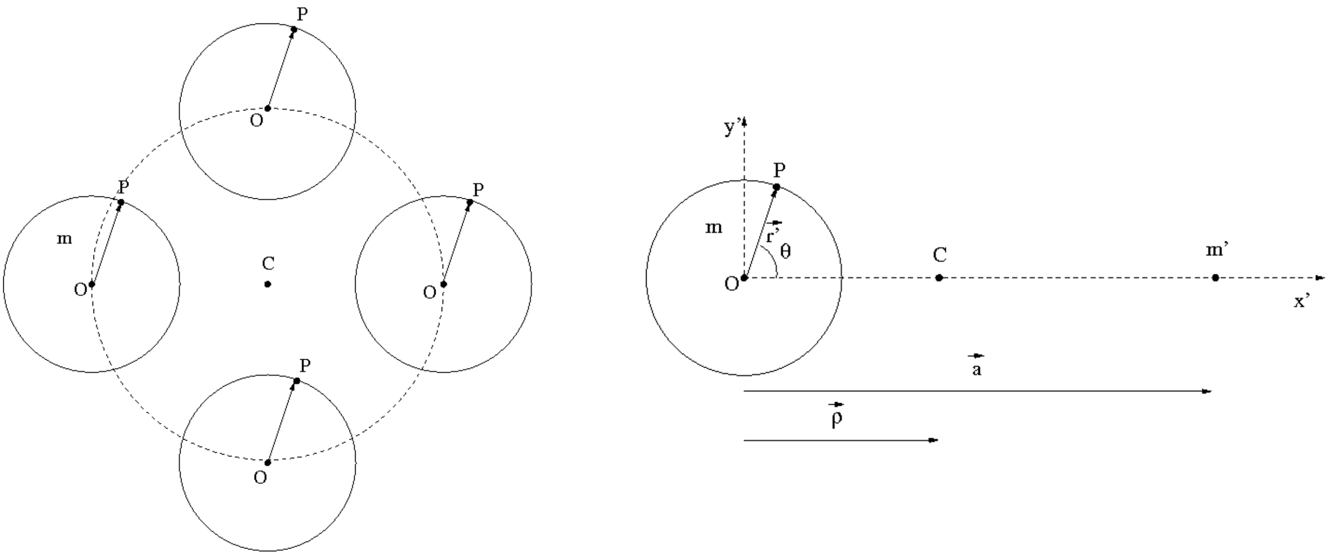


Figure 8.5: Left: orientation of the non-rotating mass m in an inertial frame of reference with respect to the center of mass C. Right: illustration of the vectors considered in the derivation of the tidal potential.

In the (x', y', z') frame of reference attached to m, a point P at the surface of m is stationary. It's position, in an inertial frame of reference is given by

$$\vec{CP} = \vec{r} = \vec{CO} + \vec{OP} = \vec{r'} - \vec{\rho}$$

where $\vec{\rho} = \frac{m'}{m+m'}\,\vec{a}$

In the inertial frame of reference, $\vec{OP} = \vec{r'}$ is a constant vector (see Fig. 8.5). This implies

$$\dot{\vec{r}} = -\vec{\omega} \wedge \vec{\rho}$$

and

$$\ddot{\vec{r}} = -\vec{\omega} \wedge (\vec{\omega} \wedge \vec{\rho}) = \omega^2\,\vec{\rho}$$

where $\vec{\omega} = \sqrt{\frac{G\,(m+m')}{a^3}}\vec{e_{z'}}$. Newton's equation implies

$$\ddot{\vec{r}} = \omega^2\,\vec{\rho} = \frac{G\,m'}{|\vec{a}-\vec{r'}|^3}\,(\vec{a}-\vec{r'}) = -\vec{\nabla}\left(\frac{-G\,m'}{|\vec{a}-\vec{r'}|}\right) \tag{8.45}$$

We note that

$$\vec{\rho} = \rho\,\vec{e_{x'}} = \rho\,\vec{\nabla}(x') = \rho\,\vec{\nabla}(r'\,\cos\theta)$$

and

$$\frac{G\,m'}{|\vec{a}-\vec{r'}|} \simeq \frac{G\,m'}{a}\left(1 + \frac{r'}{a}\,P_1(\cos\theta) + \frac{r'^2}{a^2}\,P_2(\cos\theta)\right)$$

The latter relation is valid to second order in r'/a. Let us recall that $P_1(\cos\theta) = \cos\theta$ and $P_2(\cos\theta) = \frac{1}{2}\,(3\,\cos^2\theta - 1)$. Equation 8.45 can thus be written

$$\vec{\nabla} U = \vec{0} \tag{8.46}$$

with

$$\begin{aligned} U &= \omega^2\,\rho\,r'\,\cos\theta - \frac{G\,m'}{a}\left(1 + \frac{r'}{a}\,P_1(\cos\theta) + \frac{r'^2}{a^2}\,P_2(\cos\theta)\right) \\ &= \frac{G\,(m+m')}{a^3}\,\frac{m'}{m+m'}\,a\,r'\,\cos\theta - \frac{G\,m'}{a} - \frac{G\,m'}{a^2}\,r'\,\cos\theta - \frac{G\,m'\,r'^2}{a^3}\,P_2(\cos\theta) \\ &= -\frac{G\,m'}{a}\left(1 + \frac{r'^2}{a^2}\,P_2(\cos\theta)\right) \end{aligned} \tag{8.47}$$

This tidal potential leads to an elongation of the body of mass m along the x' axis which is called tidal elongation. We stress that the part of the potential that varies with the position of the point P has an a^{-3} dependence. Therefore, the tidal force has an a^{-3} dependence.
The radius of m now depends on θ and can be described as a spheroid around the x'-axis (see equation 5.37):

$$R(\theta) = R\,\left(1 - \frac{2}{3}\,\epsilon\,P_2(\cos\theta)\right) \tag{8.48}$$

ϵ is called the ellipticity. For a spheroid, the radius at $\theta = 0$ or $\theta = \pi$ is $(1 - 2\,\epsilon/3)\,R$, whilst the radius at $\theta = \pm\frac{\pi}{2}$ is $(1 + \epsilon/3)\,R$, such that

$$\epsilon = \frac{R(\frac{\pi}{2}) - R(0)}{R}$$

If $\epsilon > 0$, then the spheroid is said to be oblate, whilst it is prolate for $\epsilon < 0$.

The actual deformation depends upon the elasticity of the body of mass m. In the case of the Earth, the rigidity of the constituent material along with the above theory allow to estimate that the Moon should produce a tidal elongation of the Earth of 31 cm, whereas the true value is close to 35 cm. The tidal pull of the Moon results in a prolate tidal bulge with $\epsilon = -4.8 \times 10^{-8}$ (Fitzpatrick 2012). The elongation due to the Sun is about half that of the Moon. Whilst the Earth's mantle responds elastically to the tidal potential, the oceans at the Earth surface deform like a liquid and are thus subject to larger tidal elongations than the underlying land. The ocean's tidal bulge remains about stationary with respect to the Moon. As the Earth rotates underneath, the level of the ocean at a specific location rises and falls down with a period near 12h25. At a given location, the amplitude of the tides depends on the shape of the shoreline and the topology of the ocean's basin.
The largest amplitudes are obtained when the Moon and the Sun lie approximately along a straight line which happens at new or full moon. This situation produces so-called spring tides. When the Moon and the Sun are located at 90° of each other, one has low amplitude tides which are called neap tides.

Let us now consider the effect of the rotation of mass m. Because of internal friction in the Earth crust and friction between the oceans and the land, there is a time lag of about 12 min between the time when the Moon passes overhead and the maximum of the tidal elongation. Because the Earth's rotational velocity Ω is larger than ω, the tidal elongation is carried ahead of the Earth-Moon axis by an angle δ (see Fig. 8.6).
The potential created by the Earth and accounting for the tidal bulge can then be expressed as (see equation 5.40)

$$U' = -\frac{G\,m}{a}\left(1 - \frac{2\,\epsilon\,R^2}{5\,a^2}\,P_2(\cos(\theta - \delta))\right) \tag{8.49}$$

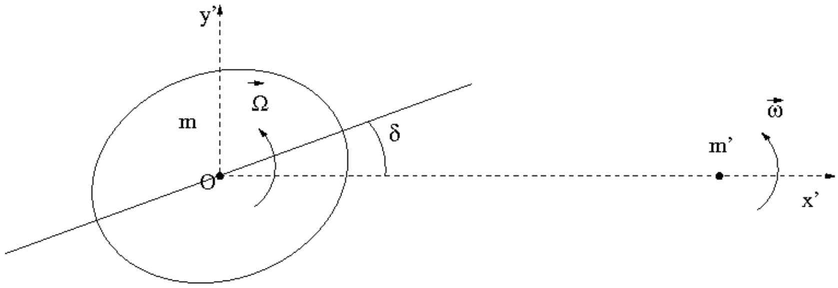


Figure 8.6: The tidal bulge lacks behind the Moon's position by an angle δ.

The torque about the Earth's center that the Moon experiences because of this gravitational field is given by

$$\vec{\mathcal{M}} = -m'\,\vec{a} \wedge \vec{\nabla}(U') = -m'\,a\,\vec{e_{x'}} \wedge \frac{1}{a}\,\frac{\partial U'}{\partial \theta}\Big)_{\theta=0}\,\vec{e_\theta} \tag{8.50}$$

$$= -G\,m\,m'\,\frac{6\,\epsilon}{5}\,\frac{R^2}{a^3}\,\cos\delta\,\sin\delta\,\vec{e_{z'}}$$

$$\simeq -G\,m\,m'\,\frac{6\,\epsilon}{5}\,\frac{R^2}{a^3}\,\delta\,\vec{e_{z'}} \tag{8.51}$$

Considering the Moon as a point mass on a circular orbit about the Earth, the orbital angular momentum is $m'a^2\,\vec{\omega}$. Since $\epsilon < 0$ and the lag $\delta > 0$, the net effect is a positive torque that increases the Moon's orbital angular momentum. As $\vec{\omega} = \sqrt{\frac{G\,(m+m')}{a^3}}\vec{e_{z'}}$ the increase in angular momentum translates into an increase of the semi-major axis of the Moon's orbit as $\frac{\dot{a}}{a} \simeq 4.3 \times 10^{-18}\,\mathrm{s}^{-1}$. Hence the Moon is slowly receding from the Earth.

We have seen above that as a result of the time lag, the torque increases the Moon's orbital angular momentum. However, considering the Earth-Moon system as an isolated system, subject to no external forces, the total angular momentum (orbital + rotational) is conserved. This then implies an equal and opposite torque acting upon the Earth and decreasing the Earth's rotational angular momentum via the relation

$$\mathcal{I}_{z'z'}\,\dot{\Omega} = -\mathcal{M}$$

Applying this relation to the case of the Earth-Moon system, one finds that $\frac{\dot{\Omega}}{\Omega} \simeq -8.7 \times 10^{-18}\,\mathrm{s}^{-1}$ (Fitzpatrick 2012). The length of the day should thus increase by 2.3×10^{-3} s per century. The observed value is somewhat lower (1.7×10^{-3} s per century).

8.6 Spin - orbit coupling

An important question related to the rotation of celestial bodies concerns the coupling between the rotational and orbital angular momentum. We have already briefly touched upon this issue in our discussion of the Cassini states (Sect. 8.4) and the tidal interactions (Sect. 8.5). Here, we will investigate under what circumstances this kind of coupling can lead to resonances.

Let us thus consider an aspherical body that orbits about a much more massive spherical object on a Keplerian orbit of semi-major axis a and eccentricity $e << 1$. This situation can describe a planet orbiting a star, a moon orbiting a planet,... Hereafter, we will call the mass of the more massive body m_0 and that of the less massive one m_1, with $m_0 >> m_1$. We express the position of the center of mass of m_0 using the principal axes of inertia of m_1 which rotate with m_1: $\vec{r} = (x, y, z)$ with $(\mathcal{I}_{xx} = A) \leq (\mathcal{I}_{yy} = B) \leq (\mathcal{I}_{zz} = C)$.

In Sect. 5.5, we have seen that the gravitational potential outside an aspherical mass can be expressed in terms of its moments of inertia. In the principal axes of inertia, equation 5.26 can be reformulated as MacCullagh's formula:

$$U = -\frac{G\,m_1}{r} - \frac{G\,(A+B+C)}{2\,r^3} + \frac{3\,G\,(A\,x^2 + B\,y^2 + C\,z^2)}{2\,r^5} \tag{8.52}$$

The force that m_1 exerts on m_0 is then obtained from

$$\vec{F} = -m_0\,\vec{\nabla} U$$

and the corresponding torque, exerted by m_1 upon m_0 is

$$\vec{\mathcal{M}} = \vec{r} \wedge \vec{F}$$

We assume that the system is isolated and thus its total angular momentum is conserved and the net torque on the system is zero. Hence, the torque exerted by m_0 upon m_1 must be equal to $-\vec{\mathcal{M}}$. To simplify the problem, let us assume that the rotation of the mass m_1 occurs around the z axis and that the orbital plane coincides with the (x, y) plane. Under these assumptions, we have $\vec{\omega} = (0, 0, \dot{\phi}) = \dot{\phi}\,\vec{e_z}$ where ϕ is the rotation angle around $\vec{e_z}$. Euler's third equation of rotational motion (equation 8.6) hence becomes

$$C\,\ddot{\phi} = -\mathcal{M}_z = \frac{3\,G\,m_0\,(B-A)\,x\,y}{r^5} \tag{8.53}$$

Let us call ξ the true anomaly of m_1 on its orbit about m_0. In the principal axes of inertia, we have $\vec{r} = r\,(\cos(\xi-\phi), \sin(\xi-\phi), 0)$. Thus equation 8.53 becomes

$$\begin{aligned} \ddot{\phi} &= \frac{3\,G\,m_0}{2\,a^3}\,\frac{B-A}{C}\left[\frac{a^3}{r^3}\,\sin(2\,\xi)\,\cos(2\,\phi) - \frac{a^3}{r^3}\,\cos(2\,\xi)\,\sin(2\,\phi)\right] \\ &= 8\,\beta\,n^2\left[\frac{a^3}{r^3}\,\sin(2\,\xi)\,\cos(2\,\phi) - \frac{a^3}{r^3}\,\cos(2\,\xi)\,\sin(2\,\phi)\right] \end{aligned} \tag{8.54}$$

with $n = \sqrt{\frac{G\,(m_0+m_1)}{a^3}}$, the mean orbital motion, and $\beta = \frac{3}{16}\,\frac{m_0}{m_0+m_1}\,\frac{B-A}{C}$ as defined previously.
According to equations 6.90 and 6.91, we can develop $\frac{a^3}{r^3}\,\cos(2\,\xi)$ and $\frac{a^3}{r^3}\,\sin(2\,\xi)$ to first order in e as

$$\begin{aligned} \left(\frac{a}{r}\right)^3 \cos(2\,\xi) &= -\frac{e}{2}\,\cos M + \cos(2\,M) + \frac{7\,e}{2}\,\cos(3\,M) + \mathcal{O}(e^2) \\ \left(\frac{a}{r}\right)^3 \sin(2\,\xi) &= -\frac{e}{2}\,\sin M + \sin(2\,M) + \frac{7\,e}{2}\,\sin(3\,M) + \mathcal{O}(e^2) \end{aligned}$$

where $M = n\,t$ is the mean anomaly. We can then insert these results into equation 8.54, which leads to

$$\ddot{\phi} = 8\,\beta\,n^2\left[\frac{-e}{2}\,\sin(n\,t - 2\,\phi) + \sin(2\,n\,t - 2\,\phi) + \frac{7\,e}{2}\,\sin(3\,n\,t - 2\,\phi)\right] \tag{8.55}$$

Equation 8.55 is highly non-linear and has no analytical solution. This equation can be solved numerically though. An interesting way to represent the result of this numerical resolution is to use a plot in the plane $(n^{-1}\,\dot{\phi}, \phi)$ and to represent the system in this plane each time we have $M = n\,t = k\,2\,\pi$ where k is an integer, i.e. each time m_1 crosses its pericenter. In this way, one obtains two different types of behaviours: either the curves that represent the evolution of the spin are open and extend over the full range of ϕ (from $-\pi$ to π), or they are closed and confined to a limited range of ϕ. In the former case, there is no correlation between the spin and the orbital motion. In

the latter case, the curves surround loci where an integer number (k_s) times the spin angular velocity $\dot{\phi}$ is equal to another integer number (k_o) times the mean orbital motion n. In these cases, there is a resonance between spin and orbital motion. Equation 8.55 can generate three different resonances for k_o/k_s = 1/2, 1/1 or 3/2.
To illustrate the most important features of these resonances, let us introduce the quantity $\eta = \phi - q\,n\,t$ where $q = k_o/k_s$ is a constant equal to 0.5, 1 or 1.5. Inserting this into equation 8.55, we find

$$\begin{aligned}\ddot{\eta} &= -8\,\beta\,n^2 \left\{ \left[\frac{-e}{2} \cos\left[(2\,q-1)\,n\,t\right] + \cos\left[(2\,q-2)\,n\,t\right] + \frac{7\,e}{2} \cos\left[(2\,q-3)\,n\,t\right] \right] \sin(2\,\eta) \right. \\ &+ \left. \left[\frac{-e}{2} \sin\left[(2\,q-1)\,n\,t\right] + \sin\left[(2\,q-2)\,n\,t\right] + \frac{7\,e}{2} \sin\left[(2\,q-3)\,n\,t\right] \right] \cos(2\,\eta) \right\} \qquad (8.56)\end{aligned}$$

If we average equation 8.56 over k_s orbital periods, assuming that η is small, we obtain the following results:

$$q = 0.5 \quad \Rightarrow \quad \ddot{\eta} = 4\,\beta\,e\,n^2 \sin(2\,\eta) \qquad (8.57)$$
$$q = 1.0 \quad \Rightarrow \quad \ddot{\eta} = -8\,\beta\,n^2 \sin(2\,\eta) \qquad (8.58)$$
$$q = 1.5 \quad \Rightarrow \quad \ddot{\eta} = -28\,\beta\,e\,n^2 \sin(2\,\eta) \qquad (8.59)$$

which, after multiplication by $\dot{\eta}$, and accounting for the fact that $\dot{\eta} = \dot{\phi} - q\,n$ can be integrated into

$$q = 0.5 \quad \Rightarrow \quad (n^{-1}\dot{\phi} - 0.5)^2 + 4\,\beta\,e \cos(2\,\phi - n\,t) = E \qquad (8.60)$$
$$q = 1.0 \quad \Rightarrow \quad (n^{-1}\dot{\phi} - 1)^2 - 8\,\beta \cos(2\,\phi - 2\,n\,t) = E \qquad (8.61)$$
$$q = 1.5 \quad \Rightarrow \quad (n^{-1}\dot{\phi} - 1.5)^2 - 28\,\beta\,e \cos(2\,\phi - 3\,n\,t) = E \qquad (8.62)$$

where E is a constant.

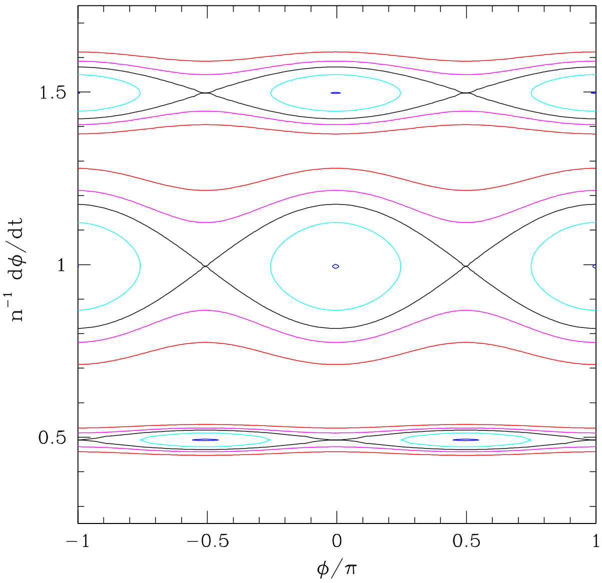


Figure 8.7: Spin - orbit resonance for $\beta = 0.002$ and $e = 0.05$. The minimum values of E (E_{min}) for the $q = 0.5$, 1.0 or 1.5 resonances are $-4\,e\,\beta$, $-8\,\beta$ and $-28\,e\,\beta$, respectively. The different contours correspond to these minimum values (blue), $E = 0.0$ (cyan), $-E_{\rm min}$ (black), $-2\,E_{min}$ (magenta) and $-4\,E_{min}$ (red).

Figure 8.7 illustrates the contours of equal E in the $(n^{-1}\,\dot{\phi}, \phi/\pi)$ plane for $M = n\,t = k\,2\,\pi$ (i.e. pericenter passage). One can clearly distinguish the closed and open contours. The Moon ($\beta \simeq 0.000425$, $e = 0.055$) constitutes an example of a $q = 1$ resonance. Due to the tidal interaction with the Earth, the Moon's rotation was slowed until it got trapped in the resonance. An example of the $q = 1.5$ resonance is provided by planet Mercury.

When the β parameter increases (i.e. for highly non-spherical bodies), the separation between the closed contours around the different resonances decreases. This situation can lead to resonance overlap that destroys the closed curves and thus prevents spin-orbit resonance.

Bibliography

[1] Batygin, K., & Brown, M.E. 2016, The Astronomical Journal, 151, 22

[2] Duriez, L. 2002, *Cours de Mécanique Céleste classique*, Lecture Notes, Université de Lille

[3] Farquhar, R.W. 1967, Journal of Spacecraft and Rockets, vol. 4, 1383

[4] Fitzpatrick, R. 2012, *An Introduction to Celestial Mechanics*, Cambridge University Press

[5] Henrard, J. 2004, *Mécanique Céleste*, Lecture Notes, Facultés Universitaires Notre Dame de la Paix, Namur

[6] Holsapple, K.A., & Michel, P. 2006, Icarus, 183, 331

[7] Kovalevsky, J. 1963, *Introduction à la Mécanique Céleste*, Collection Armand Colin, Paris Lancaster, E.R., & Blanchard, R.C. 1969, *A unified form of Lambert's Theorem*, NASA Technical Note, D-5368

[8] Lo, K.-H., Young, K., & Lee, B.Y.P. 2013, Am. J. Phys., 81, 695

[9] Nimmo, F. 2008, *Planetary Interioris*, Lecture Notes, University of California, Santa Cruz

[10] Robe, H. 1989, *Mécanique Céleste*, Lecture Notes, Université de Liège

[11] Saha, P. 1992, Icarus, 100, 434

[12] Showman, A.P. 1997, Icarus, 127, 93

[13] Siegel, C.L., & Moser, J.K. 1971, *Lectures on Celestial Mechanics*, Springer Verlag, Berlin Heidelberg New York

[14] Stevin, G. 2008, *Ciel & Terre* 124, 34

[15] Wikipedia, The Free Encyclopedia, `http://en.wikipedia.org/wiki/`

[16] Wilson, C. 2001, *Celestial Mechanics in the Eighteenth and Ninetheenth Centuries*, in *Encyclopedia of Astronomy and Astrophysics*