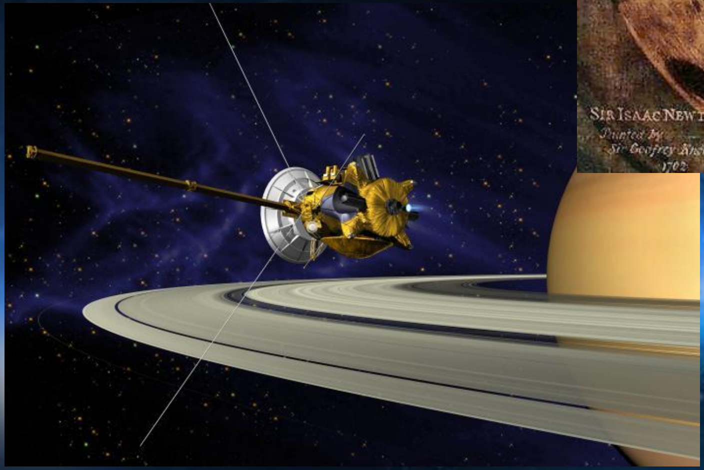
Celestial Mechanics and Space Trajectories

Master in

Space Sciences

2020 – 2021

Gregor Rauw
Professor,
AGO Department
University of Liège

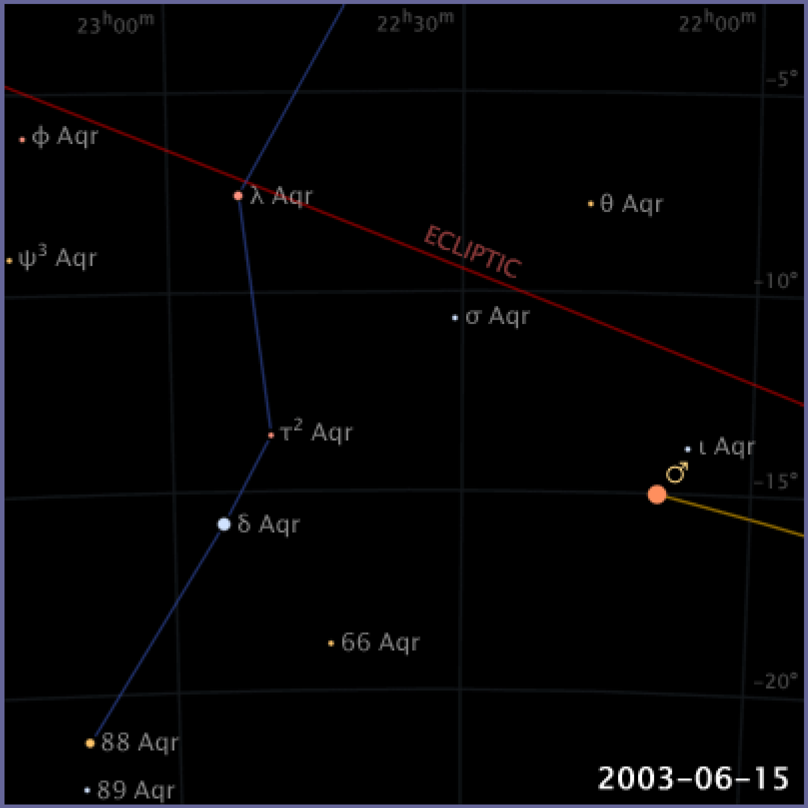


If you don't understand these cartoons, ask yourself whether you are in the right classroom!!! 🙂

Chapter I: A brief history of celestial mechanics

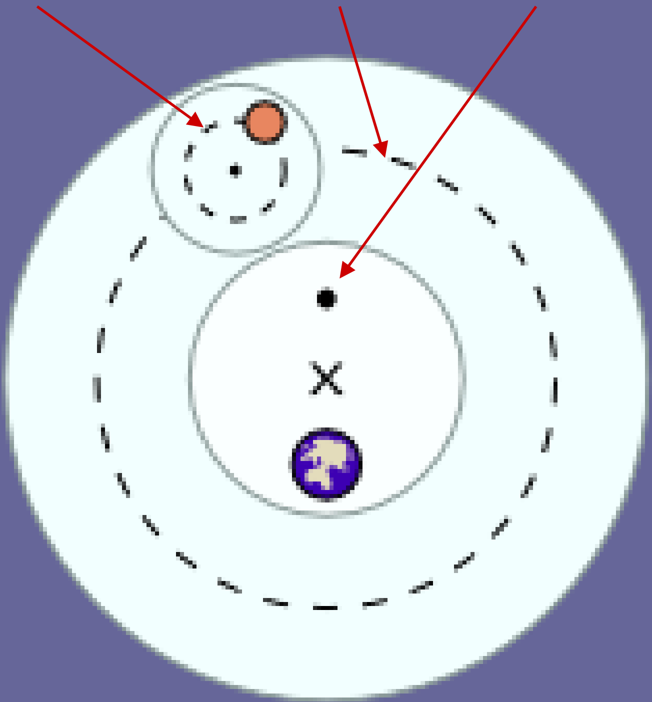
That gravity should be innate, inherent, and essential to matter, so that one body may act upon another at a distance, through a vacuum, without the mediation of anything else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man who has in philosophical matters a competent faculty of thinking, can ever fall into it.

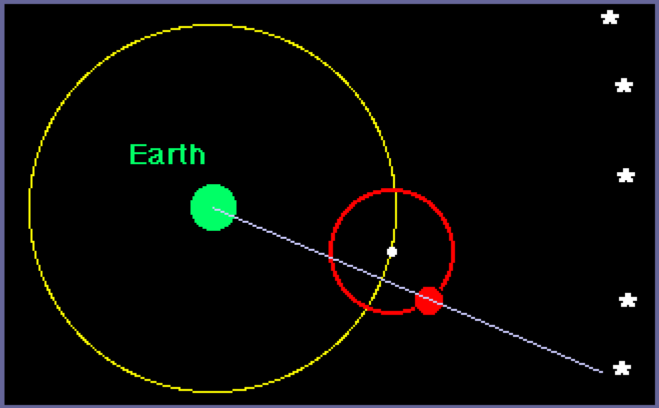
Sir Isaac Newton (Third letter to Bentley, 25 Feb 1693)



Chapter I: A brief history of celestial mechanics

- One of the main problems of ancient astronomy was to predict the position of « wandering stars » (Sun, Moon, planets).
- Until the 16th century: geocentric model described by Claudius Ptolemeus (85 – 165).
- Epicycles, deferents and equants.



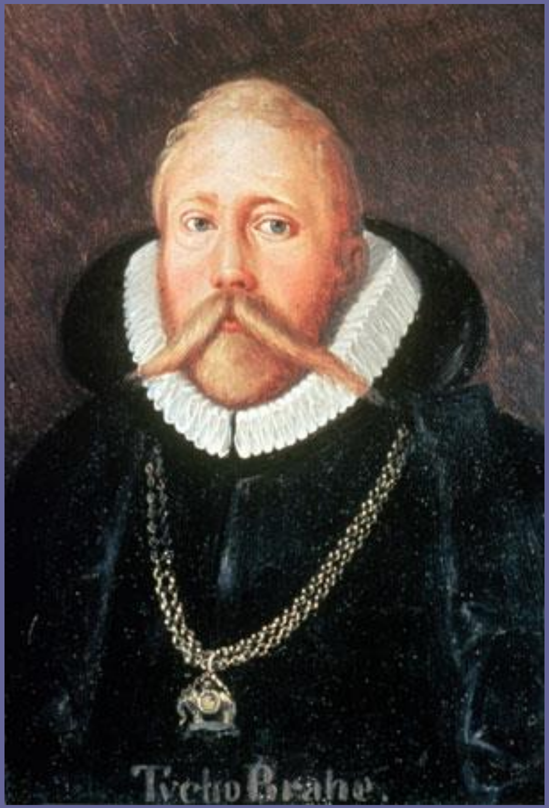


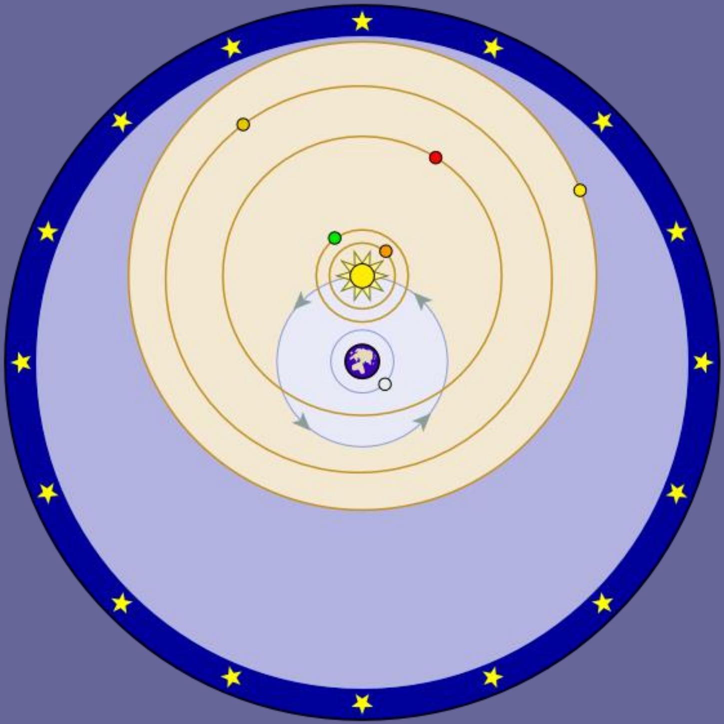
Chapter I: A brief history of celestial mechanics

- 1543: publication of *De revolutionibus orbium coelestium* by Nicolas Copernicus (1473 – 1543).
- Heliocentric model but still based on epicycles and deferents.

Chapter I: A brief history of celestial mechanics

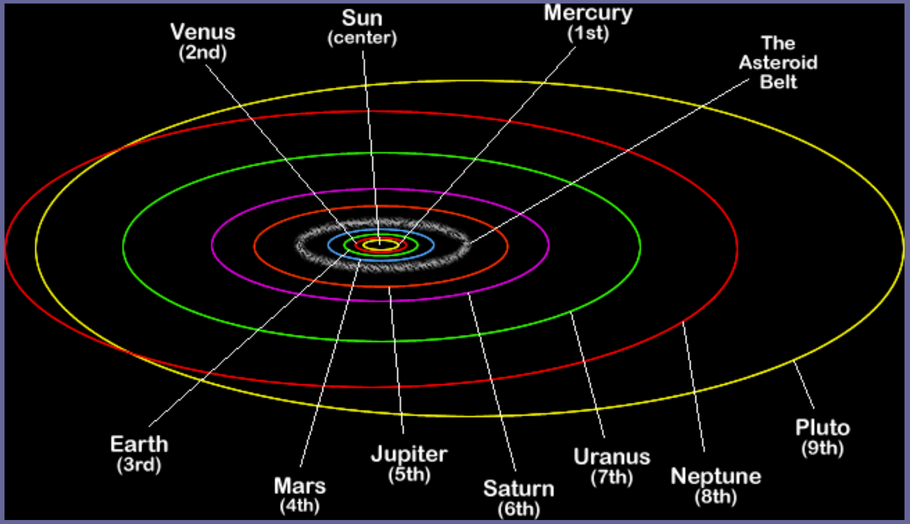
- Tycho Brahe (1546 – 1601) collected very accurate (for that time) astrometric observations and proposed an alternative geo-heliocentric model.





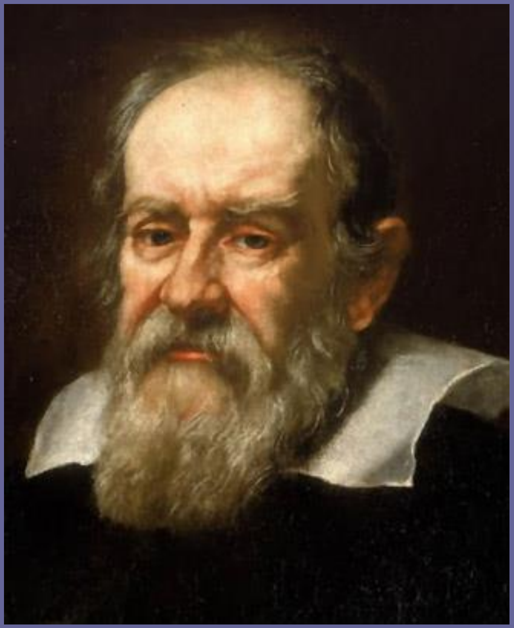
Chapter I: A brief history of celestial mechanics

- Brahe was looking for a skilled mathematician able to use and interpret his observations.
- Johannes Kepler (1571 – 1630) used Brahe's observations of planet Mars and deduced the three laws of planetary motion summarized in *Astronomia Nova* (1609) and *Harmonices mundi* (1619).
- At first these books received little attention…



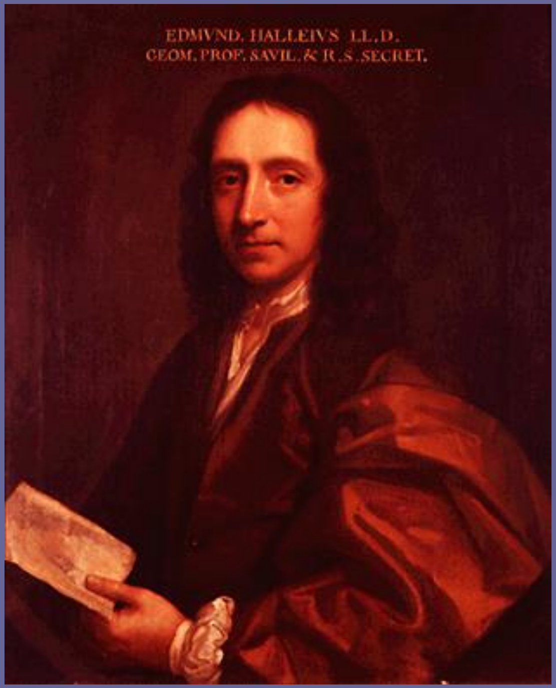
Chapter I: A brief history of celestial mechanics

- Galileo Galilei (1564 – 1642) discovered the theory of free fall.
- 1687: Isaac Newton (1643 – 1727) combined all these results in his theory of gravitation in *Philosophiae Naturalis Principia Mathematica*.



Chapter I: A brief history of celestial mechanics

- Newton's theory enabled Edmund Halley (1665 – 1742) to predict that the comet of 1682 would return in 1758.
- The comet returned, although with some delay due to the effects of Jupiter and Saturn.

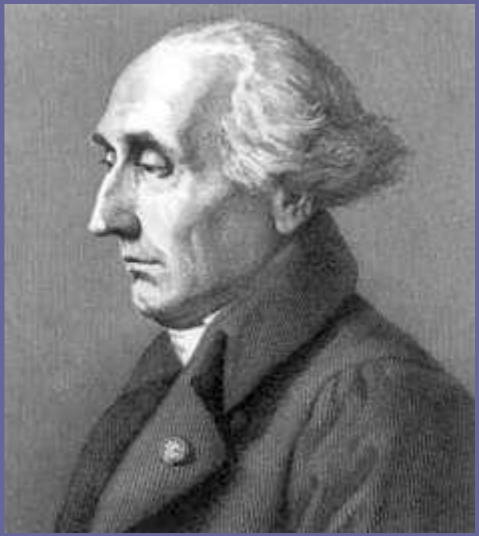


Chapter I: A brief history of celestial mechanics

- The perturbation of the comet’s motion due to Jupiter and Saturn requires to deal with the problem of more than two bodies.
- The first formulation of this problem was due to Leonhard Euler (1707 – 1783) who developed several mathematical tools for the solution of mechanical problems.
- The N-body problem became the main research topic in mechanics.

Chapter I: A brief history of celestial mechanics

- Joseph-Louis Lagrange (1736 – 1813) obtained special solutions of the 3-body problem.



- Lagrange and Pierre Simon Laplace (1749 – 1827) developed the theory of perturbed motions.

Chapter I: A brief history of celestial mechanics

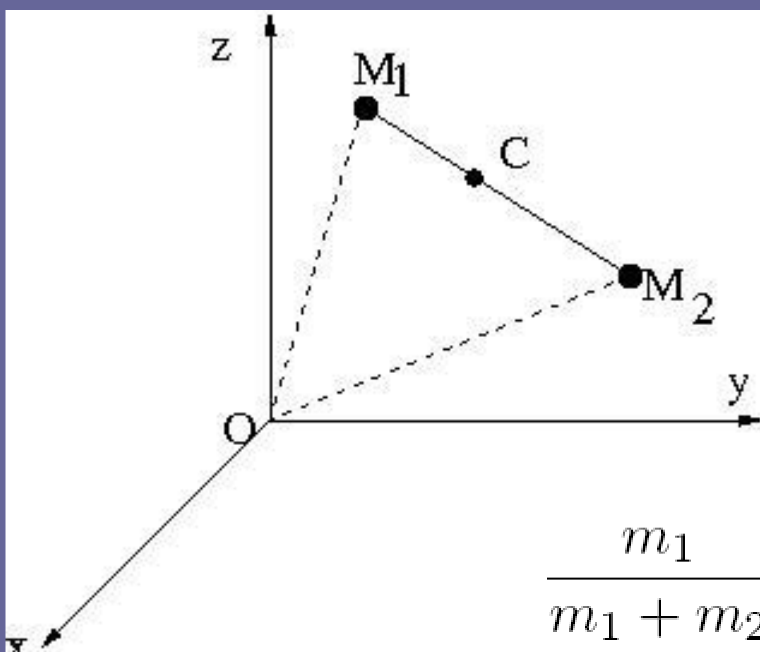
- The motion of Uranus (discovered in 1781) was perturbed by an unknown planet.
- Based on the recorded positions of Uranus, Urbain Le Verrier (1811 – 1877) predicted the position of the unknown planet (Neptune) that was discovered in September 1846 by Galle and d'Arrest at the predicted position.

Chapter I: A brief history of celestial mechanics

- And the story isn't over…
- Although General Relativity has replaced Newton's theory for the description of strong gravitational interactions, celestial mechanics remains a highly important research topic: the long-term stability of the Solar System, planetary and exo-planetary migrations, resonances, the study of rotation… are only a few examples of topics that are still at the forefront of research.

Chapter II: The two-body problem

- Inertial frame of reference: *Oxyz*
- Newton's equations ($d = |M_1M_2|$)



$$m_1\,\ddot{\vec{OM_1}} = \frac{G\,m_1\,m_2}{d^3}\,\vec{M_1M_2}$$

$$m_2\,\ddot{\vec{OM_2}} = -\frac{G\,m_1\,m_2}{d^3}\,\vec{M_1M_2}$$

$$\frac{m_1}{m_1+m_2}\,\vec{OM_1} + \frac{m_2}{m_1+m_2}\,\vec{OM_2} = \vec{OC} = \vec{a}\,t + \vec{b}$$

Chapter II: The two-body problem

- ***Inertial*** frame of reference tied to the centre of mass C:

$$\vec{r_1} = C\vec{M}_1$$

$$\vec{r_2} = C\vec{M}_2$$

$$\ddot{\vec{r_1}} = -\frac{G\, m_2^3}{(m_1 + m_2)^2}\, \frac{\vec{r_1}}{r_1^3}$$

$$\ddot{\vec{r_2}} = -\frac{G\, m_1^3}{(m_1 + m_2)^2}\, \frac{\vec{r_2}}{r_2^3}$$

- Relative motion: M_2 with respect to M_1

$$M_1\ddot{\vec{M}}_2 = -\frac{G\,(m_1 + m_2)}{d^3}\, M_1\vec{M}_2$$

- In each case:

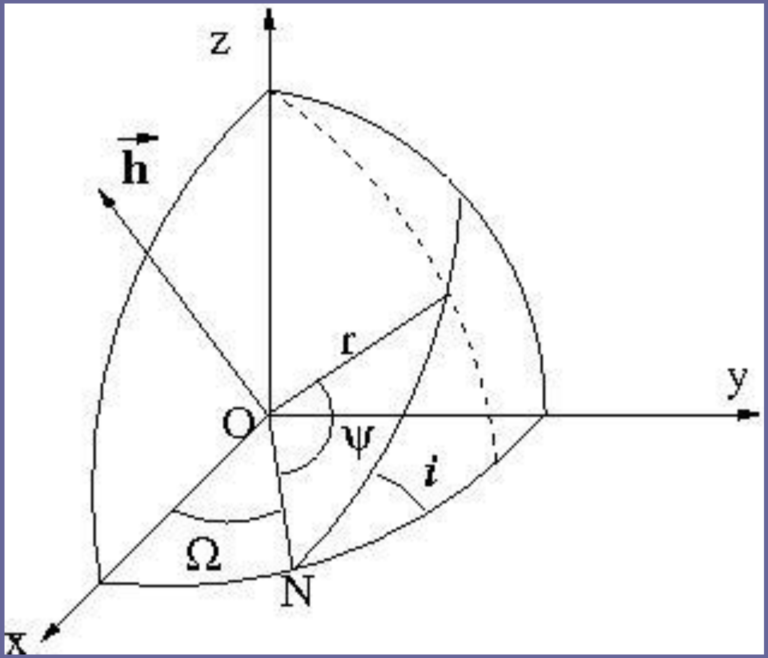
$$\ddot{\vec{r}} = -\mu\, \frac{\vec{r}}{r^3} = -\vec{\nabla}\left(-\frac{\mu}{r}\right)$$

Chapter II: The two-body problem

- Newton's equation for a unit mass under the influence of a force that derives from the -μ/r potential: $\ddot{\vec{r}} = -\mu \frac{\vec{r}}{r^3} = -\vec{\nabla}\left(-\frac{\mu}{r}\right)$

$$\Rightarrow \quad \vec{r} \wedge \ddot{\vec{r}} = \vec{0} \quad \Rightarrow \quad \vec{r} \wedge \dot{\vec{r}} = \vec{h} = \vec{Cte}$$

- Motion in a plane (perpendicular to $\vec{h}$)



Chapter II: The two-body problem

- Polar coordinates in the plane of the motion:

$$r^2\,\dot\psi = h = Cte$$

Kepler's 2nd law

$$\ddot{\vec r} = -\mu\,\frac{\vec r}{r^3} = -\vec\nabla\left(-\frac{\mu}{r}\right) \quad \Rightarrow \quad \frac{dT}{dt} = -\frac{\mu}{r^3}\,\vec r\cdot\dot{\vec r}$$

- Conservation of the total energy:

$$\frac{1}{2}(\dot r^2 + r^2\,\dot\psi^2) - \frac{\mu}{r} = \mathcal{E}$$

- Laplace-Runge-Lenz integral:

$$\frac{\dot{\vec r}\wedge\vec h}{\mu} - \vec{e_r} = \vec l \quad \Rightarrow \quad \frac{h^2}{\mu}\,\dot{\vec r} = \vec h\wedge(\vec l + \vec{e_r})$$

Chapter II: The two-body problem

- Only variable vector $\vec{e}_r$. Hodograph = locus of the end Q of the velocity vector: $$\frac{h^2}{\mu}\dot{\vec{r}} = \vec{h} \wedge (\vec{l} + \vec{e_r})$$
- For a Keplerian motion, the hodograph is a circle.
- Equation of the trajectory:

$$\frac{1}{2}(\dot{r}^2 + r^2\,\dot{\psi}^2) - \frac{\mu}{r} = \mathcal{E} \quad \Rightarrow \quad \left(\frac{d\,r}{d\psi}\right)^2 \frac{h^2}{r^4} + \frac{h^2}{r^2} - \frac{2\,\mu}{r} = 2\,\mathcal{E}$$

$$u = 1/r \Rightarrow \left(\frac{d\,u}{d\psi}\right)^2 + u^2 - \frac{2\,\mu\,u}{h^2} = \frac{2\,\mathcal{E}}{h^2}$$

$$v = u - \mu/h^2 \Rightarrow \left(\frac{d\,v}{d\psi}\right)^2 + v^2 = \frac{\mu^2}{h^4} + \frac{2\,\mathcal{E}}{h^2} = H^2 \geq 0$$

Chapter II: The two-body problem

- Equation of the trajectory: $$\left(\frac{dv}{d\psi}\right)^2 + v^2 = \frac{\mu^2}{h^4} + \frac{2\,\mathcal{E}}{h^2} = H^2 \geq 0$$

$$\Rightarrow \quad \frac{dv}{\sqrt{H^2 - v^2}} = \pm d\psi$$

$$\Rightarrow \quad r = \frac{1}{\frac{\mu}{h^2} + H\ \cos\left(\psi - \omega\right)}$$

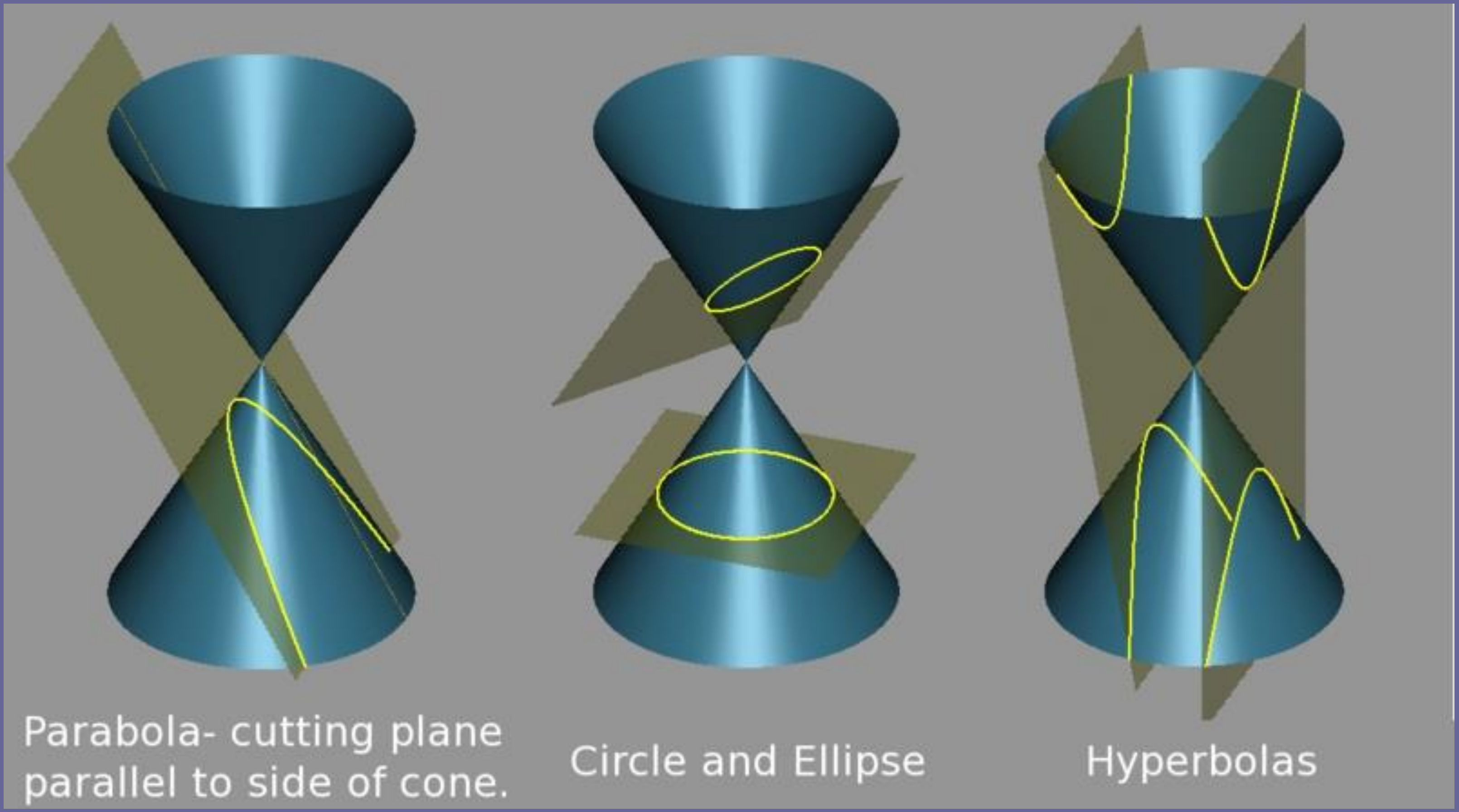
- Equation of a conic section:

$$r = \frac{C}{1 + e\ \cos\left(\psi - \omega\right)}$$

with $e = \dfrac{H\,h^2}{\mu}$ $\quad C = \dfrac{h^2}{\mu}$

Chapter II: The two-body problem

- Trajectory = conic section : Kepler's 1st law



Chapter II: The two-body problem

- Trajectory = conical section :
  - Circle if $e = 0$, $\mathcal{E} = -\mu^2/(2h^2)$
  - Ellipse if $0 < e < 1$, $-\mu^2/(2h^2) < \mathcal{E} < 0$
  - Parabola if $e = 1$, $\mathcal{E} = 0$
  - Hyperbola if e > 1, $\mathcal{E} > 0$
- Whatever the nature of the conic: $|\vec{l}| = e$

$$r = \frac{h^2}{\mu}\frac{1}{1 + e\cos\phi}$$

$\Rightarrow$ distance at pericentre: $r_P = \frac{h^2}{\mu\,(1+e)}$

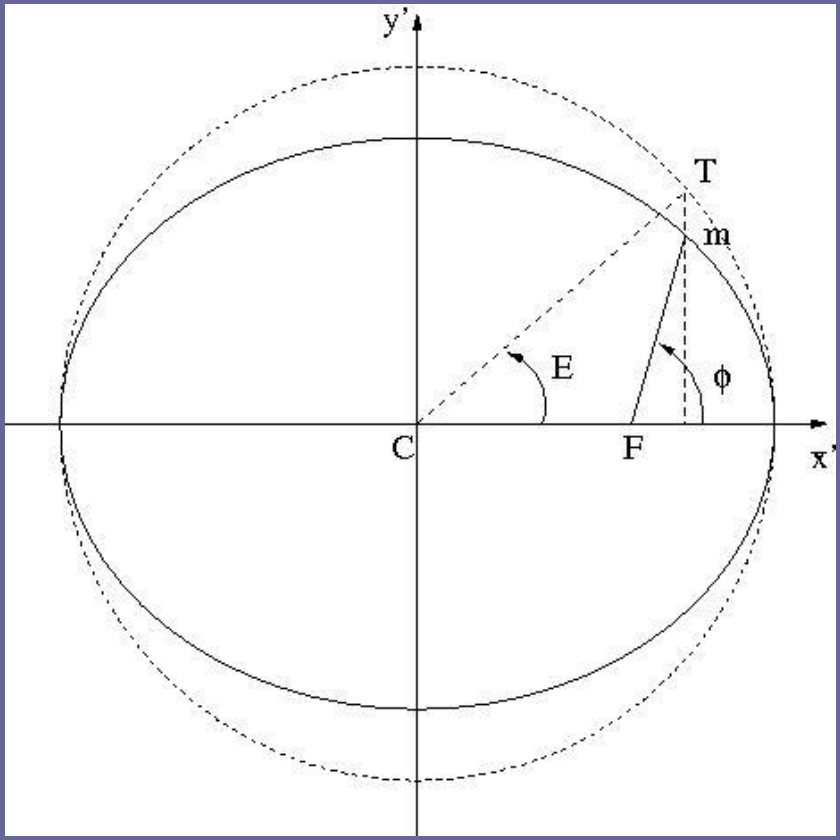
$$\Rightarrow \quad r = r_P \frac{1+e}{1 + e\cos\phi}$$

$$\mathcal{E} = \frac{1}{2}\frac{h^2}{r_P^2} - \frac{\mu}{r_P} = \frac{\mu}{2\,r_P}(e-1)$$

2.1 Case of the ellipse

- Trajectory = ellipse : $r_P = a\,(1-e)$ $\Rightarrow$ $h = \sqrt{\mu\, a\,(1-e^2)}$

$$\Rightarrow \quad r = \frac{a\,(1-e^2)}{1+e\,\cos\phi}$$



2.1 Case of the ellipse

- Trajectory = ellipse :

$h = \sqrt{\mu\, a\,(1-e^2)}$ $\Rightarrow$ $$\frac{1}{2}\left(\dot{r}^2 + \frac{\mu\, a\,(1-e^2)}{r^2}\right) - \frac{\mu}{r} = -\frac{\mu}{2\,a}$$

$$\Rightarrow \quad \left(r\,\frac{d\,r}{dt}\right)^2 = \frac{\mu}{a}\left[a^2\,e^2 - (r-a)^2\right]$$

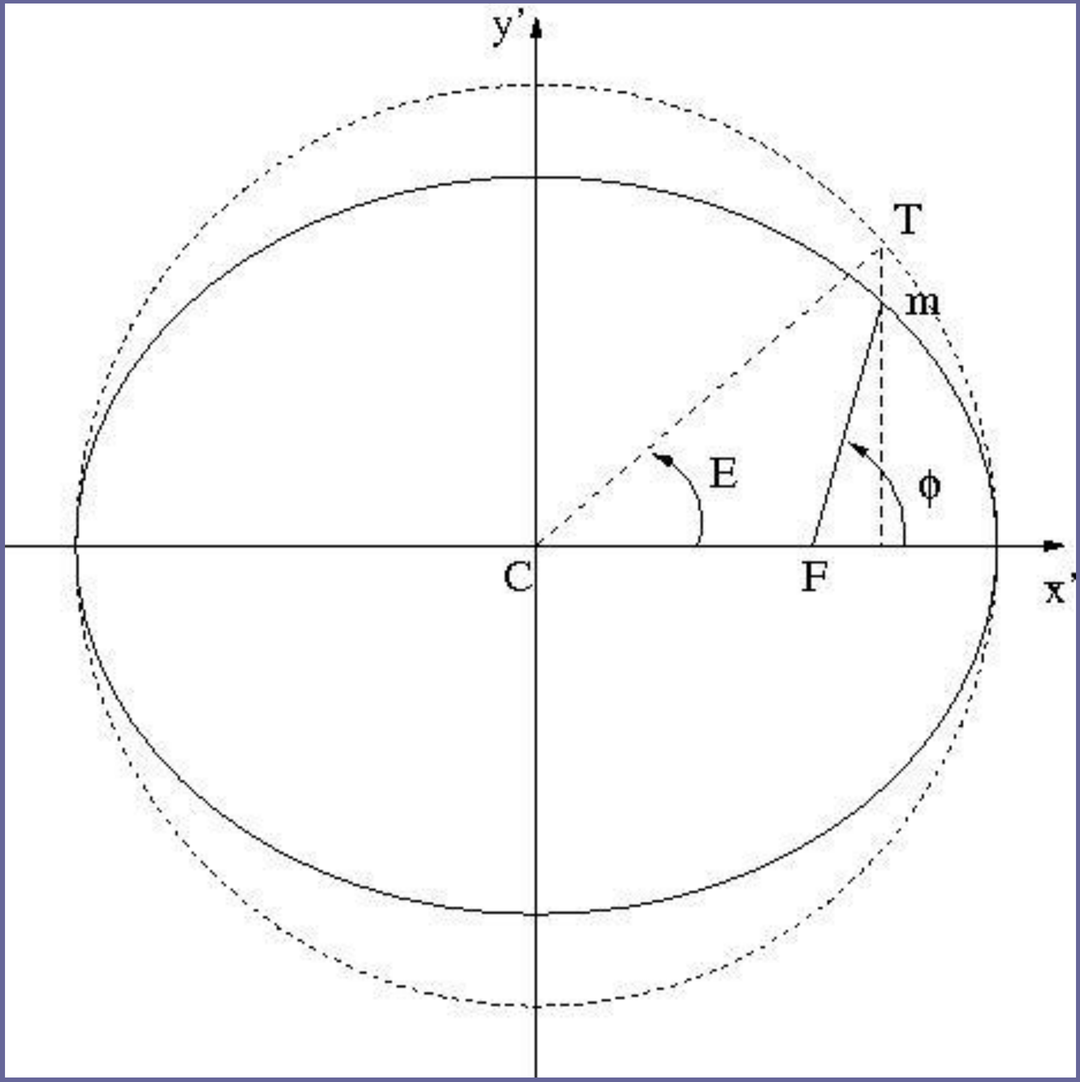
$r = a(1 - e\cos E) \Rightarrow$ Kepler's equation:

$$E - e\,\sin E = \sqrt{\frac{\mu}{a^3}}\,(t - t_0)$$

Kepler's 3rd law : $$\left(\frac{2\,\pi}{P}\right)^2 a^3 = \mu$$

2.1 Case of the ellipse

- $r = a(1 - e \cos E)$: eccentric anomaly



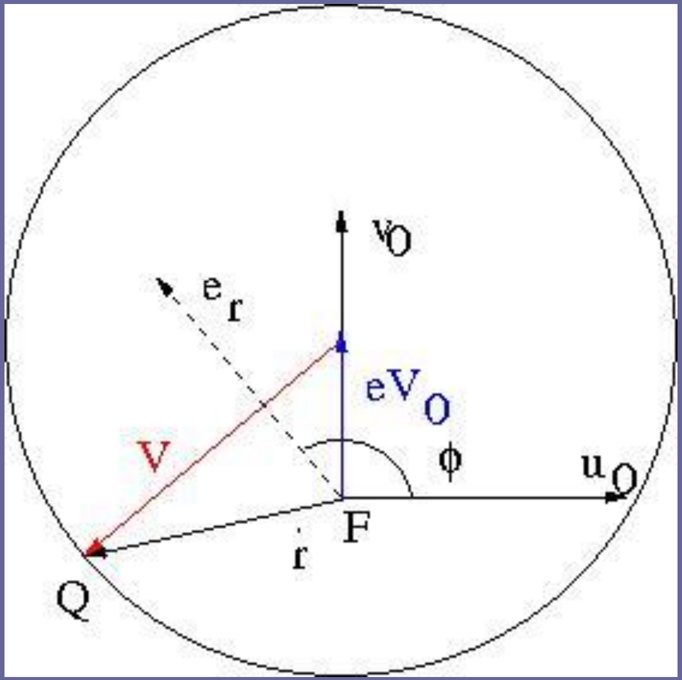
$$\cos\phi = \frac{\cos E - e}{1 - e\,\cos E}$$

$$\sin\phi = \frac{\sqrt{1-e^2}\,\sin E}{1 - e\,\cos E}$$

$$\tan\frac{\phi}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{E}{2}$$

2.1 Case of the ellipse

- Hodograph: centre of force inside the circle



$$\frac{h^2}{\mu}\dot{\vec{r}} = \vec{h} \wedge (\vec{l} + \vec{e_r})$$

2.2 Case of the parabola

- Trajectory = parabola $$r = \frac{2\,r_P}{1+\cos\phi} \quad \Rightarrow \quad h = \sqrt{2\,\mu\,r_P}$$

- Motion on the parabola:

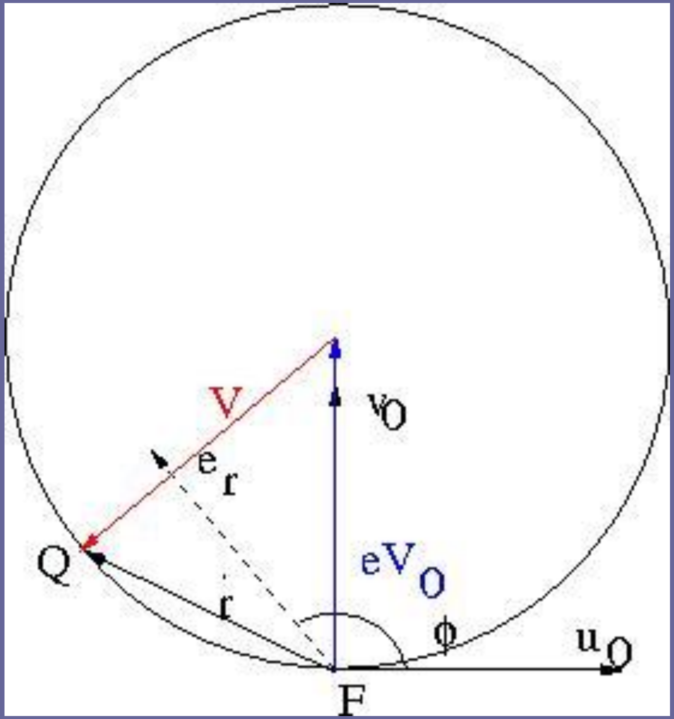
$$\dot{r}^2 + r^2\,\dot{\phi}^2 = \frac{2\,\mu}{r}$$

$$\Rightarrow \quad r^2\,\dot{r}^2 - 2\,\mu\,r + 2\,\mu\,r_P = 0$$

$$\Rightarrow \quad s + \frac{s^3}{3} = \sqrt{\frac{\mu}{2\,r_P^3}}\,(t - t_0)$$

2.2 Case of the parabola

- Hodograph: centre of force located on the circle (φ between 0 and π)



$$\frac{h^2}{\mu}\dot{\vec{r}} = \vec{h} \wedge (\vec{l} + \vec{e_r})$$

2.3 Case of the hyperbola

- Trajectory = hyperbola $r_P = a\,(e-1)$ $\Rightarrow$ $r = \dfrac{a\,(e^2-1)}{1+e\,\cos\phi}$

$$h = \sqrt{\mu\, a\,(e^2-1)}$$

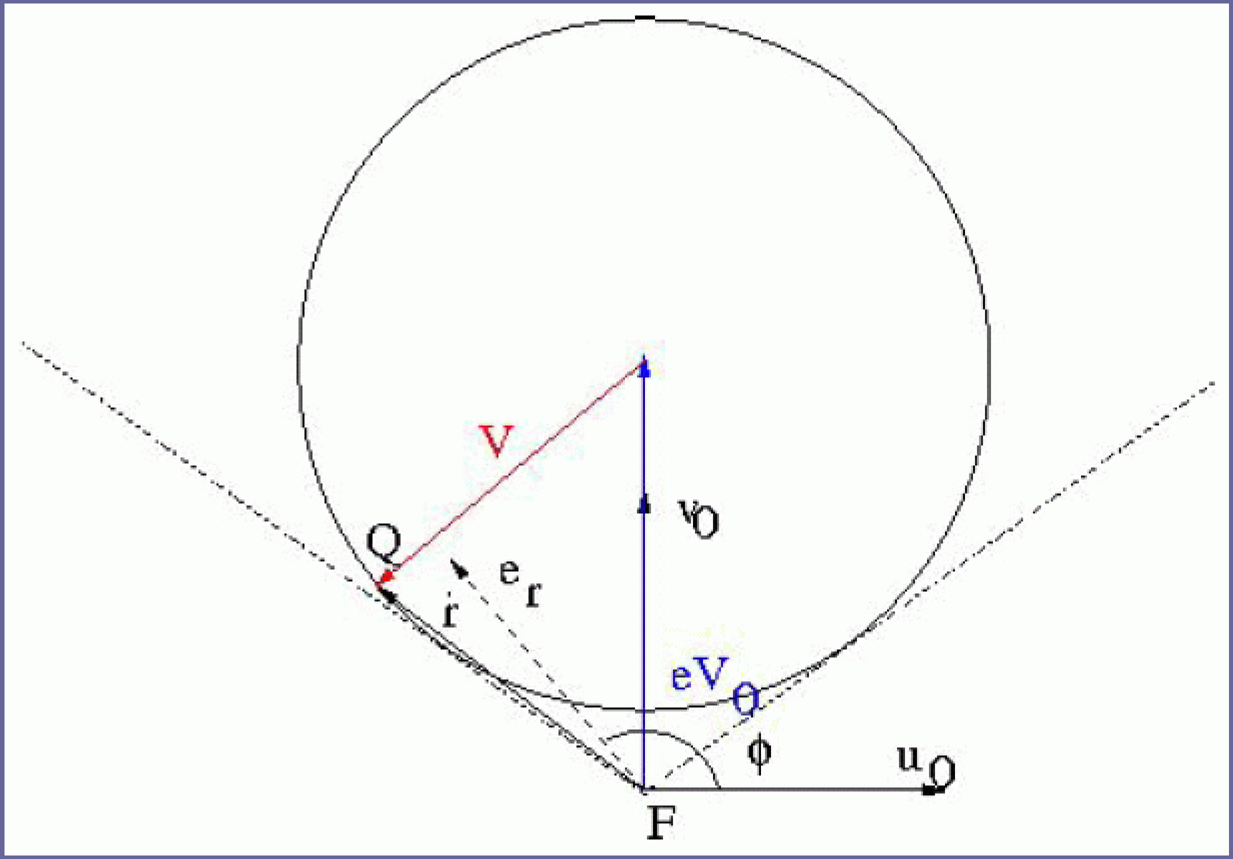
- Motion on the trajectory: $\dot{r}^2 + r^2\,\dot{\phi}^2 = \dfrac{2\,\mu}{r} + \dfrac{\mu}{a}$

$$\Rightarrow \quad r^2\,\dot{r}^2 = \frac{\mu}{a}\,(r^2 + 2\,r\,a + a^2) - \mu\,a\,e^2$$

ch F = (r + a)/(ae) $\Rightarrow$ $e\,\sinh F - F = \sqrt{\dfrac{\mu}{a^3}}\,(t-t_0)$

2.3 Case of the hyperbola

- Hodograph: centre of force located outside the circle
- φ inside $]\arccos -1/e, 2\pi - \arccos -1/e[$



$$\frac{h^2}{\mu}\dot{\vec{r}} = \vec{h} \wedge (\vec{l} + \vec{e_r})$$

2.4 Elements of the orbit

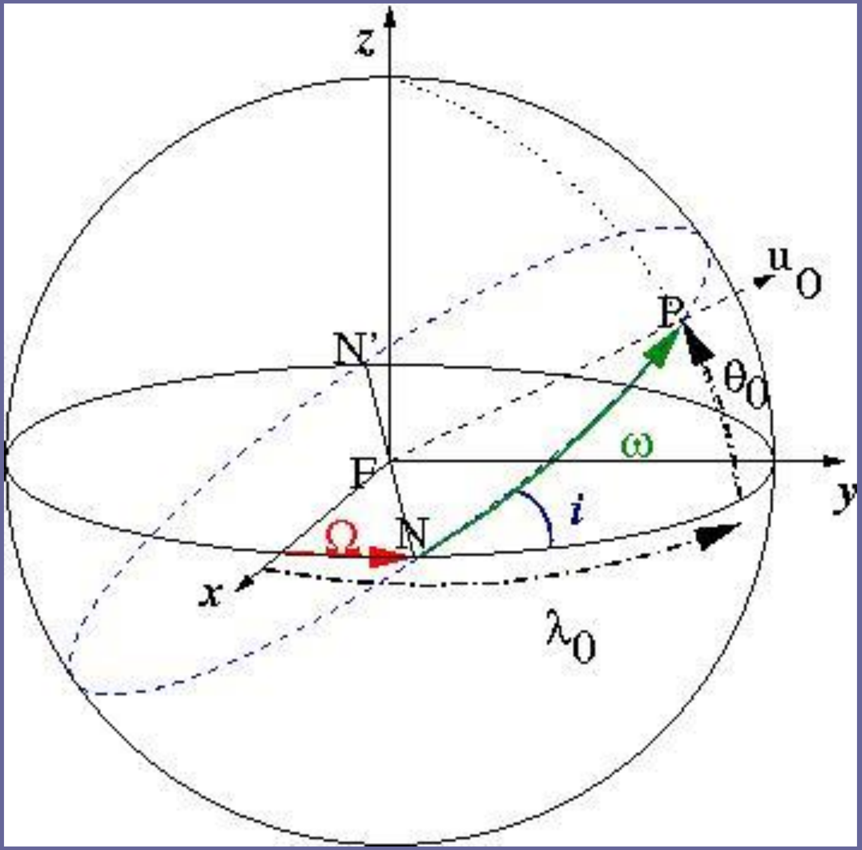
- Solution can be described by 6 + 1 parameters:

$$(\vec{h}, \vec{l}, t_0, \mu)$$

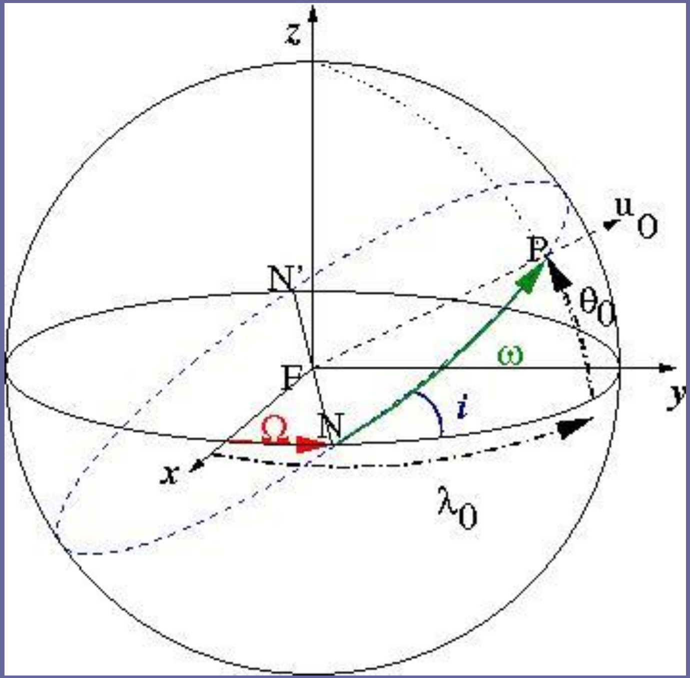
with $\vec{h} \cdot \vec{l} = 0$

- Equivalent to the elements

$$(\Omega, i, \omega, a, e, t_0, \mu)$$



2.4 Elements of the orbit



$$
\begin{aligned}
x &= r\left[\cos\Omega\,\cos(\omega+\phi) - \sin\Omega\,\sin(\omega+\phi)\,\cos i\right] \\
y &= r\left[\sin\Omega\,\cos(\omega+\phi) + \cos\Omega\,\sin(\omega+\phi)\,\cos i\right] \\
z &= r\left[\sin i\,\sin(\omega+\phi)\right]
\end{aligned}
$$

$$
\begin{aligned}
\dot{x} &= \dot{r}\left[\cos\Omega\,\cos(\omega+\phi) - \sin\Omega\,\sin(\omega+\phi)\,\cos i\right] - \frac{h}{r}\left[\cos\Omega\,\sin(\omega+\phi) + \sin\Omega\,\cos(\omega+\phi)\,\cos i\right] \\
\dot{y} &= \dot{r}\left[\sin\Omega\,\cos(\omega+\phi) + \cos\Omega\,\sin(\omega+\phi)\,\cos i\right] - \frac{h}{r}\left[\sin\Omega\,\sin(\omega+\phi) - \cos\Omega\,\cos(\omega+\phi)\,\cos i\right] \\
\dot{z} &= \dot{r}\left[\sin i\,\sin(\omega+\phi)\right] + \frac{h}{r}\left[\sin i\,\cos(\omega+\phi)\right]
\end{aligned}
$$

2.4 Elements of the orbit

- Planets of the Solar System:

Planet	a (AU)	e	i (°)	Ω (°)	ϖ (°)	L_0 (°)	n (arcsec day^{-1})
Mercury	0.3871	0.2056	7.00	48.33	77.46	252.25	14732.42
Venus	0.7233	0.0068	3.39	76.68	131.56	181.98	5767.67
Earth	1.0000	0.0167	–	–	102.94	100.47	3548.19
Mars	1.5237	0.0934	1.85	49.56	336.06	355.43	1886.52
Jupiter	5.2028	0.0485	1.30	100.46	14.33	34.35	299.128
Saturn	9.5388	0.0555	2.49	113.66	93.06	50.08	120.455
Uranus	19.182	0.0463	0.77	74.01	173.00	314.05	42.231
Neptune	30.058	0.0090	1.77	131.78	48.12	304.39	21.534
Pluto	39.44	0.2485	17.33	110.7	224.6	237.7	14.3

$$\varpi = \Omega + \omega$$

$$L_0 = \varpi + n\,(t_0' - t_0)$$

$$L = L_0 + n\,(t - t_0')$$

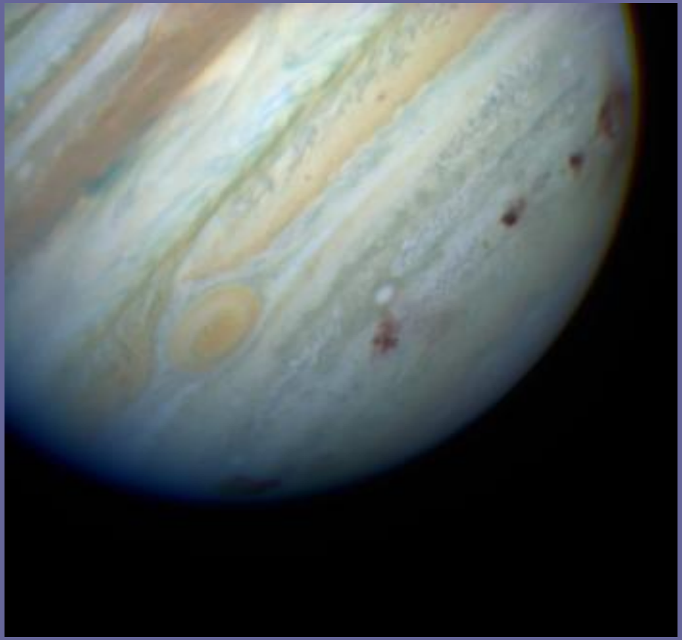
2.5 Example of the limits of the 2-body problem: the Roche limit

- In the two-body problem, the masses are assumed to behave as point-like masses. To what extent is this a valid assumption?
- In principle, a (solid) spherically-symmetric body behaves as a point-like mass.
- There are two problems though: non-spherical distribution of the matter inside the body and tidal forces that overcome the body's self-gravity.
- The issue of non-spherical mass distributions will be dealt with in chapter V. Here we focus on the problem of the competition between tidal forces and self-gravity.

2.5 Example of the limits of the 2-body problem: the Roche limit

- Consider two bodies of masses M (planet) and m (moon) with $M >> m$.
- The Roche radius is the minimum distance between the masses below which the body of mass m will break down under the influence of the tidal force produced by the body of mass M that overcomes the self gravity of the smaller body.

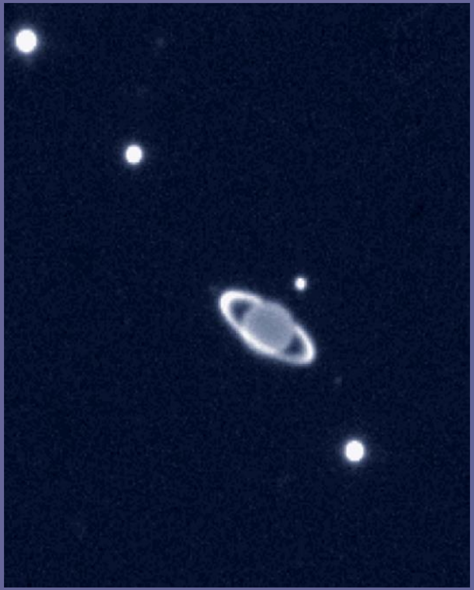
2.5 Example of the limits of the 2-body problem: the Roche limit



- The Roche limit explains the disintegration of comet Shoemaker-Levy 9 in July 1992 (followed by the impacts of the fragments in July 1994):

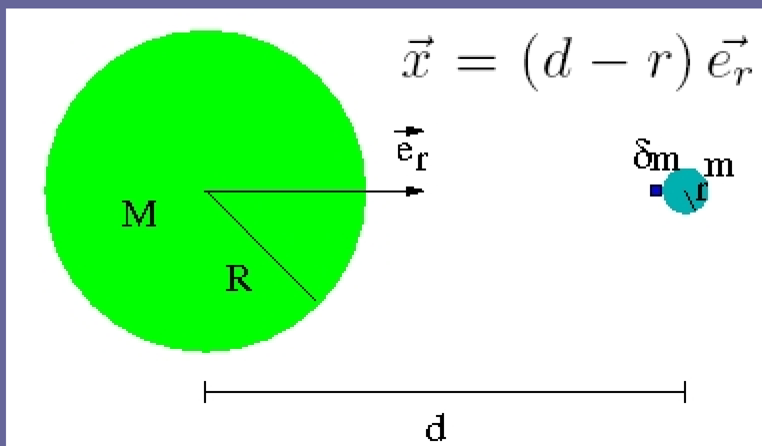
2.5 Example of the limits of the 2-body problem: the Roche limit

- The Roche limit also explains the existence of rings around giant planets: the debris could not assemble into a bigger moon or come from a moon that was disintegrated.



2.5 Example of the limits of the 2-body problem: the Roche limit

- Let's assume a rigid satellite (it does not change its shape) held together by self-gravity, and rotating synchronously.



$$\omega^2 = \frac{G\,M}{d^3}$$

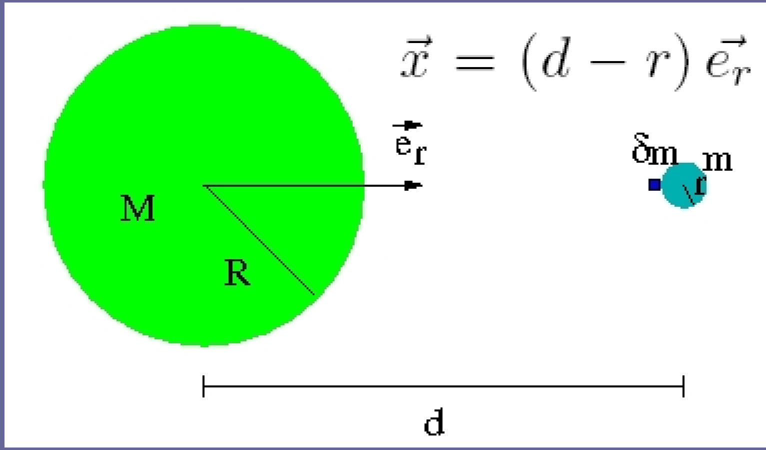
$$\cancel{\frac{\delta^2 \vec{x}}{\delta t^2}} = \ddot{\vec{x}} - \left(2\,\vec{\omega} \wedge \cancel{\frac{\delta \vec{x}}{\delta t}} + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{x})\right)$$

- As long as the small mass δm remains attached to the satellite, its relative acceleration and velocity are both zero.

2.5 Example of the limits of the 2-body problem: the Roche limit

- Newton's equation:

$$\delta m\,\ddot{\vec{x}} = -\frac{G\,M\,\delta m}{d^3}\,\vec{x} = -\frac{G\,M\,\delta m}{x^2}\,\vec{e_r} + \frac{G\,m\,\delta m}{r^2}\,\vec{e_r} + \vec{N}$$



$$\begin{aligned}\frac{N}{G\,\delta m} &= \frac{M}{d^3}(d-r) - \frac{M}{(d-r)^2} + \frac{m}{r^2}\\ &\simeq \frac{M}{d^2}(1-\frac{r}{d}) - \frac{M}{d^2}(1+2\,\frac{r}{d}) + \frac{m}{r^2}\\ &= -\frac{3\,M\,r}{d^3} + \frac{m}{r^2}\end{aligned}$$

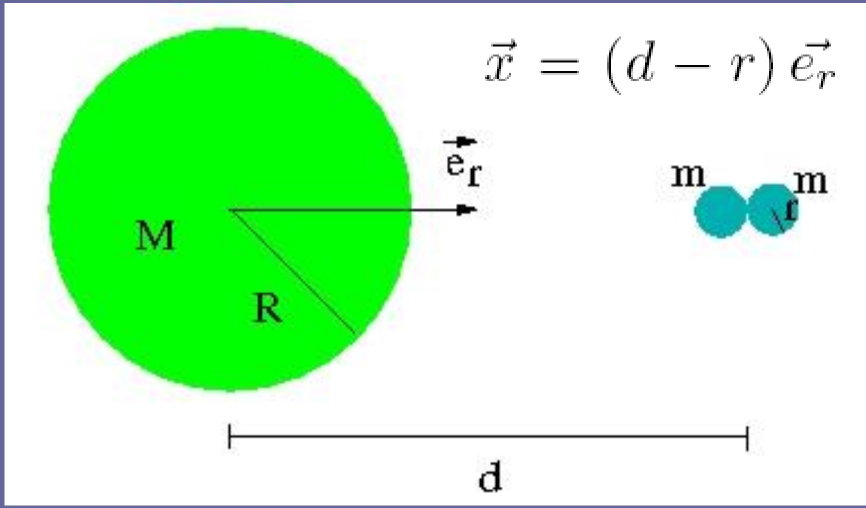
- The normal reaction form $\vec{N}$ vanishes for

$$\frac{d}{r} = \left(\frac{3\,M}{m}\right)^{1/3}$$

$$d_{\mathrm{RL}} = \left(\frac{3\,\rho_M}{\rho_m}\right)^{1/3} R = 1.44\,\left(\frac{\rho_M}{\rho_m}\right)^{1/3} R$$

2.5 Example of the limits of the 2-body problem: the Roche limit

- "Fluid" satellite (it changes its shape under the effect of the tidal attraction of *M*) represented by two spheres of radius *r* and mass *m*, rotating synchronously.

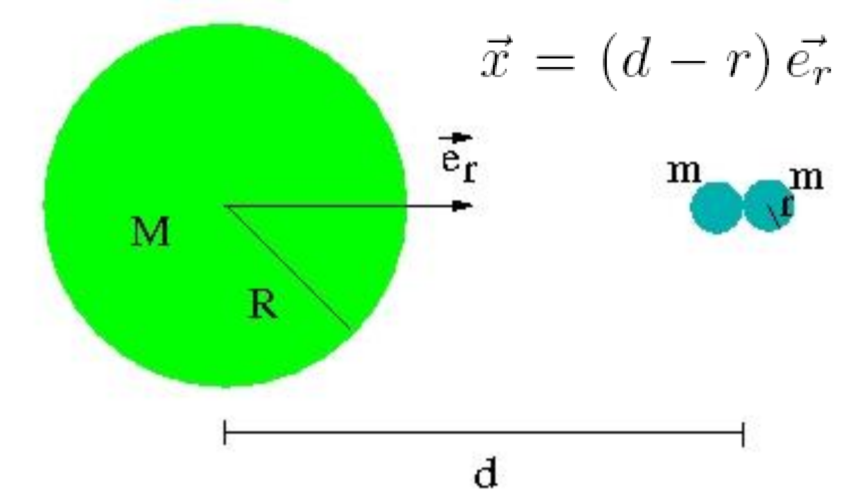


$$\omega^2 = \frac{G\,M}{d^3}$$

$$m\,\ddot{\vec{x}} = -\frac{G\,M\,m}{d^3}\,\vec{x} = -\frac{G\,M\,m}{x^2}\,\vec{e_r} + \frac{G\,m\,m}{4\,r^2}\,\vec{e_r} + \vec{N}$$

2.5 Example of the limits of the 2-body problem: the Roche limit

$$m\,\ddot{\vec{x}} = -\frac{G\,M\,m}{d^3}\,\vec{x} = -\frac{G\,M\,m}{x^2}\,\vec{e_r} + \frac{G\,m\,m}{4\,r^2}\,\vec{e_r} + \vec{N}$$



$$\vec{x} = (d - r)\,\vec{e_r}$$

$$\begin{aligned}
\frac{N}{G\,m} &= \frac{M}{d^3}\,(d-r) - \frac{M}{(d-r)^2} + \frac{m}{4\,r^2} \\
&\simeq \frac{M}{d^2}\,(1 - \frac{r}{d}) - \frac{M}{d^2}\,(1 + 2\,\frac{r}{d}) + \frac{m}{4\,r^2} \\
&= -\frac{3\,M\,r}{d^3} + \frac{m}{4\,r^2}
\end{aligned}$$

$$d_{\mathrm{RL}} = \left(\frac{12\,\rho_M}{\rho_m}\right)^{1/3} R = 2.29\,\left(\frac{\rho_M}{\rho_m}\right)^{1/3} R$$

Chapter III: Lagrangian and Hamiltonian mechanics

- The Lagrangian formalism
  - q_j $j=1,\ldots,n$ generalized coordinates
  - Principle of virtual works: $$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0$$
  - Lagrangian: $\mathcal{L} = T - U$
  - 2nd order differential equations in a space of dimension n

Chapter III: Lagrangian and Hamiltonian mechanics

- The Hamiltonian formalism
  - First order differential equations in a space of dimension *2 n*
  - p_j generalized momentum associated with q_j $\quad j=1,\ldots,n$

$$p_j = \frac{\partial \mathcal{L}}{\partial \dot{q}_j}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = 0 \quad \Rightarrow \quad \frac{d}{dt} p_j = \frac{\partial \mathcal{L}}{\partial q_j}$$

  - Hamiltonian:

$$\mathcal{H} = \sum_{j=1}^{n} p_j \, \dot{q}_j - \mathcal{L}$$

Chapter III: Lagrangian and Hamiltonian mechanics

$$\frac{d}{dt}p_j = \frac{\partial \mathcal{L}}{\partial q_j} \quad \& \quad \mathcal{H} = \sum_{j=1}^{n} p_j\,\dot{q}_j - \mathcal{L} \quad \Rightarrow \quad \begin{aligned} \frac{\partial \mathcal{H}}{\partial p_j} &= \dot{q}_j \\ \frac{\partial \mathcal{H}}{\partial q_j} &= -\dot{p}_j \\ \frac{\partial \mathcal{H}}{\partial t} &= -\frac{\partial \mathcal{L}}{\partial t} \end{aligned}$$

- Hamilton's canonical equations:
- If the forces derive from a potential U (independent of $\dot{q}_j$):

$$\mathcal{H} = T + U$$

- If the Hamiltonian does not depend explicitly on certain variables, the integration of the canonical equations relative to the conjugated variables is straightforward.

Chapter III: Lagrangian and Hamiltonian mechanics

- The Hamilton Jacobi method:
  - Find a new set of variables that preserves the canonical form of Hamilton's equations and simplifies the expression of the Hamiltonian.
  - New set of variables: $$q_j = f_j(x_1, ..., x_n, y_1, ..., y_n, t)$$ $$p_j = g_i(x_1, ..., x_n, y_1, ..., y_n, t)$$
  - This new set is canonical if: $\frac{\partial \mathcal{H}'}{\partial y_j} = \dot{x}_j$ & $\frac{\partial \mathcal{H}'}{\partial x_j} = -\dot{y}_j$
  - These conditions are met if and only if

$$[x_j, y_k] = -[y_j, x_k] = \delta_{jk}$$
$$[x_j, x_k] = [y_j, y_k] = 0$$

& $[t, \alpha] = -\frac{\partial F^*}{\partial \alpha} \quad \forall \alpha$

with $$[u, v] = \sum_{i=1}^{n} \left(\frac{\partial f_i}{\partial u} \frac{\partial g_i}{\partial v} - \frac{\partial f_i}{\partial v} \frac{\partial g_i}{\partial u} \right)$$

Chapter III: Lagrangian and Hamiltonian mechanics

- The Hamilton Jacobi method:
  - New Hamiltonian: $\mathcal{H}'(x_i, y_i, t) = \mathcal{H} + F^*$
  - The function F^* must be such that $\sum_{j=1}^{n}(p_j\, dq_j + x_j\, dy_j) + F^*\, dt = dG$

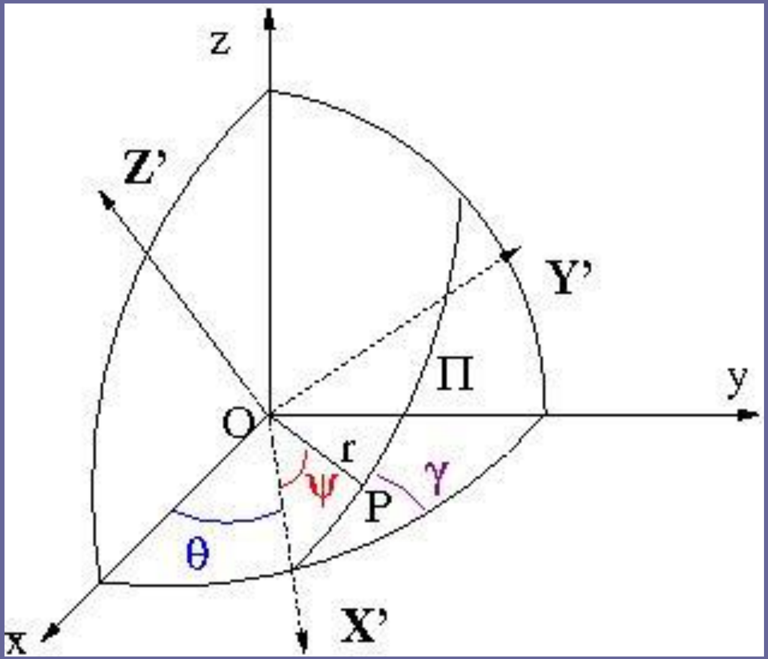
$$\Rightarrow \quad \sum_{j=1}^{n}(p_j\, dq_j + x_j\, dy_j) + (\mathcal{H}' - \mathcal{H})\, dt = dG$$

  - If we choose the new Hamiltonian to be $\mathcal{H}' = 0$
    then, the generating function $G(q_j,y_j,t)$ must be such that

$$\begin{aligned} \mathcal{H}(q_j, \frac{\partial G}{\partial q_j}, t) + \frac{\partial G}{\partial t} &= 0 \\ \frac{\partial G}{\partial q_j} &= p_j \\ \frac{\partial G}{\partial y_j} &= x_j \end{aligned} \quad \Rightarrow \quad \frac{\partial \mathcal{H}'}{\partial y_j} = \dot{x}_j = 0 \quad \& \quad \frac{\partial \mathcal{H}'}{\partial x_j} = -\dot{y}_j = 0$$

Chapter III: Lagrangian and Hamiltonian mechanics

- Application to the two-body problem:



Let Π be the instantaneous plane of the motion, perpendicular to the OZ' axis. The position of P is described by the polar coordinates in the plane.

Absolute position given by (r, θ, γ, ψ)

⇒ Angular velocity vector:

$$\vec{\Omega} = \dot{\theta}\,\vec{e_z} + \dot{\gamma}\,\vec{e_{X'}} + \dot{\psi}\,\vec{e_{Z'}} = \dot{\gamma}\,\vec{e_{X'}} + \dot{\theta}\,\sin\gamma\,\vec{e_{Y'}} + (\dot{\theta}\,\cos\gamma + \dot{\psi})\,\vec{e_{Z'}}$$

$$\Rightarrow \quad \dot{\vec{r}} = \dot{r}\,\vec{e_r} + r\,(\dot{\theta}\,\cos\gamma + \dot{\psi})\,\vec{e_\psi} + r\,(\dot{\gamma}\,\sin\psi - \dot{\theta}\,\sin\gamma\,\cos\psi)\,\vec{e_{Z'}}$$

$$\Rightarrow \quad \dot{\gamma}\,\sin\psi - \dot{\theta}\,\sin\gamma\,\cos\psi = 0$$

Chapter III: Lagrangian and Hamiltonian mechanics

$$\dot{\vec{r}} = \dot{r}\,\vec{e_r} + r\,(\dot{\theta}\,\cos\gamma + \dot{\psi})\,\vec{e_\psi} + r\,(\dot{\gamma}\,\sin\psi - \dot{\theta}\,\sin\gamma\,\cos\psi)\,\vec{e_{Z'}}$$

$$\Rightarrow \quad T = \frac{1}{2}\left[\dot{r}^2 + r^2\,(\dot{\theta}\,\cos\gamma + \dot{\psi})^2 + r^2\,(\dot{\gamma}\,\sin\psi - \dot{\theta}\,\sin\gamma\,\cos\psi)^2\right]$$

- Lagrangian: $\mathcal{L} = T + \frac{\mu}{r}$

$$\Rightarrow \quad \begin{aligned} \frac{d}{dt}\left(\frac{\partial\,\mathcal{L}}{\partial\dot{r}}\right) - \frac{\partial\,\mathcal{L}}{\partial\,r} &= 0 \\ \frac{d}{dt}\left(\frac{\partial\,\mathcal{L}}{\partial\dot{\psi}}\right) - \frac{\partial\,\mathcal{L}}{\partial\,\psi} &= 0 \\ \frac{d}{dt}\left(\frac{\partial\,\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\,\mathcal{L}}{\partial\,\theta} &= 0 \\ \frac{d}{dt}\left(\frac{\partial\,\mathcal{L}}{\partial\dot{\gamma}}\right) - \frac{\partial\,\mathcal{L}}{\partial\,\gamma} &= 0 \end{aligned}$$

Chapter III: Lagrangian and Hamiltonian mechanics

- Canonical variables:

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \dot{r} = R$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\psi}} = r^2 \left(\dot{\psi} + \dot{\theta}\, \cos\gamma\right) = \Psi$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = r^2 \left(\dot{\theta}\, \cos^2\gamma + \dot{\psi}\, \cos\gamma\right) - r^2 \left(\dot{\gamma}\, \sin\psi - \dot{\theta}\, \sin\gamma\, \cos\psi\right)\, \sin\gamma\, \cos\psi = \Theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\gamma}} = r^2 \left(\dot{\gamma}\, \sin\psi - \dot{\theta}\, \sin\gamma\, \cos\psi\right)\, \sin\psi = \Gamma = 0$$

$$\Rightarrow \quad \mathcal{H} = \frac{1}{2}\left[R^2 + \frac{\Psi^2}{r^2} + \frac{(\Theta - \Psi\, \cos\gamma)^2}{r^2\, \sin^2\gamma} + \frac{\Gamma^2}{r^2}\right] - \frac{\mu}{r}$$

$$\frac{d\,r}{dt} = \frac{\partial\,\mathcal{H}}{\partial R} = R$$

$$\frac{d\,R}{dt} = -\frac{\partial\,\mathcal{H}}{\partial r} = \frac{(\Psi^2 + \Gamma^2)\ \sin^2\gamma + (\Theta - \Psi\ \cos\gamma)^2}{r^3\ \sin^2\gamma} - \frac{\mu}{r^2}$$

$$\frac{d\,\psi}{dt} = \frac{\partial\,\mathcal{H}}{\partial\Psi} = \frac{\Psi}{r^2} - \frac{(\Theta - \Psi\ \cos\gamma)\ \cos\gamma}{r^2\ \sin^2\gamma}$$

$$\frac{d\,\Psi}{dt} = -\frac{\partial\,\mathcal{H}}{\partial\psi} = 0$$

$$\frac{d\,\theta}{dt} = \frac{\partial\,\mathcal{H}}{\partial\Theta} = \frac{\Theta - \Psi\ \cos\gamma}{r^2\ \sin^2\gamma}$$

$$\frac{d\,\Theta}{dt} = -\frac{\partial\,\mathcal{H}}{\partial\theta} = 0$$

$$\frac{d\,\gamma}{dt} = \frac{\partial\,\mathcal{H}}{\partial\Gamma} = \frac{\Gamma}{r^2} = 0$$

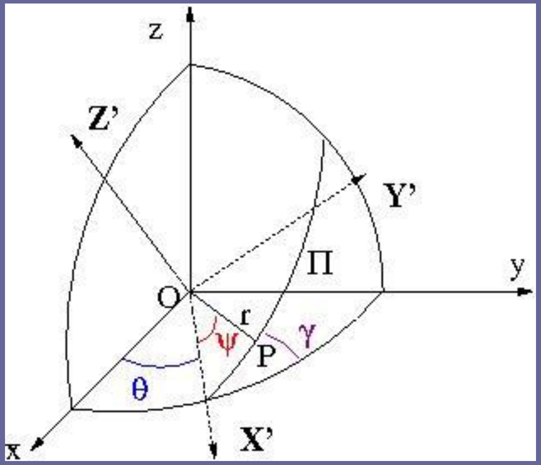
$$\frac{d\,\Gamma}{dt} = -\frac{\partial\,\mathcal{H}}{\partial\gamma} = \frac{(\Theta - \Psi\ \cos\gamma)^2\ \cos\gamma}{r^2\ \sin^3\gamma} - \frac{(\Theta - \Psi\ \cos\gamma)\ \Psi}{r^2\ \sin\gamma}$$

Chapter III: Lagrangian and Hamiltonian mechanics

$$\frac{d\gamma}{dt} = \frac{\partial \mathcal{H}}{\partial \Gamma} = 0 \quad \Rightarrow \quad \gamma = i = Cte$$

$$\dot{\gamma}\,\sin\psi - \dot{\theta}\,\sin\gamma\,\cos\psi = 0 \quad \Rightarrow \quad \dot{\theta} = 0$$

$$\theta = \Omega = Cte$$



$$\frac{d\theta}{dt} = \frac{\partial \mathcal{H}}{\partial \Theta} = \frac{\Theta - \Psi\,\cos\gamma}{r^2\,\sin^2\gamma} = 0 \quad \Rightarrow \quad \Theta = \Psi\,\cos\gamma.$$

$$\Rightarrow \quad \mathcal{H} = \frac{1}{2}\left[R^2 + \frac{\Psi^2}{r^2}\right] - \frac{\mu}{r}$$

Chapter III: Lagrangian and Hamiltonian mechanics

$$\mathcal{H} = \frac{1}{2}\left[R^2 + \frac{\Psi^2}{r^2}\right] - \frac{\mu}{r}$$

The Hamiltonian does not depend on ψ, θ, γ and *t*.

$$\Rightarrow \quad \Psi = r^2\,\dot{\psi} = Cte = h$$

$$\Theta = h' = h\,\cos i$$

$$\Gamma = 0$$

$$\mathcal{H} = \varepsilon = Cte(t)$$

The variables γ and Γ are not needed to describe the status of the system and can thus be discarded.

- The Delaunay canonical elements:
  - We search a set of canonical variables $(q_1, q_2, q_3, p_1, p_2, p_3)$ such that $q_1 = t - t_0$ and all other variables are constants.
  - This implies: $\mathcal{H}' = p_1 = \varepsilon$ & $\dot{q}_1 = \frac{\partial \mathcal{H}'}{\partial p_1} = 1$

$$dG = R\,dr + \Psi\,d\psi + \Theta\,d\theta + q_1\,dp_1 + q_2\,dp_2 + q_3\,dp_3$$

$$p_1 = \mathcal{H} = \frac{1}{2}\left[\left(\frac{\partial G}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial G}{\partial \psi}\right)^2\right] - \frac{\mu}{r}$$

We wish to preserve those variables that are constants:

$q_3 = \frac{\partial G}{\partial p_3} = \theta = \Omega$, $p_2 = \frac{\partial G}{\partial \psi} = \Psi = h$ $p_3 = \frac{\partial G}{\partial \theta} = \Theta = h'$

$$\Rightarrow \quad G = h\,\psi + h'\,\Omega + G'(r, -, -, \varepsilon, h, -)$$

Chapter III: Lagrangian and Hamiltonian mechanics

$$G = h\,\psi + h'\,\Omega + G'(r, -, -, \varepsilon, h, -)$$

with $\mathcal{H}(q_j, \frac{\partial G}{\partial q_j}, t) + \frac{\partial G}{\partial t} = 0$ $\Rightarrow$ $\varepsilon - \frac{1}{2}\left(\frac{\partial G'}{\partial r}\right)^2 - \frac{h^2}{2\,r^2} + \frac{\mu}{r} = 0$

$$\Rightarrow \quad G'(r, -, -, \varepsilon, h, -) = \pm \int_{r_0}^{r} \sqrt{2\,\varepsilon + \frac{2\,\mu}{s} - \frac{h^2}{s^2}}\, ds$$

The condition $q_1 = \frac{\partial G}{\partial p_1} = \frac{\partial G'}{\partial \varepsilon} = t - t_0$ implies

$$\frac{\partial G'}{\partial \varepsilon} = \pm \int_{r_0}^{r} \frac{ds}{\sqrt{2\,\varepsilon + \frac{2\,\mu}{s} - \frac{h^2}{s^2}}} \mp \frac{\partial r_0}{\partial \varepsilon} \boxed{\sqrt{2\,\varepsilon + \frac{2\,\mu}{r_0} - \frac{h^2}{r_0^2}}} = t - t_0$$

$$\Rightarrow \quad r_0 = \frac{h^2/\mu}{1 \pm \sqrt{1 + 2\,\varepsilon\, h^2/\mu^2}}$$

Chapter III: Lagrangian and Hamiltonian mechanics

$$\dot{r} = R = \frac{\partial G'}{\partial r} = \sqrt{2\,\varepsilon + \frac{2\,\mu}{r} - \frac{h^2}{r^2}}$$

$\Rightarrow$ t_0 is the time of pericenter or apocenter passage. Here we choose the pericenter.

$$r_0 = \frac{h^2/\mu}{1 + \sqrt{1 + 2\,\varepsilon\,h^2/\mu^2}}$$

$$\frac{\partial \mathcal{H}'}{\partial p_2} = \frac{\partial \mathcal{H}'}{\partial h} = \dot{q}_2 = 0$$

$$q_2 = \frac{\partial G}{\partial p_2} = \psi + \frac{\partial G'}{\partial h} = \psi \mp \int_{r_0}^{r} \frac{h\,ds}{s^2\sqrt{2\,\varepsilon + \frac{2\mu}{s} - \frac{h^2}{s^2}}} \mp \frac{\partial r_0}{\partial h}\sqrt{2\,\varepsilon + \frac{2\,\mu}{r_0} - \frac{h^2}{r_0^2}} = \omega$$

$$\Rightarrow \quad r = \frac{h^2/\mu}{1 + \sqrt{1 + \frac{2\,\varepsilon\,h^2}{\mu^2}}\cos(\psi - \omega)}$$

Chapter III: Lagrangian and Hamiltonian mechanics

Elliptical orbit: $r_0 = a\,(1-e)$

$$G'(r,-,-,\varepsilon,h,-) = \pm \int_{r_0}^{r} \sqrt{2\,\varepsilon + \frac{2\,\mu}{s} - \frac{h^2}{s^2}}\,ds$$

$$q_1 = \frac{\partial\,G}{\partial p_1} = \frac{\partial\,G'}{\partial \varepsilon} = t - t_0$$

$$\Rightarrow \quad M = \sqrt{\frac{\mu}{a^3}}\,(t-t_0) = E - e\,\sin E$$

⇒ Delaunay canonical elements:

$$(l,g,\theta,L,G,\Theta) = (M,\omega,\Omega,\sqrt{\mu\,a},\sqrt{\mu\,a\,(1-e^2)},\sqrt{\mu\,a\,(1-e^2)}\,\cos i)$$

Chapter III: Lagrangian and Hamiltonian mechanics

$$(l, g, \theta, L, G, \Theta) = \left(M, \omega, \Omega, \sqrt{\mu\, a}, \sqrt{\mu\, a\,(1-e^2)}, \sqrt{\mu\, a\,(1-e^2)}\,\cos i\right)$$

The Delaunay elements become ill-defined if *e* is close to 0 and/or *i* is close to 0° or 180°

$\Rightarrow$ Poincaré elements:

$$\begin{pmatrix} \Lambda \\ \xi \\ p \\ \lambda \\ \eta \\ q \end{pmatrix} = \begin{pmatrix} L \\ \sqrt{2\,(L-G)}\,\cos(g+\theta) \\ \sqrt{2\,(G-\Theta)}\,\cos\theta \\ l+g+\theta \\ -\sqrt{2\,(L-G)}\,\sin(g+\theta) \\ -\sqrt{2\,(G-\Theta)}\,\sin\theta \end{pmatrix}$$

Chapter IV: Expanding the elliptical motion in series of *e*

- The Bessel functions of the first kind:
  - Bessel's differential equation:

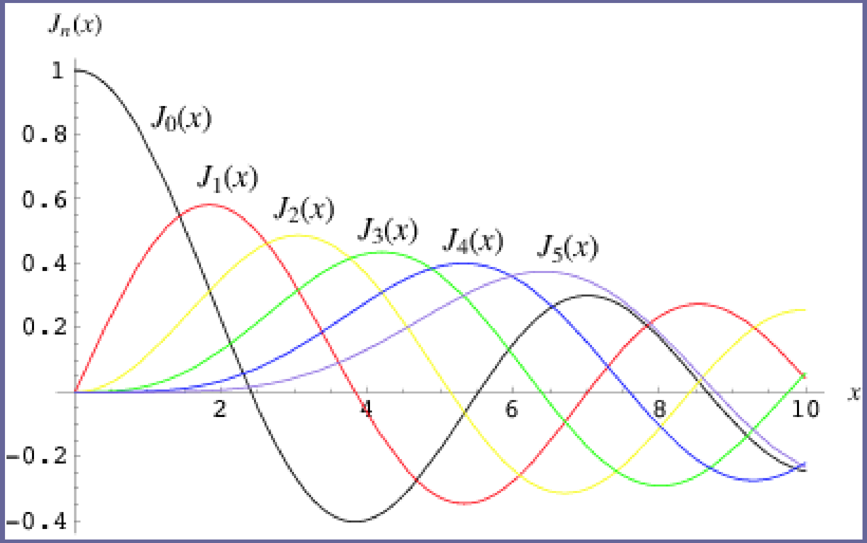
$$x^2 \frac{d^2 f(x)}{dx^2} + x \frac{d f(x)}{dx} + (x^2 - \alpha^2) f(x) = 0$$

  - Solutions that are finite in $x = 0$: Bessel functions of the first kind.

$$J_s(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+s)!} \left(\frac{x}{2}\right)^{2m+s}$$

  - Generating function:

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{s=-\infty}^{+\infty} J_s(x)\, t^s$$



Chapter IV: Expanding the elliptical motion in series of *e*

- The Bessel functions of the first kind:
  - Important properties:

$$J_s(x) = (-1)^s \, J_{-s}(x)$$

$$J_s(x) = (-1)^s \, J_s(-x)$$

$$s \, J_s(x) = [J_{s-1}(x) + J_{s+1}(x)] \, \frac{x}{2}$$

$$J'_s(x) = \frac{d \, J_s(x)}{dx} = \frac{1}{2} \, (J_{s-1}(x) - J_{s+1}(x))$$

  - Take *t = exp(j ψ)* : $J_s(x) = \frac{1}{2\,\pi} \int_0^{2\,\pi} \exp\left[j(\, x \, \sin\psi - s\,\psi)\right] d\psi$

$$J_s(x) = \frac{1}{\pi} \int_0^{\pi} \cos\left(s\,\psi - x \, \sin\psi\right) d\psi$$

4.1 Expanding into Fourier series

- The majority of important quantities of an elliptical orbit ($e < 1$) are periodic functions of φ, E or M of period 2π.
- These functions can thus be expanded into Fourier series:

$$f(u) = \frac{a_0}{2} + \sum_{k=1}^{+\infty} \left(a_k \cos(k\,u) + b_k \sin(k\,u)\right)$$

with

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \cos(k\,u)\, du \qquad k = 0, 1, 2, \ldots$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(u) \sin(k\,u)\, du \qquad k = 1, 2, \ldots$$

4.1 Expanding into Fourier series

- Examples of Fourier expansions:
  - Consider the quantity *a/r*: $\frac{a}{r} = \frac{dE}{dM} = \frac{1}{1-e\ \cos E}$
  - It's an even function of the variables φ, E or $M \Rightarrow b_k = 0$

$$\begin{aligned}\frac{a}{r} &= \frac{1}{2\pi}\int_0^{2\pi}\frac{a}{r}\,dM' + \sum_{k=1}^{+\infty}\left(\frac{2}{\pi}\int_0^{\pi}\frac{a}{r}\cos\left(k\,M'\right)dM'\right)\cos\left(k\,M\right)\\ &= 1 + \sum_{k=1}^{+\infty}\boxed{\left(\frac{2}{\pi}\int_0^{\pi}\cos\left[k\left(E - e\ \sin E\right)\right]dE\right)}\cos\left(k\,M\right)\end{aligned}$$

$$J_s(x) = \frac{1}{\pi}\int_0^{\pi}\cos\left(s\,\psi - x\ \sin\psi\right)d\psi \quad\Rightarrow\quad \boxed{\frac{a}{r} = 1 + \sum_{k=1}^{+\infty} 2\,J_k(k\,e)\ \cos\left(k\,M\right)}$$

4.1 Expanding into Fourier series

- Examples of Fourier expansions:

$$\frac{a}{r} = 1 + \sum_{k=1}^{+\infty} 2\, J_k(k\, e)\, \cos(k\, M) = \frac{d\,E}{dM}$$

$$\Rightarrow \quad E = M + \sum_{k=1}^{+\infty} \frac{2}{k}\, J_k(k\, e)\, \sin(k\, M)$$

- General expression of cos(pE) and sin(pE):

$$\cos(p\, E) = \delta_{p0} - \frac{e}{2}\, \delta_{p1} + \sum_{k=1}^{+\infty} \frac{p}{k} (J_{k-p}(k\, e) - J_{k+p}(k\, e))\, \cos(k\, M)$$

$$\sin(p\, E) = \sum_{k=1}^{+\infty} \frac{p}{k} (J_{k-p}(k\, e) + J_{k+p}(k\, e))\, \sin(k\, M)$$

4.1 Expanding into Fourier series

- Examples of Fourier expansions:

$$\cos(p\,E) \;=\; \delta_{p0} - \frac{e}{2}\,\delta_{p1} + \sum_{k=1}^{+\infty} \frac{p}{k}\left(J_{k-p}(k\,e) - J_{k+p}(k\,e)\right)\,\cos(k\,M)$$

  - *r = a(1 – e cos E) &* $\cos\phi = \dfrac{\cos E - e}{1 - e\,\cos E}$

$$\Rightarrow \quad \frac{r}{a}\,\cos\phi = -\frac{3\,e}{2} + \sum_{k=1}^{+\infty} \frac{2}{k}\,J_k'(k\,e)\,\cos(k\,M)$$

$$\sin(p\,E) \;=\; \sum_{k=1}^{+\infty} \frac{p}{k}\left(J_{k-p}(k\,e) + J_{k+p}(k\,e)\right)\,\sin(k\,M)$$

  - *r = a(1 – e cosE) &* $\sin\phi = \dfrac{\sqrt{1-e^2}\,\sin E}{1 - e\,\cos E}$

$$\Rightarrow \quad \frac{r}{a}\,\sin\phi = \sqrt{1-e^2}\sum_{k=1}^{+\infty} \frac{2}{k\,e}\,J_k(k\,e)\,\sin(k\,M)$$

4.1 Expanding into Fourier series

- Examples of Fourier expansions:

$$\cos(p\,E) \quad = \quad \delta_{p0} - \frac{e}{2}\,\delta_{p1} + \sum_{k=1}^{+\infty} \frac{p}{k}\left(J_{k-p}(k\,e) - J_{k+p}(k\,e)\right)\,\cos(k\,M)$$

& r = a(1 – e cos E)

$$\Rightarrow \quad \frac{r}{a} = 1 - e\,\cos E = 1 + \frac{e^2}{2} - 2\,e\,\sum_{k=1}^{+\infty} \frac{1}{k}\,J'_k(k\,e)\,\cos(k\,M)$$

4.2 The d'Alembert characteristics

- If e is a small quantity what can we say about the different terms of the Fourier series?
- Bessel functions: $$J_s(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+s)!} \left(\frac{x}{2}\right)^{2m+s}$$

  of the order of $x^{|s|}$ $\Rightarrow$ the *cos(sM)* and *sin(sM)* terms in the expressions of *cos (pE)* and *sin(pE)* are multiplied by terms of order $e^{|s \pm p|}$
- A Fourier series possesses the d'Alembert characteristics of degree *p*, if the dominant term of this series corresponds to the *p*th harmonics. Such a series can be written:

$$S_p(e, M) = \sum_{k=-\infty}^{+\infty} e^{|p-k|}\, s_k(e)\, \exp\left(i\, k\, M\right)$$

4.3 Development into asymptotic series of *e*

- If *e* is a small quantity, we can restrict ourselves to a limited number of terms in the Bessel functions…

$$
\begin{aligned}
J_s(s\,e) &= \sum_{i=0}^{+\infty} \frac{(-1)^i}{i!\,(i+s)!} \left(\frac{s\,e}{2}\right)^{2i+s} \\
J_1(e) &= \frac{e}{2} - \frac{e^3}{16} + \frac{e^5}{384} + \mathcal{O}(e^7) \\
J_2(2\,e) &= \frac{e^2}{2} - \frac{e^4}{6} + \frac{e^6}{48} + \mathcal{O}(e^8) \\
J_3(3\,e) &= \frac{9\,e^3}{16} - \frac{81\,e^5}{256} + \mathcal{O}(e^7) \\
J_4(4\,e) &= \frac{2\,e^4}{3} - \frac{8\,e^6}{15} + \mathcal{O}(e^8) \\
J_5(5\,e) &= \frac{625\,e^5}{768} + \mathcal{O}(e^7) \\
J_6(6\,e) &= \frac{81\,e^6}{80} + \mathcal{O}(e^8)
\end{aligned}
$$

4.3 Development into asymptotic series of *e*

- If *e* is a small quantity, we can restrict ourselves to a limited number of terms in the Bessel functions and their derivatives.

$$J_s(s\,e) = \sum_{i=0}^{+\infty} \frac{(-1)^i}{i!\,(i+s)!} \left(\frac{s\,e}{2}\right)^{2i+s}$$

$$J_1(e) = \frac{e}{2} - \frac{e^3}{16} + \frac{e^5}{384} + \mathcal{O}(e^7)$$

$$J_2(2\,e) = \frac{e^2}{2} - \frac{e^4}{6} + \frac{e^6}{48} + \mathcal{O}(e^8)$$

$$J_3(3\,e) = \frac{9\,e^3}{16} - \frac{81\,e^5}{256} + \mathcal{O}(e^7)$$

$$J_4(4\,e) = \frac{2\,e^4}{3} - \frac{8\,e^6}{15} + \mathcal{O}(e^8)$$

$$J_5(5\,e) = \frac{625\,e^5}{768} + \mathcal{O}(e^7)$$

$$J_6(6\,e) = \frac{81\,e^6}{80} + \mathcal{O}(e^8)$$

$$J'_s(s\,e) = \sum_{i=0}^{+\infty} \frac{(-1)^i\,(2\,i+s)}{2\,(i!\,(i+s)!)} \left(\frac{s\,e}{2}\right)^{2i+s-1}$$

$$J'_1(e) = \frac{1}{2} - \frac{3\,e^2}{16} + \frac{5\,e^4}{384} + \mathcal{O}(e^6)$$

$$J'_2(2\,e) = \frac{e}{2} - \frac{e^3}{3} + \frac{e^5}{16} + \mathcal{O}(e^7)$$

$$J'_3(3\,e) = \frac{9\,e^2}{16} - \frac{135\,e^4}{256} + \mathcal{O}(e^6)$$

$$J'_4(4\,e) = \frac{2\,e^3}{3} - \frac{4\,e^5}{5} + \mathcal{O}(e^7)$$

$$J'_5(5\,e) = \frac{625\,e^4}{768} + \mathcal{O}(e^6)$$

$$J'_6(6\,e) = \frac{81\,e^5}{80} + \mathcal{O}(e^7)$$

4.3 Development into asymptotic series of *e*

- Examples of developments into asymptotic series:

$$\frac{r}{a} = 1 - e\,\cos E = 1 + \frac{e^2}{2} - 2\,e \sum_{k=1}^{+\infty} \frac{1}{k}\, J'_k(k\,e)\,\cos\,(k\,M)$$

$$\Rightarrow$$

$$\begin{aligned}\frac{r}{a} \;=\;& 1 + \frac{e^2}{2} - (e - \frac{3\,e^3}{8} + \frac{5\,e^5}{192})\,\cos M - (\frac{e^2}{2} - \frac{e^4}{3})\cos\,(2\,M) - (\frac{3\,e^3}{8} - \frac{45\,e^5}{128})\,\cos\,(3\,M)\\ & - \frac{e^4}{3}\,\cos\,(4\,M) - \frac{125\,e^5}{384}\,\cos\,(5\,M) + \mathcal{O}(e^6)\end{aligned}$$

These developments are not absolutely converging for all values of $e < 1$!

4.3 Development into asymptotic series of *e*

- Examples of developments into asymptotic series:

$$\frac{a}{r} = 1 + \sum_{k=1}^{+\infty} 2\, J_k(k\,e)\, \cos(k\,M)$$

$$\Rightarrow$$

$$\begin{aligned}\frac{a}{r} &= 1 + (e - \frac{e^3}{8} + \frac{e^5}{192})\cos M + (e^2 - \frac{e^4}{3})\cos(2\,M) + (\frac{9\,e^3}{8} - \frac{81\,e^5}{128})\cos(3\,M)\\ &\quad + \frac{4\,e^4}{3}\cos(4\,M) + \frac{625\,e^5}{384}\cos(5\,M) + \mathcal{O}(e^6)\end{aligned}$$

4.3 Development into asymptotic series of *e*

- Examples of developments into asymptotic series:

$$\frac{r}{a}\cos\phi = -\frac{3\,e}{2} + \sum_{k=1}^{+\infty} \frac{2}{k}\, J_k'(k\,e)\ \cos(k\,M)$$

$$\Rightarrow$$

$$\begin{aligned}\frac{r}{a}\cos\phi \;=\;& -\frac{3\,e}{2} + \left(1 - \frac{3\,e^2}{8} + \frac{5\,e^4}{192}\right)\cos M + \left(\frac{e}{2} - \frac{e^3}{3} + \frac{e^5}{16}\right)\cos(2\,M) + \left(\frac{3\,e^2}{8} - \frac{45\,e^4}{128}\right)\cos(3\,M) \\ & + \left(\frac{e^3}{3} - \frac{2\,e^5}{5}\right)\cos(4\,M) + \frac{125\,e^4}{384}\cos(5\,M) + \frac{27\,e^5}{80}\cos(6\,M) + \mathcal{O}(e^6)\end{aligned}$$

4.3 Development into asymptotic series of *e*

- Examples of developments into asymptotic series:

$$\frac{r}{a}\,\sin\phi = \sqrt{1-e^2}\sum_{k=1}^{+\infty}\frac{2}{k\,e}\,J_k(k\,e)\,\sin\left(k\,M\right)$$

$$\Rightarrow$$

$$\begin{aligned}\frac{r}{a}\,\sin\phi \quad &= \quad (1-\frac{5\,e^2}{8}-\frac{11\,e^4}{192})\,\sin M + (\frac{e}{2}-\frac{5\,e^3}{12}+\frac{e^5}{24})\,\sin\left(2\,M\right) + (\frac{3\,e^2}{8}-\frac{51\,e^4}{128})\,\sin\left(3\,M\right)\\ &+(\frac{e^3}{3}-\frac{13\,e^5}{30})\,\sin\left(4\,M\right)+\frac{125\,e^4}{384}\,\sin\left(5\,M\right)+\frac{27\,e^5}{80}\,\sin\left(6\,M\right)+\mathcal{O}(e^6)\end{aligned}$$

Chapter V: The forces acting on a body in space

- The associated Legendre functions:
  - General Legendre differential equation:

$$\left[n\,(n+1) - \frac{p^2}{1-x^2}\right] P_n^{(p)}(x) - 2\,x\,\frac{d\,P_n^{(p)}(x)}{dx} + (1-x^2)\,\frac{d^2\,P_n^{(p)}(x)}{dx^2} = 0$$

  - Solutions for p ≥ 0: associated Legendre functions

$$P_n^{(p)}(x) = \frac{(1-x^2)^{p/2}}{2^n\,n!}\,\frac{d^{n+p}}{dx^{n+p}}(x^2-1)^n$$

  - For p = 0, we get the associated Legendre polynomials

$$P_n(x) = \frac{1}{2^n\,n!}\,\frac{d^n}{dx^n}(x^2-1)^n$$

Chapter V: The forces acting on a body in space

- Associated Legendre functions:
  - Generating function of the associated Legendre polynomials:

$$\frac{1}{\sqrt{1-2\,x\,t+t^2}} = \sum_{n=0}^{+\infty} P_n(x)\,t^n$$

$$P_0^{(0)}(x) = 1$$

$$P_1^{(0)}(x) = x \qquad P_1^{(1)}(x) = \sqrt{1-x^2}$$

$$P_2^{(0)}(x) = \frac{3\,x^2}{2} - \frac{1}{2} \qquad P_2^{(1)}(x) = 3\,x\,\sqrt{1-x^2} \qquad P_2^{(2)} = 3\,(1-x^2)$$

$$P_3^{(0)}(x) = \frac{5\,x^3}{2} - \frac{3\,x}{2} \qquad P_3^{(1)}(x) = \left(\frac{15\,x^2}{2} - \frac{3}{2}\right)\sqrt{1-x^2} \qquad P_3^{(2)} = 15\,x\,(1-x^2) \qquad P_3^{(3)}(x) = 15\,(1-x^2)^{3/2}$$

  - Recurrence relations:

$$(n+1-p)\,P_{n+1}^{(p)}(x) - (2\,n+1)\,x\,P_n^{(p)}(x) + (n+p)\,P_{n-1}^{(p)}(x) = 0$$

$$P_n^{(p+2)}(x) - \frac{2\,(p+1)\,x}{\sqrt{1-x^2}}\,P_n^{(p+1)}(x) + (n-p)\,(n+p+1)\,P_n^{(p)}(x) = 0$$

5.1 The gravitational potential

- A point-like mass (at O) produces a potential at P (different from O):

$$U(P) = -\frac{G\,m}{|\vec{OP}|}$$

- Several point-like masses:

$$U(P) = -\sum_i \frac{G\,m_i}{|\vec{O_i P}|}$$

- The resulting acceleration writes:

$$\vec{g} = -\vec{\nabla}\,U(P)$$

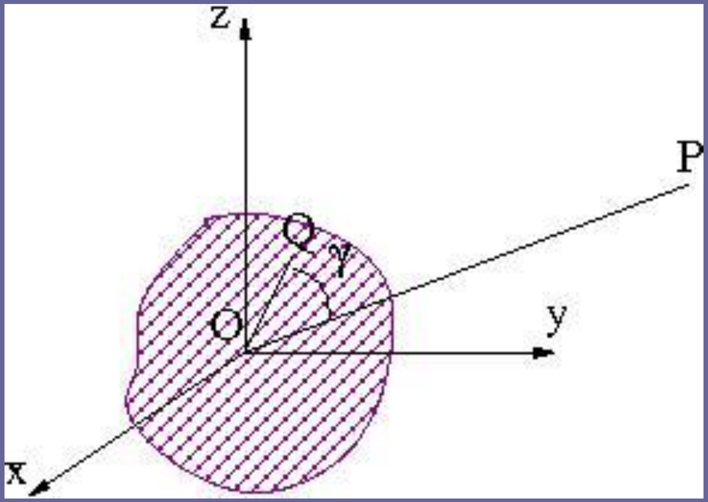
$$\Rightarrow \quad \vec{\nabla}\cdot\vec{g} = 0$$

$$\Rightarrow \begin{cases} \Delta\,U(P) = 0 \quad \forall P \neq O_i \\ \Delta\,U(O_i) = -4\,\pi\,G\,m_i \end{cases}$$

5.1 The gravitational potential

- Continuous mass-distribution of density ρ:

$$dm = \rho(Q)\,dV$$



$$\Delta U(P) = -4\,\pi\,G\,\rho(P)$$

$$\vec{g}(P) = -\int_{Q\in S} \frac{G\,\vec{QP}}{|\vec{QP}|^3}\,dm$$

$$U(P) = U(r,\theta,\lambda) = \frac{-G\,M}{r}\sum_{n=0}^{\infty}\frac{W_n(\theta,\lambda)}{r^n}$$

$$\Delta U(P) = 0$$

⇒ Spherical coordinates (r, θ, λ) with θ the latitude and λ the longitude.

$$\Delta U = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\,\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\,\cos^2\theta}\frac{\partial^2 U}{\partial\lambda^2}$$

5.1 The gravitational potential

- Poisson's equation:

$$\Delta U(P) = 0 \quad \& \quad U(P) = U(r,\theta,\lambda) = \frac{-G\,M}{r}\sum_{n=0}^{\infty}\frac{W_n(\theta,\lambda)}{r^n}$$

$$\Rightarrow \quad \Delta\left(\frac{W_n(\theta,\lambda)}{r^{n+1}}\right) = 0 \quad \Rightarrow$$

$$n\,(n+1)\,W_n(\theta,\lambda) + \frac{1}{\cos^2\theta}\frac{\partial^2 W_n(\theta,\lambda)}{\partial\lambda^2} + \frac{\partial^2 W_n(\theta,\lambda)}{\partial\theta^2} - \tan\theta\,\frac{\partial\, W_n(\theta,\lambda)}{\partial\theta} = 0$$

$W_n(\theta,\lambda)$ periodic function of λ with a period 2π:

$$W_n(\theta,\lambda) = \sum_{p=-\infty}^{+\infty} Q_n^{(p)}(\theta)\,\exp\left(j\,p\,\lambda\right)$$

5.1 The gravitational potential

$$n\,(n+1)\,W_n(\theta,\lambda) + \frac{1}{\cos^2\theta}\,\frac{\partial^2 W_n(\theta,\lambda)}{\partial\lambda^2} + \frac{\partial^2 W_n(\theta,\lambda)}{\partial\theta^2} - \tan\theta\,\frac{\partial W_n(\theta,\lambda)}{\partial\theta} = 0$$

& $$W_n(\theta,\lambda) = \sum_{p=-\infty}^{+\infty} Q_n^{(p)}(\theta)\,\exp\,(j\,p\,\lambda)$$ $\Rightarrow$

$$\sum_{p=-\infty}^{+\infty}\left\{[n\,(n+1) - \frac{p^2}{\cos^2\theta}]\,Q_n^{(p)}(\theta) + \frac{\partial^2 Q_n^{(p)}(\theta)}{\partial\theta^2} - \tan\theta\,\frac{\partial Q_n^{(p)}(\theta)}{\partial\theta}\right\}\exp\,(j\,p\,\lambda) = 0$$

$$\Rightarrow\quad [n\,(n+1) - \frac{p^2}{\cos^2\theta}]\,Q_n^{(p)}(\theta) + \frac{\partial^2 Q_n^{(p)}(\theta)}{\partial\theta^2} - \tan\theta\,\frac{\partial Q_n^{(p)}(\theta)}{\partial\theta} = 0$$

$s = \sin\theta$ $\Rightarrow$

$$[n\,(n+1) - \frac{p^2}{1-s^2}]\,Q_n^{(p)}(s) - 2\,s\,\frac{\partial Q_n^{(p)}(s)}{\partial s} + (1-s^2)\,\frac{\partial^2 Q_n^{(p)}(s)}{\partial s^2} = 0$$

5.1 The gravitational potential

$$\left[n\,(n+1) - \frac{p^2}{1-s^2}\right] Q_n^{(p)}(s) - 2\,s\,\frac{\partial\, Q_n^{(p)}(s)}{\partial s} + (1-s^2)\,\frac{\partial^2\, Q_n^{(p)}(s)}{\partial s^2} = 0$$

identical to

$$\left[n\,(n+1) - \frac{p^2}{1-x^2}\right] P_n^{(p)}(x) - 2\,x\,\frac{d\, P_n^{(p)}(x)}{dx} + (1-x^2)\,\frac{d^2\, P_n^{(p)}(x)}{dx^2} = 0$$

$$\Rightarrow \quad U = -\frac{G\,M}{r} \sum_{n=0}^{+\infty} \sum_{p=0}^{n} \frac{1}{r^n} P_n^{(p)}(\sin\theta) \left[c_{np}\,\cos(p\,\lambda) + s_{np}\,\sin(p\,\lambda)\right]$$

If O is the centre of mass of the distribution of matter: $c_{1p} = s_{1p} = 0$

$$U = -\frac{G\,M}{r} \left(1 + \sum_{n=2}^{+\infty} \left(\frac{R_e}{r}\right)^n \left\{-J_n\,P_n(\sin\theta) + \sum_{p=1}^{n} P_n^{(p)}(\sin\theta)\left[c_{np}\,\cos(p\,\lambda) + s_{np}\,\sin(p\,\lambda)\right]\right\}\right)$$

5.1 The gravitational potential

$$U = -\frac{GM}{r}\left(1 + \sum_{n=2}^{+\infty}\left(\frac{R_e}{r}\right)^n \left\{-J_n\, P_n(\sin\theta) + \sum_{p=1}^{n} P_n^{(p)}(\sin\theta)\left[c_{np}\cos(p\,\lambda) + s_{np}\sin(p\,\lambda)\right]\right\}\right)$$

	Earth	Mars	Jupiter	Saturn	Moon
GM ($\mathrm{m^3\,s^{-2}}$)	$3.986\,10^{14}$	$4.283\,10^{13}$	$1.267\,10^{17}$	$3.793\,10^{16}$	$4.903\,10^{12}$
R_e (km)	6378	3397	71398	60000	1738
J_2	0.001083	0.001964	0.01475	0.01645	0.000203
c_{22}	$1.57\,10^{-6}$	$-5.5\,10^{-5}$			$2.23\,10^{-5}$
s_{22}	$-0.90\,10^{-6}$	$3.1\,10^{-5}$			
J_3	$-2.53\,10^{-6}$	$3.6\,10^{-5}$			$6\,10^{-6}$
c_{31}	$2.19\,10^{-6}$				$2.9\,10^{-5}$
s_{31}	$0.27\,10^{-6}$	$2.6\,10^{-5}$			$4\,10^{-6}$
J_4	$-1.62\,10^{-6}$		$-5.8\,10^{-4}$	$-1.0\,10^{-3}$	

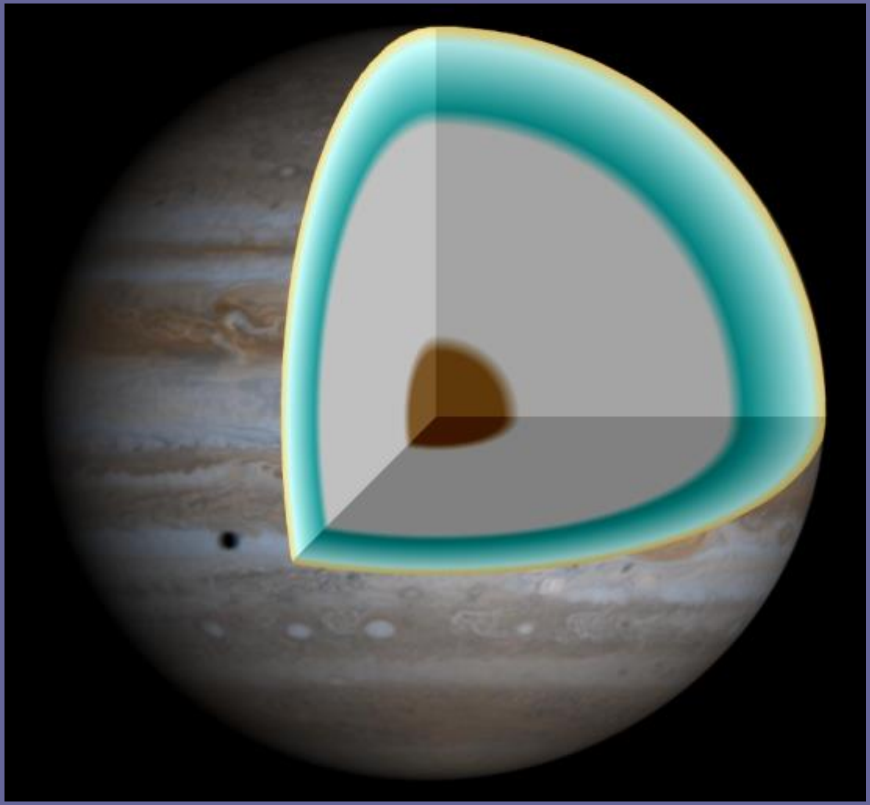
$$\vec{\nabla} U = \left(\frac{\partial U}{\partial r}, \frac{1}{r\,\cos\theta}\frac{\partial U}{\partial \lambda}, \frac{1}{r}\frac{\partial U}{\partial \theta}\right)$$

is no longer a radial vector

5.1 The gravitational potential

$$U = -\frac{G\,M}{r}\left(1+\sum_{n=2}^{+\infty}\left(\frac{R_e}{r}\right)^n\left\{-J_n\,P_n(\sin\theta)+\sum_{p=1}^{n}P_n^{(p)}(\sin\theta)\left[c_{np}\,\cos(p\,\lambda)+s_{np}\,\sin(p\,\lambda)\right]\right\}\right)$$

For a given planet, the coefficients of this expression depend upon the internal distribution of matter (moments of inertia).



$$J_2 = \frac{2\mathcal{I}_{zz}-\mathcal{I}_{xx}-\mathcal{I}_{yy}}{2\,M\,R_e^2}$$

$$\mathcal{I}_{ii} = \sum_{j\neq i,j=1}^{3}\int x_j^2\,\rho\,dV$$

$$\mathcal{I}_{ij} = -\int(x_i\,x_j)\,\rho\,dV$$

$$c_{21} = \frac{\mathcal{I}_{xz}}{M\,R_e^2}$$

$$s_{21} = \frac{\mathcal{I}_{yz}}{M\,R_e^2}$$

$$c_{22} = \frac{\mathcal{I}_{yy}-\mathcal{I}_{xx}}{4\,M\,R_e^2}$$

$$s_{22} = \frac{\mathcal{I}_{xy}}{2\,M\,R_e^2}$$

5.1 The gravitational potential

Let us assume that the planet rotates as a solid body.

The flattening of the planet depends on J_2 and the rotational velocity.

$$U = -\frac{G\,M}{r} + \frac{G\,M\,R_e^2}{r^3}\,J_2\,(\frac{3}{2}\sin^2\theta - \frac{1}{2})$$

In the frame of reference rotating with the planet: potential due to centrifugal force: $\frac{1}{2}\,\omega^2\,r^2\,\cos^2\theta$

$$\Rightarrow \quad U = -\frac{G\,M}{r} + \frac{G\,M\,R_e^2}{r^3}\,J_2\,(\frac{3}{2}\sin^2\theta - \frac{1}{2}) - \frac{1}{2}\,\omega^2\,r^2\,\cos^2\theta$$

The surface of the planet is an equipotential of *U*.

Let's define the flattening: $f = \frac{R_e - R_p}{R_e}$

$$\Rightarrow \quad f \simeq \frac{3}{2}\,J_2 + \frac{1}{2}\,\frac{\omega^2\,R_e^3}{G\,M}$$

5.1 The gravitational potential

For a body with rotational symmetry about the *z'* axis, c_{np} and s_{np} are zero, and

$$J_n = -\frac{2\,\pi\,R_m^3}{M} \int_{-\pi/2}^{\pi/2} P_n(\sin\theta')\,\cos\theta' \left(\int_0^{R(\theta')} \left(\frac{r'}{R_m}\right)^{2+n} \rho(r',\theta')\,\frac{dr'}{R_m} \right) d\theta'$$

Special case: spheroid: $R(\theta) = R_m \left(1 - \frac{2\,\epsilon}{3}\,P_2(\sin\theta)\right)$

$\epsilon = \frac{R_e - R_p}{R_m}$ is the ellipticity.

Since $\frac{2\,z^2 - x^2 - y^2}{2} = r^2\,P_2(\sin\theta)$ we obtain

$$\begin{aligned} J_2 &= -\frac{1}{M\,R_e^2} \int_V \rho\,\frac{2\,z^2 - x^2 - y^2}{2}\,dV \\ &= -\frac{2\,\pi}{M\,R_e^2} \int_{-\pi/2}^{\pi/2} P_2(\sin\theta)\,\cos\theta\,(\int_0^{R(\theta)} \rho\,r^4\,dr)\,d\theta \end{aligned}$$

For a uniform density, this yields:

$$J_2 = -\frac{3}{10\,R_e^2\,R_m^3} \int_{-\pi/2}^{\pi/2} P_2(\sin\theta)\,\cos\theta\,R(\theta)^5\,d\theta \simeq -\frac{3}{10} \int_{-\pi/2}^{\pi/2} P_2(\sin\theta)\,\cos\theta\,[1 - \frac{10}{3}\,\epsilon\,P_2(\sin\theta)]\,d\theta$$

5.1 The gravitational potential

Legendre polynomials are orthogonal, hence

$$J_2 \simeq -\frac{3}{10} \int_{-\pi/2}^{\pi/2} P_2(\sin\theta)\ \cos\theta\,[1 - \frac{10}{3}\,\epsilon\, P_2(\sin\theta)]\, d\theta$$

implies $J_2 \simeq \frac{2\,\epsilon}{5}$

Therefore, we find that the potential outside a spheroid of uniform density becomes

$$U = -\frac{G\,M}{r}\left(1 - \frac{J_2\,R_e^2}{r^2}\,P_2(\sin\theta)\right) + \mathcal{O}(\epsilon^2)$$

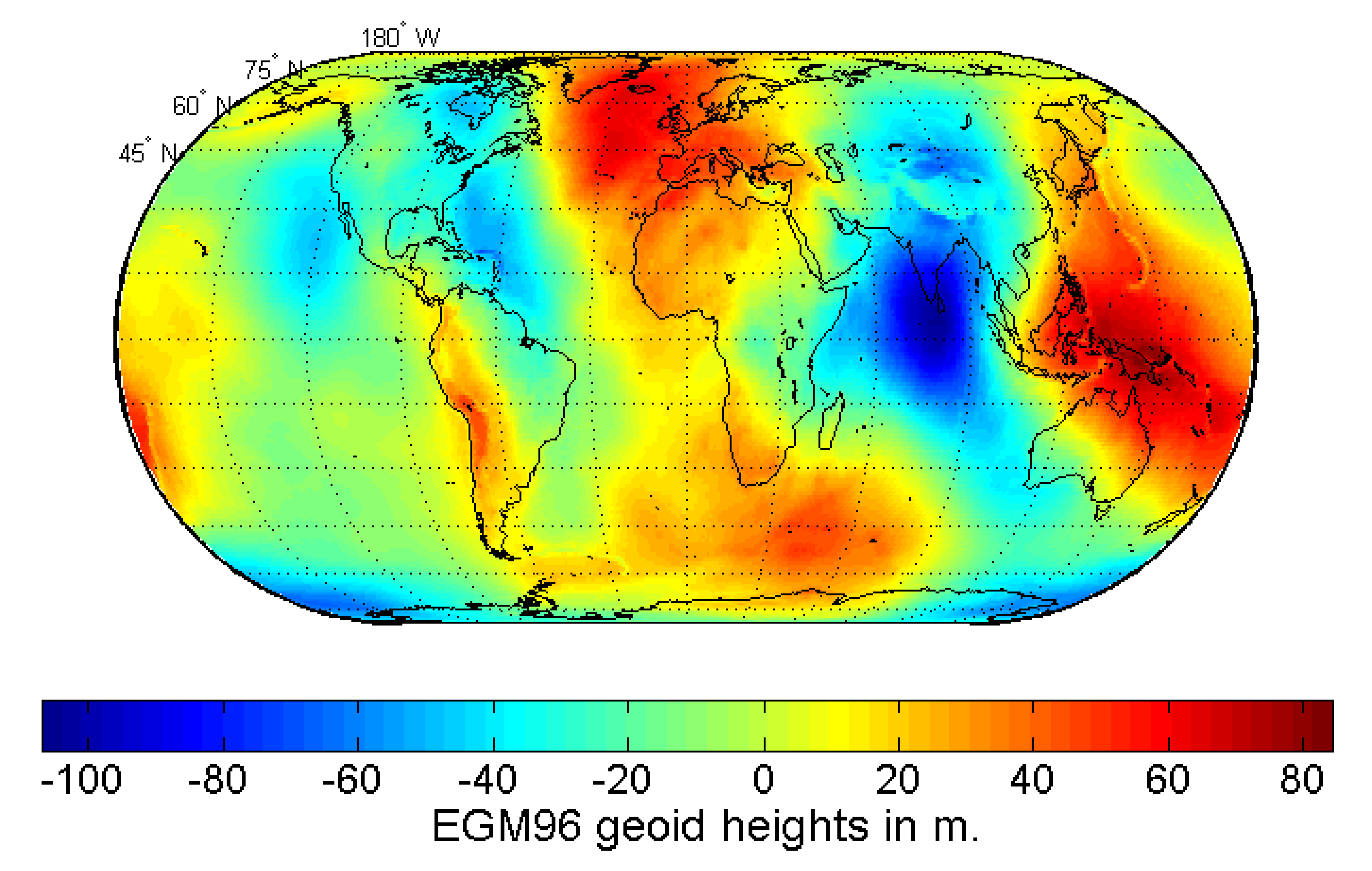
For a non-rotating spheroid, the potential at its surface is given by

$$U(R(\theta)) = -\frac{G\,M}{R(\theta)}\left(1 - \frac{J_2\,R_e^2}{R(\theta)^2}\,P_2(\sin\theta)\right) \simeq -\frac{G\,M}{R_m}\left[1 + \left(\frac{2\,\epsilon}{3} - J_2\right)\,P_2(\sin\theta)\right]$$

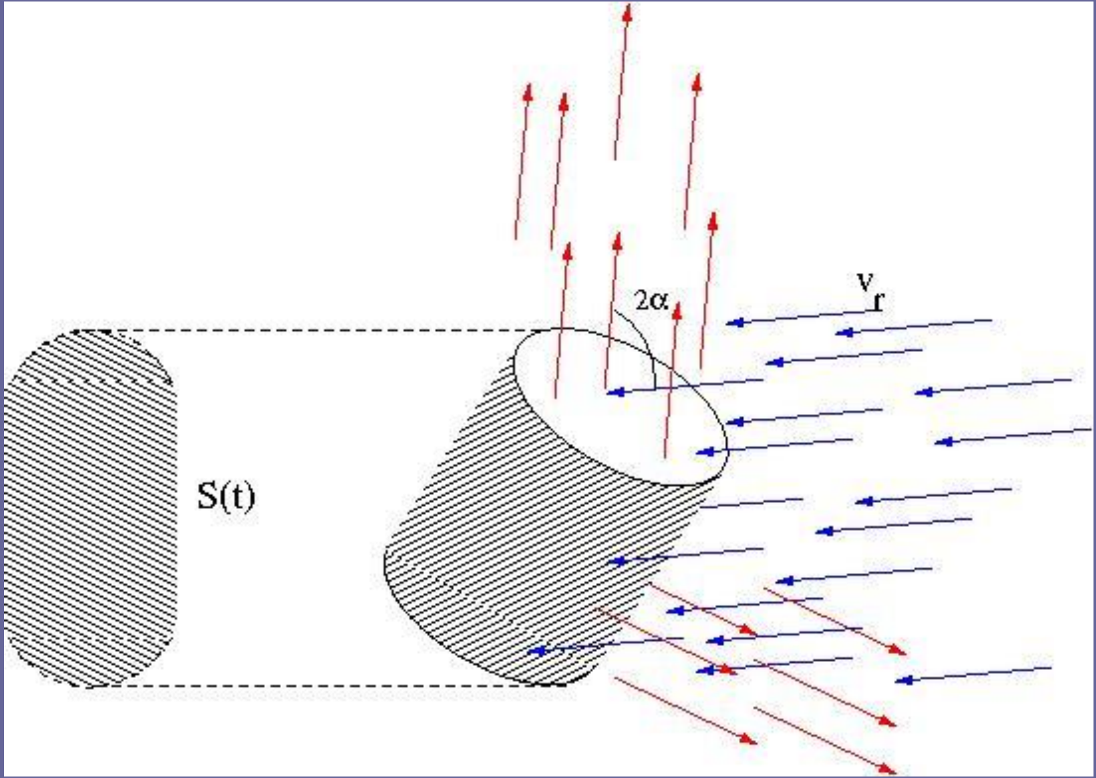
Equilibrium (*U*=*Cst* over the surface) implies a spherical shape for a fluid, non-rotating body!

5.1 The gravitational potential

Accurate predictions of the motions of satellites require a large number of terms in the expression of the potential: n = 360 in the EGM96 model.



5.2 The drag force due to the residual atmosphere



Body moving at relative velocity $\vec{v_r} = \dot{\vec{r}} - \vec{v_p}$

$$dV = S(t)\,|\vec{v_r}|\,dt$$

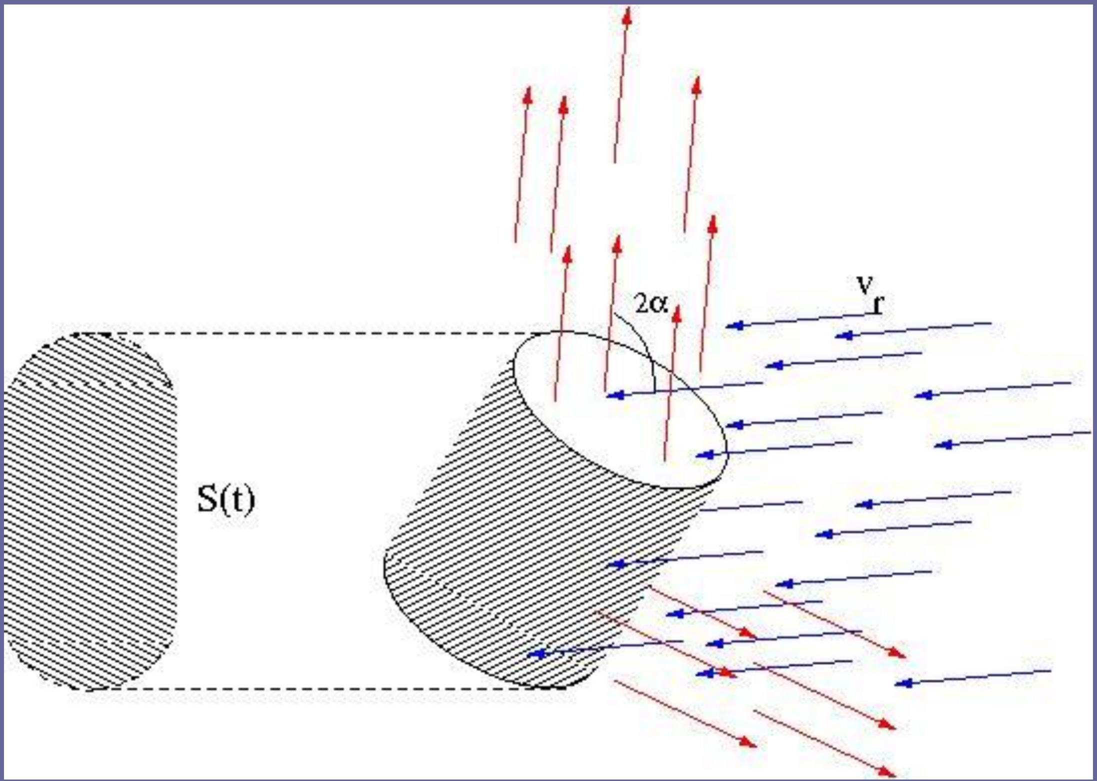
$$m\,d\dot{\vec{r}} = -\sum_p m_p\,d\vec{v_p}$$

$$|d\vec{v_p}| = |\vec{v_r}|\,(1+\cos(2\,\alpha))$$

$$\vec{F_D} = -\frac{C_D}{2}\,S(t)\,\rho\,|\vec{v_r}|\,\vec{v_r}$$

$$\frac{|\vec{F_D}|\,r^2}{G\,M\,m} = \frac{C_D}{2}\frac{S}{m}\,r\,\rho$$

5.3 Radiation pressure

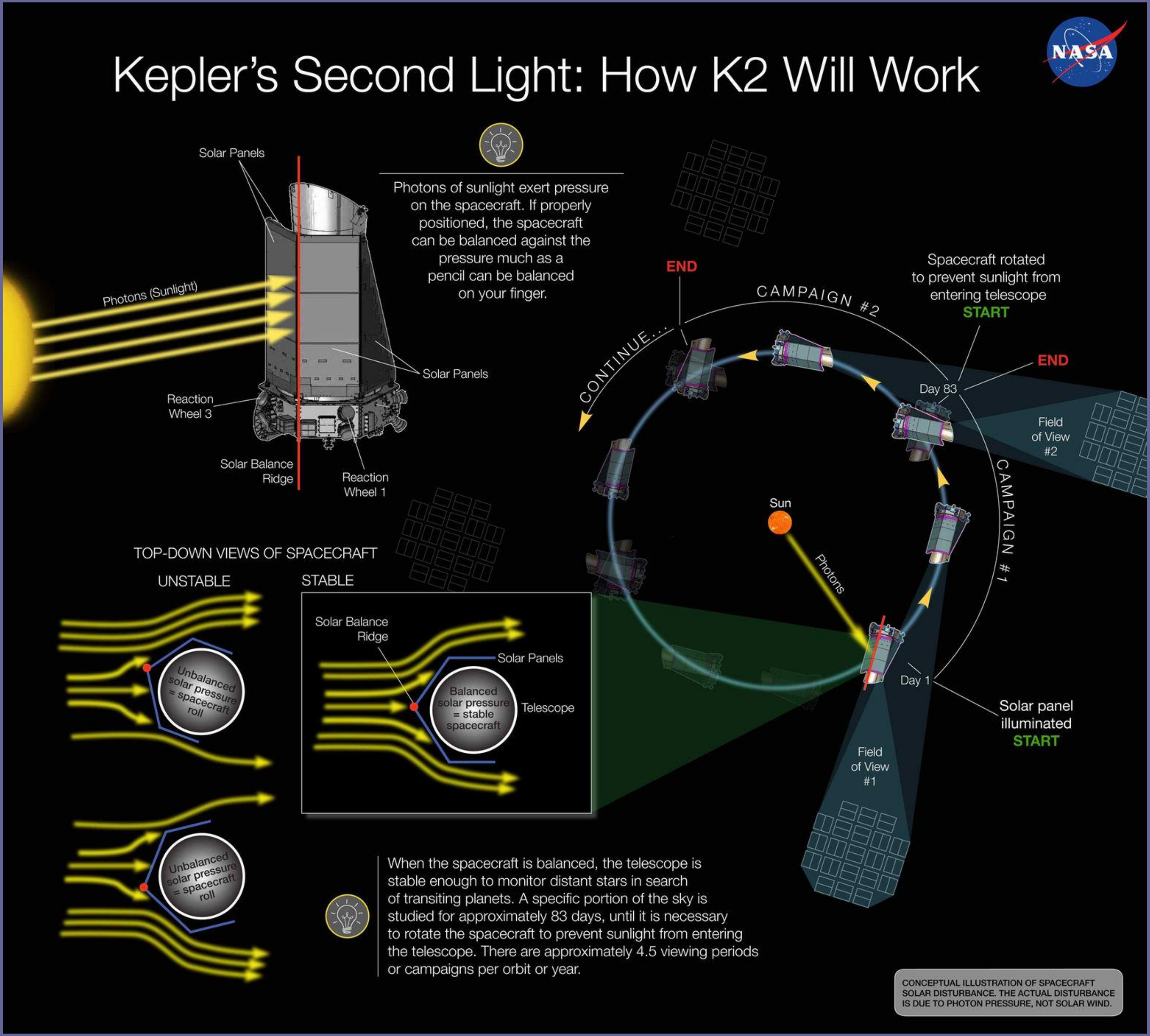


Similar reasoning as for the drag force:

$$\vec{F_R} = -\frac{C_R\, I}{c}\, S\, \vec{e_R}$$

Direct solar light + reflection by the planet

5.3 Radiation pressure



Chapter VI: Perturbations of the Keplerian motion

- Consider the following equation:

$$\ddot{\vec{r}} = -\frac{\mu\,\vec{r}}{r^3} + \vec{P}(\vec{r}, \dot{\vec{r}}, t)$$

- The solution of a Keplerian problem can be written as either
  - a combination of a conical section and a hodograph (6-dimensional space of the geometrical coordinates and the coordinates of the velocity vector). Representation of the orbit = 2 curves in two 3-dimensional subspaces.
  - the elements of the orbit (e.g. 6-dimensional space of Delaunay). Representation of the orbit = single point in a 6-dimensional space.

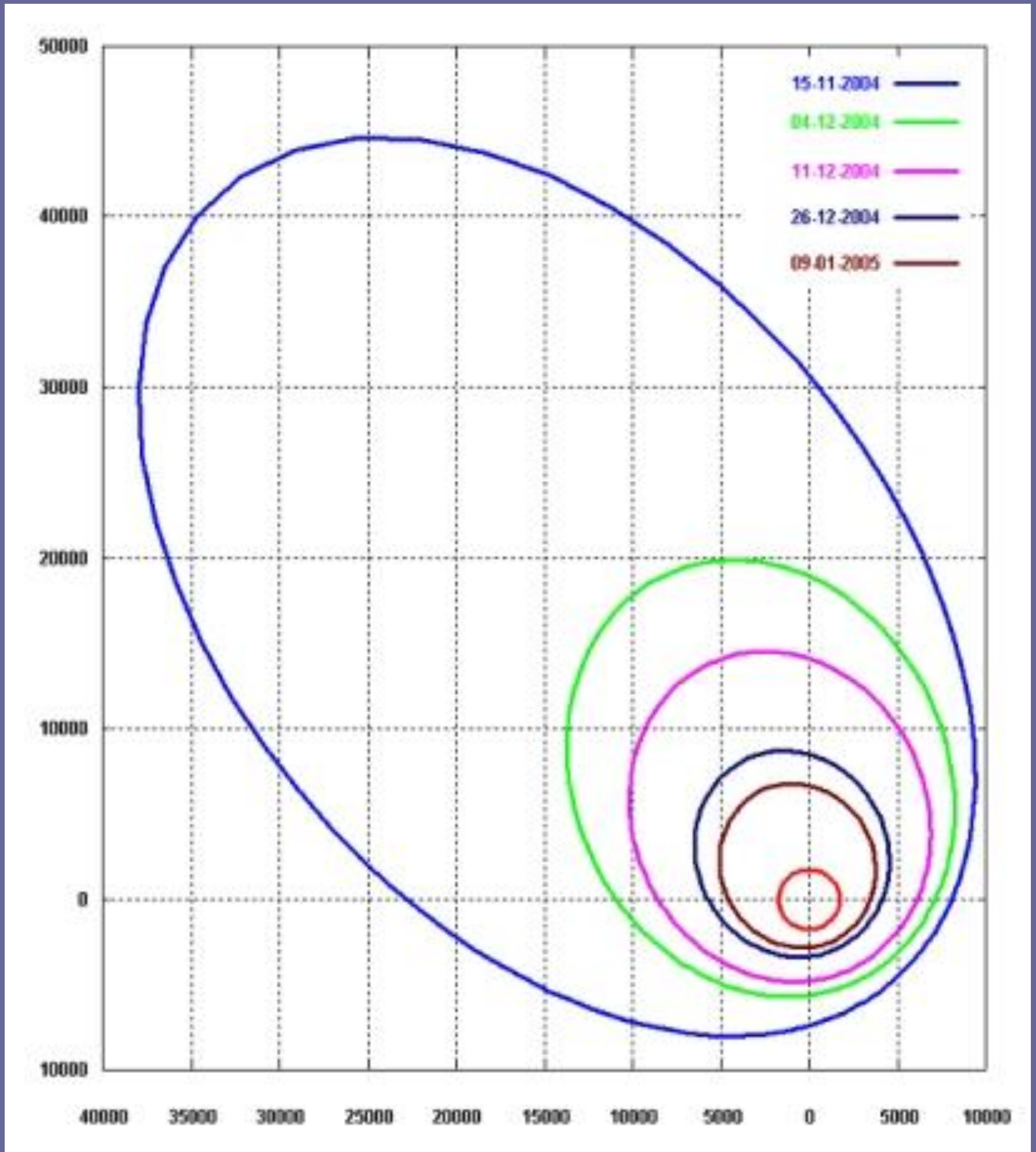
$$(l, g, \theta, L, G, \Theta) = (M, \omega, \Omega, \sqrt{\mu\,a}, \sqrt{\mu\,a\,(1-e^2)}, \sqrt{\mu\,a\,(1-e^2)}\,\cos i)$$

Chapter VI: Perturbations of the Keplerian motion

- When the problem becomes non-Keplerian, we define an ***osculating orbit*** at time t = the Keplerian trajectory with the same velocity and same position as the real trajectory at time t. The mass would follow this orbit if the non-Keplerian forces would disappear at time t.
- At each moment, one can define the 6 elements of the osculating orbit. However, these elements are no longer constant as a function of time and the point that describes the osculating orbit in the Delaunay space changes with time $\Rightarrow$ Method of the variations of the parameters.

Chapter VI: Perturbations of the Keplerian motion

Example: osculating orbits of Smart 1 upon its approach to the Moon.



6.1 Method of the variations of the parameters and …

- For a Keplerian trajectory, we express the coordinates and the velocities as a function of the elements of the orbit:

$$x_j = x_j(a, e, i, \Omega, \omega, t_0, t) \quad \& \quad u_j = u_j(a, e, i, \Omega, \omega, t_0, t)$$

- With $u_j = \dfrac{\partial x_j}{\partial t}$ & $\dfrac{\partial u_j}{\partial t} = -\dfrac{\mu x_j}{r^3}$ for a Keplerian motion

$$\frac{d x_j}{dt} = \frac{\partial x_j}{\partial a}\frac{d a}{dt} + \frac{\partial x_j}{\partial e}\frac{d e}{dt} + \ldots + \frac{\partial x_j}{\partial t_0}\frac{d t_0}{dt} + \frac{\partial x_j}{\partial t} = u_j$$

$$\frac{d u_j}{dt} = \frac{\partial u_j}{\partial a}\frac{d a}{dt} + \frac{\partial u_j}{\partial e}\frac{d e}{dt} + \ldots + \frac{\partial u_j}{\partial t_0}\frac{d t_0}{dt} + \frac{\partial u_j}{\partial t} = -\frac{\mu x_j}{r^3} + P_j(x_j, u_j, t)$$

6.1 Method of the variations of the parameters and …

- System of 6 equations allowing to determine the time derivatives of the 6 osculating elements:

$$
\begin{aligned}
\frac{\partial x_j}{\partial a}\frac{d\,a}{dt} + \frac{\partial x_j}{\partial e}\frac{d\,e}{dt} + \ldots + \frac{\partial x_j}{\partial t_0}\frac{d\,t_0}{dt} &= 0 \\
\frac{\partial u_j}{\partial a}\frac{d\,a}{dt} + \frac{\partial u_j}{\partial e}\frac{d\,e}{dt} + \ldots + \frac{\partial u_j}{\partial t_0}\frac{d\,t_0}{dt} &= P_j(x_j, u_j, t)
\end{aligned}
$$

The partial derivatives need to be computed following the relations of the Keplerian motion.

6.1… Gauss equations

- Alternative approach: distinguish between the Keplerian and non-Keplerian part of the temporal variation of a quantity *X*:

$$\frac{d\,X}{dt} = \left(\frac{d\,X}{dt}\right)_{\text{Kepler}} + \frac{\delta\,X}{\delta t}$$

- For the osculating orbit, we get: $\frac{\delta \vec{r}}{\delta t} = \vec{0}$ & $\frac{\delta \dot{\vec{r}}}{\delta t} = \vec{P}$

- For the angular momentum, Laplace integral and energy, we get:

$$\vec{r} \wedge \dot{\vec{r}} = \vec{h}$$

$$\frac{\dot{\vec{r}} \wedge \vec{h}}{\mu} - \vec{e_r} = \vec{l}$$

$$\frac{1}{2}(\dot{r}^2 + r^2\,\dot{\psi}^2) - \frac{\mu}{r} = \mathcal{E}$$

$$\begin{aligned}
\frac{d\,\vec{h}}{dt} &= \vec{r} \wedge \frac{\delta \dot{\vec{r}}}{\delta t} = \vec{r} \wedge \vec{P} \\
\mu\,\frac{d\,\vec{l}}{dt} &= 2\,(\dot{\vec{r}} \cdot \vec{P})\,\vec{r} - (\dot{\vec{r}} \cdot \vec{r})\,\vec{P} - (\vec{r} \cdot \vec{P})\,\dot{\vec{r}} \\
\frac{d\,\varepsilon}{dt} &= \dot{\vec{r}} \cdot \vec{P}
\end{aligned}$$

6.1 … Gauss equations

- Instantaneous angular velocity of the osculating motion:

$$\vec{\Omega} = \dot{\Omega}\,\vec{e_z} + \frac{d\,i}{dt}\,\vec{e_{X'}} + \dot{\omega}\,\vec{e_{Z'}} = \frac{d\,i}{dt}\,\vec{e_{X'}} + \dot{\Omega}\,\sin i\,\vec{e_{Y'}} + (\dot{\Omega}\,\cos i + \dot{\omega})\,\vec{e_{Z'}}$$

- Other formulation of the derivatives of the angular momentum and Laplace integral:

$$\frac{d\,\vec{h}}{dt} = \dot{h}\,\vec{e_{Z'}} + \vec{\Omega}\wedge\vec{h}$$

$$\frac{d\,\vec{l}}{dt} = \frac{d\,(e\,\vec{u_0})}{dt} = \dot{e}\,\vec{u_0} + \vec{\Omega}\wedge\vec{l}$$

$\Rightarrow$

$$\begin{aligned}
\dot{h} &= (\vec{r}\wedge\vec{P})\cdot\vec{e_{Z'}} = r\,\vec{e_\phi}\cdot\vec{P} \\
h\,\frac{d\,i}{dt} &= -(\vec{r}\wedge\vec{P})\cdot\vec{e_{Y'}} = r\,\cos(\phi+\omega)\,\vec{e_{Z'}}\cdot\vec{P} \\
h\,\dot{\Omega}\sin i &= (\vec{r}\wedge\vec{P})\cdot\vec{e_{X'}} = r\,\sin(\phi+\omega)\,\vec{e_{Z'}}\cdot\vec{P} \\
\mu\,\dot{e} &= 2\,(\dot{\vec{r}}\cdot\vec{P})\,(\vec{r}\cdot\vec{u_0}) - (\dot{\vec{r}}\cdot\vec{r})\,(\vec{P}\cdot\vec{u_0}) - (\vec{r}\cdot\vec{P})\,(\dot{\vec{r}}\cdot\vec{u_0}) \\
\mu\,e\,(\dot{\omega} + \dot{\Omega}\,\cos i) &= 2\,(\dot{\vec{r}}\cdot\vec{P})\,(\vec{r}\cdot\vec{v_0}) - (\dot{\vec{r}}\cdot\vec{r})\,(\vec{P}\cdot\vec{v_0}) - (\vec{r}\cdot\vec{P})\,(\dot{\vec{r}}\cdot\vec{v_0}) \\
\frac{\mu}{2\,a^2}\,\dot{a} &= \dot{\vec{r}}\cdot\vec{P}
\end{aligned}$$

6.1 … Gauss equations

- Variation of the mean anomaly, Keplerian part

$$n(t) = n(t_0) - \int_{t_0}^{t} \frac{3}{2} \sqrt{\mu}\, a^{-5/2}\, \dot{a}\, dt'$$

- and non-Keplerian part:

$$\frac{\delta M}{\delta t} = -\sqrt{1-e^2}\,(\dot{\omega} + \dot{\Omega}\cos i) - \frac{2\,r}{\sqrt{\mu\, a}}(\vec{P}\cdot\vec{e_r})$$

$$\Rightarrow \quad \dot{M} = n(t) - \sqrt{1-e^2}\,(\dot{\omega} + \dot{\Omega}\cos i) - \frac{2\,r}{\sqrt{\mu\, a}}(\vec{P}\cdot\vec{e_r})$$

- We express the force in cylindrical coordinates:

$$\vec{P} = R\,\vec{e_r} + T\,\vec{e_\phi} + W\,\vec{e_{Z'}}$$

6.1 … Gauss equations

- Gauss equations: $\vec{P} = R\,\vec{e_r} + T\,\vec{e_\phi} + W\,\vec{e_{Z'}}$

$$\dot{a} = \sqrt{\frac{a^3}{\mu}}\frac{2}{\sqrt{1-e^2}}\left[R\,(e\,\sin\phi) + T\,(1 + e\,\cos\phi)\right]$$

$$\dot{e} = \sqrt{\frac{a\,(1-e^2)}{\mu}}\left\{R\,\sin\phi + T\left[\cos\phi + \frac{r}{a\,(1-e^2)}\,(e + \cos\phi)\right]\right\}$$

$$\frac{d\,i}{dt} = \frac{r\,\cos(\omega+\phi)}{\sqrt{a\,\mu\,(1-e^2)}}\,W$$

$$\dot{\Omega} = \frac{r\,\sin(\omega+\phi)}{\sqrt{a\,\mu\,(1-e^2)}\,\sin i}\,W$$

$$\dot{\omega} = \sqrt{\frac{a\,(1-e^2)}{\mu\,e^2}}\left[-R\,\cos\phi + T\,\sin\phi\,(1 + \frac{r}{a\,(1-e^2)})\right] - \frac{r\,\sin(\omega+\phi)}{\sqrt{a\,\mu\,(1-e^2)}}\,\cot i\,W$$

$$\dot{M} = n(t) - \frac{1-e^2}{e}\sqrt{\frac{a}{\mu}}\left[-R\,\cos\phi + T\,\sin\phi\,(1 + \frac{r}{a\,(1-e^2)})\right] - \frac{2\,r}{\sqrt{a\,\mu}}\,R$$

6.1.1 Application to atmospheric drag

- Atmospheric drag force: $\vec{F_D} = -\frac{C_D}{2}\, S(t)\, \rho\, |\vec{v_r}|\, \vec{v_r}$

$\Rightarrow$ $\vec{P} = -k\,\dot{\vec{r}}$ & $k = \frac{C_D\, S}{2\, m}\, \rho\, \frac{n\, a}{\sqrt{1-e^2}}\, \sqrt{1 + e^2 + 2\, e\, \cos\phi}$

$\Rightarrow$ $R = -k\,\dot{r},\; T = -k\, r\, \dot{\phi}$ & $W = 0$

$\Rightarrow$ $\frac{d\,i}{dt} = \dot{\Omega} = 0$

$$\dot{a} = -C_D\, \frac{S}{m}\, \rho\, n\, a^2 \left[\frac{1 + e^2 + 2\, e\, \cos\phi}{1 - e^2}\right]^{3/2}$$

$$\dot{e} = -C_D\, \frac{S}{m}\, \rho\, \frac{n\, a}{\sqrt{1 - e^2}}\, \sqrt{1 + e^2 + 2\, e\, \cos\phi}\, (e + \cos\phi)$$

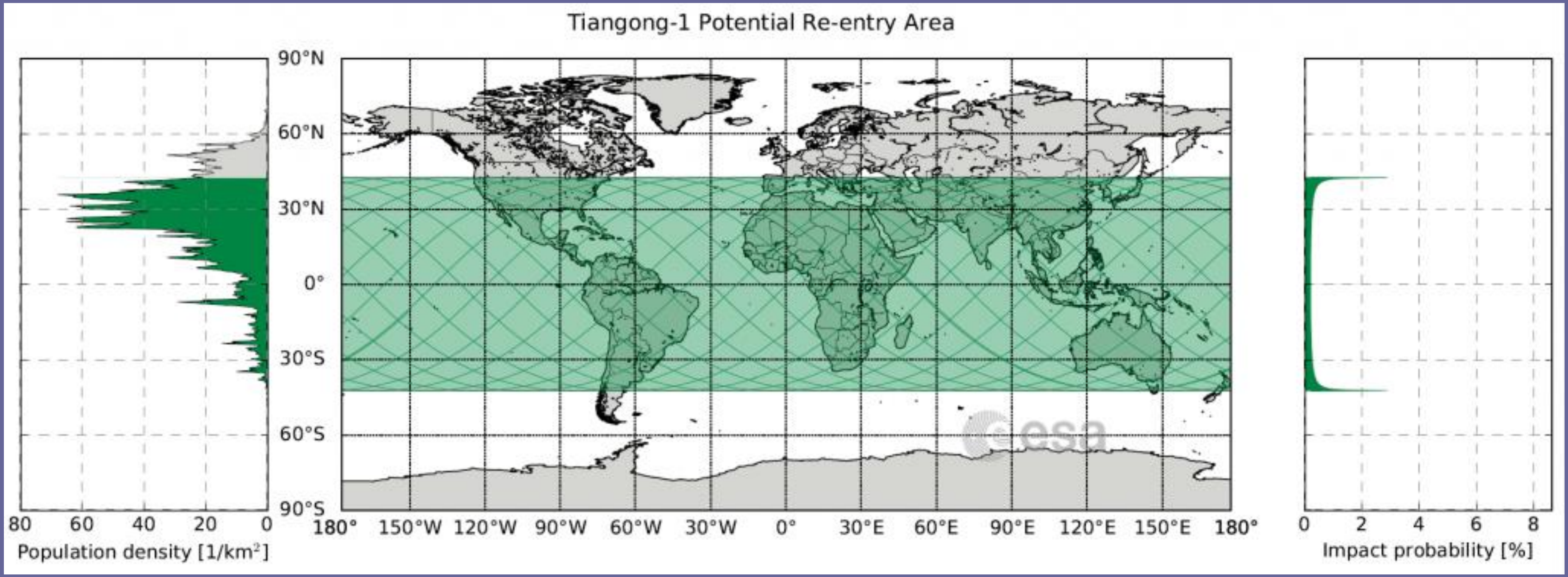
$$\dot{\omega} = -C_D\, \frac{S}{m}\, \rho \frac{n\, a}{e\, \sqrt{1 - e^2}}\, \sqrt{1 + e^2 + 2\, e\, \cos\phi}\, \sin\phi$$

$$\dot{M} = n(t) + C_D\, \frac{S}{m}\, \rho\, n\, a \sqrt{1 + e^2 + 2\, e\, \cos\phi}\, \frac{\sin\phi}{e}\, \frac{1 + e^2 + e\, \cos\phi}{1 + e\, \cos\phi}$$

6.1.1 Application to atmospheric drag

$$\frac{di}{dt} = \dot{\Omega} = 0$$

- This delimits the zone for atmospheric re-entry of space debris e.g. defunct Chinese space station Tiangong-1 in April 2018:



6.1.1 Application to the J_2 term of the geopotential

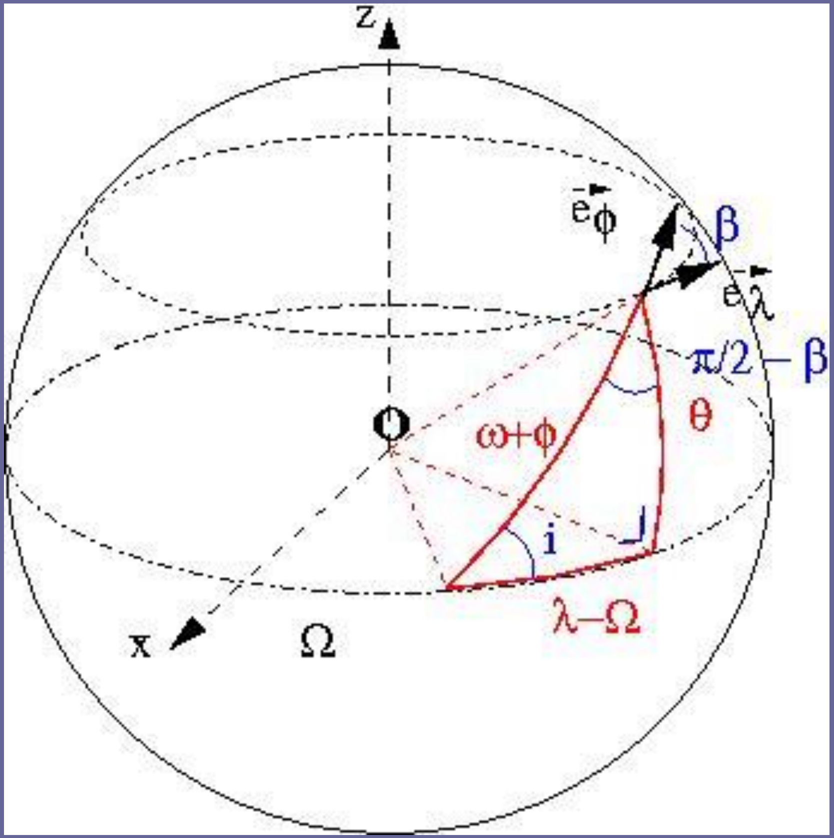
- Conservative force: $\vec{P} = -\vec{\nabla} U'$ with

$$U' = J_2\,\mu \frac{R_e^2}{r^3}\,(\frac{3}{2}\,\sin^2\theta - \frac{1}{2})$$

$$\Rightarrow \quad \begin{aligned} \vec{P} &= -(\frac{\partial U'}{\partial r}, \frac{1}{r\,\cos\theta}\frac{\partial U'}{\partial \lambda}, \frac{1}{r}\frac{\partial U'}{\partial \theta}) \\ &= (J_2\,\mu \frac{R_e^2}{r^4}\,(\frac{9}{2}\,\sin^2\theta - \frac{3}{2}), 0, -3\,J_2\,\mu \frac{R_e^2}{r^4}\,\sin\theta\,\cos\theta) \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} R &= J_2\,\mu \frac{R_e^2}{r^4}\,(\frac{9}{2}\,\sin^2\theta - \frac{3}{2}) \\ T &= -3\,J_2\,\mu \frac{R_e^2}{r^4}\,\sin\theta\,\cos\theta\,\sin\beta \\ W &= -3\,J_2\,\mu \frac{R_e^2}{r^4}\,\sin\theta\,\cos\theta\,\cos\beta \end{aligned}$$

6.1.1 Application to the J_2 term of the geopotential



$\Rightarrow$

$$\sin\theta = \sin i \, \sin(\omega + \phi)$$

$$\sin\theta \, \cos\theta \, \sin\beta = \sin^2 i \, \sin(\omega + \phi) \, \cos(\omega + \phi)$$

$$\sin\theta \, \cos\theta \, \cos\beta = \sin i \, \cos i \, \sin(\omega + \phi)$$

6.1.1 Application to the J_2 term of the geopotential

$$\sin\theta = \sin i\ \sin(\omega+\phi)$$

$$\sin\theta\ \cos\theta\ \sin\beta = \sin^2 i\ \sin(\omega+\phi)\ \cos(\omega+\phi)$$

$$\sin\theta\ \cos\theta\ \cos\beta = \sin i\ \cos i\ \sin(\omega+\phi)$$

$$R = J_2\,\mu\frac{R_e^2}{r^4}\,(\frac{9}{2}\ \sin^2\theta - \frac{3}{2})$$

$$T = -3\,J_2\,\mu\frac{R_e^2}{r^4}\ \sin\theta\ \cos\theta\ \sin\beta$$

$$W = -3\,J_2\,\mu\frac{R_e^2}{r^4}\ \sin\theta\ \cos\theta\ \cos\beta$$

$\Rightarrow$

$$R = \frac{3}{2}\,J_2\,\mu\frac{R_e^2}{r^4}\,(3\ \sin^2 i\ \sin^2(\omega+\phi) - 1)$$

$$T = -3\,J_2\,\mu\frac{R_e^2}{r^4}\ \sin^2 i\ \sin(\omega+\phi)\ \cos(\omega+\phi)$$

$$W = -3\,J_2\,\mu\frac{R_e^2}{r^4}\ \sin i\ \cos i\ \sin(\omega+\phi)$$

6.1.1 Application to the J_2 term of the geopotential

- Using the Gauss equations:

$$\dot{a} = \frac{3\,J_2\,n\,a}{4\sqrt{1-e^2}}\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^4\Big[e\,\sin^2 i\,(6\,\sin\phi - 5\,\sin(2\,\omega+3\,\phi)+\sin(2\,\omega+\phi))$$

$$-4\,e\,\sin\phi - 4\,\sin^2 i\,\sin(2\,\omega+2\,\phi)\Big]$$

$$\dot{e} = \frac{3}{8}J_2\,n\sqrt{1-e^2}\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^4\left[6\,\sin^2 i\,\sin\phi - 5\,\sin^2 i\,\sin(2\,\omega+3\,\phi)+\sin^2 i\,\sin(2\,\omega+\phi) - 4\,\sin\phi\right]$$

$$-\frac{3}{4}\frac{J_2\,n}{\sqrt{1-e^2}}\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^3\sin^2 i\,\left[2\,e\,\sin(2\,\omega+2\,\phi)+\sin(2\,\omega+\phi)+\sin(2\,\omega+3\,\phi)\right]$$

$$\frac{d\,i}{dt} = \frac{-3\,J_2\,n}{2\sqrt{1-e^2}}\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^3\sin i\,\cos i\,\sin(2\,\omega+2\,\phi)$$

$$\dot{\Omega} = \frac{-3\,J_2\,n}{2\sqrt{1-e^2}}\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^3\cos i\,(1-\cos(2\,\omega+2\,\phi))$$

$$\dot{\omega} = \frac{3}{8}J_2\,n\frac{\sqrt{1-e^2}}{e}\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^4\left[4\,\cos\phi - \sin^2 i\,(6\,\cos\phi - 5\,\cos(2\,\omega+3\,\phi) - \cos(2\,\omega+\phi))\right]$$

$$+\frac{3}{4}\frac{J_2\,n}{e\sqrt{1-e^2}}\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^3\sin^2 i\,\left[\cos(2\,\omega+3\,\phi)-\cos(2\,\omega+\phi)\right]+\dot{\Omega}\,\cos i$$

$$\dot{M} = n(t) - \sqrt{1-e^2}\,(\dot{\omega}+\dot{\Omega}\,\cos i) + 3\,J_2\,n\left(\frac{R_e}{a}\right)^2\left(\frac{a}{r}\right)^3\left[1-\frac{3}{2}\sin^2 i\,(1-\cos(2\,\omega+2\,\phi))\right]$$

6.2 The Lagrange equations

- Hamiltonian of the Keplerian problem with the canonical Delaunay elements:

$$\mathcal{H} = -\frac{\mu^2}{2\,L^2}$$

$$(l, g, \theta, L, G, \Theta) = \left(M, \omega, \Omega, \sqrt{\mu\, a}, \sqrt{\mu\, a\,(1-e^2)}, \sqrt{\mu\, a\,(1-e^2)}\, \cos i\right)$$

- If the force $\vec{P}$ is conservative (potential *U'*), then the new Hamiltonian can be expressed as:

$$\mathcal{H}' = -\frac{\mu^2}{2\,L^2} + U'(l, g, \theta, L, G, \Theta)$$

- Hamilton's canonical equations yield:

$$\frac{d\,L}{dt} = -\frac{\partial\,U'}{\partial l} \qquad \frac{d\,l}{dt} = \frac{\mu^2}{L^3} + \frac{\partial\,U'}{\partial L}$$

$$\frac{d\,G}{dt} = -\frac{\partial\,U'}{\partial g} \qquad \frac{d\,g}{dt} = \frac{\partial\,U'}{\partial G}$$

$$\frac{d\,\Theta}{dt} = -\frac{\partial\,U'}{\partial \theta} \qquad \frac{d\,\theta}{dt} = \frac{\partial\,U'}{\partial \Theta}$$

6.2 The Lagrange equations

$$(l, g, \theta, L, G, \Theta) = (M, \omega, \Omega, \sqrt{\mu\, a}, \sqrt{\mu\, a\,(1-e^2)}, \sqrt{\mu\, a\,(1-e^2)}\,\cos i)$$

$$\begin{aligned}
\mu\, a = L^2 \quad &\Rightarrow \quad \mu\, da = 2\, L\, dL \\
e^2 = 1 - \frac{G^2}{L^2} \quad &\Rightarrow \quad e\, de = \frac{G^2}{L^3}\, dL - \frac{G}{L^2}\, dG \\
\cos i = \frac{\Theta}{G} \quad &\Rightarrow \quad \sin i\, di = \frac{\Theta}{G^2}\, dG - \frac{1}{G}\, d\Theta
\end{aligned}$$

&

$$\begin{aligned}
\frac{d\,L}{dt} &= -\frac{\partial\, U'}{\partial l} \\
\frac{d\,G}{dt} &= -\frac{\partial\, U'}{\partial g} \\
\frac{d\,\Theta}{dt} &= -\frac{\partial\, U'}{\partial \theta}
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\frac{d\,a}{dt} &= \frac{2\,L}{\mu}\frac{d\,L}{dt} = -\frac{2}{n\,a}\frac{\partial\, U'}{\partial M} \\
\frac{d\,e}{dt} &= \frac{1}{e}\left[\frac{h^2}{(\mu\, a)^{3/2}}\frac{d\,L}{dt} - \frac{h}{\mu\, a}\frac{d\,G}{dt}\right] = \frac{1}{e}\left[\frac{e^2-1}{n\,a^2}\frac{\partial\, U'}{\partial M} + \frac{\sqrt{1-e^2}}{n\,a^2}\frac{\partial\, U'}{\partial \omega}\right] \\
\frac{d\,i}{dt} &= \frac{1}{n\,a^2\sqrt{1-e^2}\,\sin i}\left[\frac{\partial\, U'}{\partial \Omega} - \cos i\,\frac{\partial\, U'}{\partial \omega}\right]
\end{aligned}$$

6.2 The Lagrange equations

$$(l, g, \theta, L, G, \Theta) = (M, \omega, \Omega, \sqrt{\mu\, a}, \sqrt{\mu\, a\,(1-e^2)}, \sqrt{\mu\, a\,(1-e^2)}\,\cos i)$$

$$\begin{aligned} \mu\, a = L^2 \quad &\Rightarrow \quad \mu\, da = 2\,L\,dL \\ e^2 = 1 - \frac{G^2}{L^2} \quad &\Rightarrow \quad e\,de = \frac{G^2}{L^3}\,dL - \frac{G}{L^2}\,dG \\ \cos i = \frac{\Theta}{G} \quad &\Rightarrow \quad \sin i\,di = \frac{\Theta}{G^2}\,dG - \frac{1}{G}\,d\Theta \end{aligned}$$

&

$$\begin{aligned} \frac{d\,l}{dt} &= \frac{\mu^2}{L^3} + \frac{\partial\,U'}{\partial L} \\ \frac{d\,g}{dt} &= \frac{\partial\,U'}{\partial G} \\ \frac{d\,\theta}{dt} &= \frac{\partial\,U'}{\partial \Theta} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \frac{d\,\Omega}{dt} &= \frac{-1}{n\,a^2\,\sqrt{1-e^2}\,\sin i}\,\frac{\partial\,U'}{\partial i} \\ \frac{d\,\omega}{dt} &= \frac{\partial\,U'}{\partial i}\,\frac{\partial\,i}{\partial G} + \frac{\partial\,U'}{\partial e}\,\frac{\partial\,e}{\partial G} = \frac{\cos i}{n\,a^2\,\sqrt{1-e^2}\,\sin i}\,\frac{\partial\,U'}{\partial i} - \frac{\sqrt{1-e^2}}{n\,a^2\,e}\,\frac{\partial\,U'}{\partial e} \\ \frac{d\,M}{dt} &= \frac{\mu^2}{(\mu\,a)^{3/2}} + \frac{\partial\,U'}{\partial a}\,\frac{\partial\,a}{\partial L} + \frac{\partial\,U'}{\partial e}\,\frac{\partial\,e}{\partial L} = n(t) + \frac{2}{n\,a}\,\frac{\partial\,U'}{\partial a} + \frac{1-e^2}{n\,a^2\,e}\,\frac{\partial\,U'}{\partial e} \end{aligned}$$

6.2 The Lagrange equations

- More compact formulation of the Lagrange equations (antisymmetric matrix):

$$\begin{bmatrix} \frac{da}{dt} \\ \frac{dM}{dt} \\ \frac{de}{dt} \\ \frac{d\omega}{dt} \\ \frac{di}{dt} \\ \frac{d\Omega}{dt} \end{bmatrix} = \begin{bmatrix} 0 \\ n(t) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{n\,a^2} \begin{bmatrix} 0 & -2\,a & 0 & 0 & 0 & 0 \\ 2\,a & 0 & \frac{1-e^2}{e} & 0 & 0 & 0 \\ 0 & -\frac{1-e^2}{e} & 0 & \frac{\sqrt{1-e^2}}{e} & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{1-e^2}}{e} & 0 & \frac{\cos i}{\sqrt{1-e^2}\,\sin i} & 0 \\ 0 & 0 & 0 & -\frac{\cos i}{\sqrt{1-e^2}\,\sin i} & 0 & \frac{1}{\sqrt{1-e^2}\,\sin i} \\ 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{1-e^2}\,\sin i} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial U'}{\partial a} \\ \frac{\partial U'}{\partial M} \\ \frac{\partial U'}{\partial e} \\ \frac{\partial U'}{\partial \omega} \\ \frac{\partial U'}{\partial i} \\ \frac{\partial U'}{\partial \Omega} \end{bmatrix}$$

6.3 Differential equations depending on a small parameter

- Let η be a small parameter and consider the differential equation

$$\frac{dx}{dt} = f(x, \eta)$$

- If $\eta = 0$, this equation has the solution $x_0(t)$. We seek solutions of the kind

$$x(t,\eta) = x_0(t) + \eta\, x_1(t) + \eta^2\, x_2(t) + \ldots$$

- The differential equation can be written to 2^{nd} order in η:

$$\begin{aligned}
\frac{dx_0}{dt} + \eta\,\frac{dx_1}{dt} + \eta^2\,\frac{dx_2}{dt} + \ldots &= f(x_0 + \eta\, x_1 + \eta^2\, x_2 + \ldots, t, 0 + \eta) \\
&= f(x_0, t, 0) + \frac{\partial f}{\partial x_0}(\eta\, x_1 + \eta^2\, x_2 + \ldots) + \frac{1}{2}\frac{\partial^2 f}{\partial x_0^2}(\eta\, x_1 + \eta^2\, x_2 + \ldots)^2 \\
&\quad + \left.\frac{\partial f}{\partial \eta}\right)_{\eta=0} \eta + \frac{\partial^2 f}{\partial x_0\, \partial \eta}(\eta\, x_1 + \eta^2\, x_2 + \ldots)\,\eta + \left.\frac{1}{2}\frac{\partial^2 f}{\partial \eta^2}\right)_{\eta=0} \eta^2 \quad (
\end{aligned}$$

6.3 Differential equations depending on a small parameter

- Comparing the coefficients of the different powers of η, we find that:

$$\begin{aligned} \frac{d\,x_0}{dt} &= f(x_0, t, 0) \\ \frac{d\,x_1}{dt} &= \left.\frac{\partial f}{\partial x_0} x_1 + \frac{\partial f}{\partial \eta}\right)_{\eta=0} \\ \frac{d\,x_2}{dt} &= \left.\frac{\partial f}{\partial x_0} x_2 + \frac{1}{2}\frac{\partial^2 f}{\partial x_0^2} x_1^2 + \frac{\partial^2 f}{\partial x_0\,\partial \eta} x_1 + \frac{1}{2}\frac{\partial^2 f}{\partial \eta^2}\right)_{\eta=0} \\ \ldots \end{aligned}$$

- These equations depend on n + 1 constants of integration, whereas the original equation depends on a single constant of integration.

6.3 Differential equations depending on a small parameter

- Determination of the constants of integration.

$$x(t,\eta) = x_0(t,\alpha_0) + \eta\, x_1(t,\alpha_1) + \eta^2\, x_2(t,\alpha_2) + \ldots + \eta^n\, x_n(t,\alpha_n) + \mathcal{O}(\eta^{n+1})$$

1. We set $x_0(t_0,\alpha_0) = X_0 = x(t_0)$ and $x_j(t_0,\alpha_j) = 0 \quad \forall j = 1,\ldots,n$
2. We only add a single constant to the solution of the first equation

$$x(t) = x_0(t,\alpha_0) + \eta\, x_1(t) + \eta^2\, x_2(t) + \ldots + \eta^n\, x_n(t) + \mathcal{O}(\eta^{n+1})$$

and we require that:

$$x(t_0) = X_0 = x_0(t_0,\alpha_0) + \eta\, x_1(t_0) + \eta^2\, x_2(t_0) + \ldots + \eta^n\, x_n(t_0) + \mathcal{O}(\eta^{n+1})$$

- Both methods are equivalent to each other at the order η^n.

6.4 Secular, periodic and mixed terms

- Suppose that the force $\overrightarrow{P}$ is conservative (potential *U'*) and represents a small perturbation (parameter η):

$$U' = \eta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[U_{jkl}^{(1)}(a,e,i) + \eta\, U_{jkl}^{(2)}(a,e,i) + \ldots \right] \cos\left(j\,\Omega + k\,\omega + l\,M\right)$$

- We express the variations of the metric elements *(a, e, i)* through the Lagrange equations:

$$\dot{\alpha} = \eta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[A_{jkl}^{(1)}(a,e,i) + \eta\, A_{jkl}^{(2)}(a,e,i) + \ldots \right] \sin\left(j\,\Omega + k\,\omega + l\,M\right)$$

- In the same way for the angular elements *(Ω, ω, M):*

$$\dot{\beta} = n(t)\,\delta_{\beta M} + \eta \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \left[B_{jkl}^{(1)}(a,e,i) + \eta\, B_{jkl}^{(2)}(a,e,i) + \ldots \right] \cos\left(j\,\Omega + k\,\omega + l\,M\right)$$

$$n(t) = n_0(a_0) + \frac{d\,n}{da_0}\left(a_1\,\eta + a_2\,\eta^2 + \ldots\right) + \frac{1}{2}\,\frac{d^2\,n}{da_0^2}\left(a_1\,\eta + a_2\,\eta^2 + \ldots\right)^2 + \ldots$$

6.4 Secular, periodic and mixed terms

- We are searching solutions of the kind

$$\alpha(t,\eta) = \alpha_0 + \eta\,\alpha_1(t) + \eta^2\,\alpha_2(t) + \ldots$$

- At 0th order in η:

$$\begin{aligned} \dot{\alpha_0} &= 0 \\ \dot{\beta_0} &= n_0\,\delta_{\beta M} \end{aligned}$$

At 0th order, the osculating elements are either constants or linear function of time (*M*): $M = n_0\,t + m_0$

- At 1st order for the metric elements:

$$\dot{\alpha_1} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} A^{(1)}_{jkl}(a_0, e_0, i_0)\ \sin\left(j\,\Omega_0 + k\,\omega_0 + l\,M_0\right)$$

$$\Rightarrow \quad \alpha_1 = -\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=1}^{\infty} \frac{A^{(1)}_{jkl}(a_0, e_0, i_0)\ \cos\left(j\,\Omega_0 + k\,\omega_0 + l\,M_0\right)}{l\,n_0}$$

6.4 Secular, periodic and mixed terms

- At 1st order for the angular elements:

$$\dot{\beta}_1 = \frac{d\,n}{d a_0}\, a_1\, \delta_{\beta M} + B^{(1)}_{000}(a_0, e_0, i_0) + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} B^{(1)}_{jkl}(a_0, e_0, i_0)\ \cos\left(j\,\Omega_0 + k\,\omega_0 + l\,M_0\right)$$

Given the expression of a_1, we obtain:

$$\dot{\beta}_1 = B^{(1)}_{000}(a_0, e_0, i_0) + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} B'^{(1)}_{jkl}(a_0, e_0, i_0)\ \cos\left(j\,\Omega_0 + k\,\omega_0 + l\,M_0\right)$$

$$\Rightarrow$$

$$\beta_1 = B^{(1)}_{000}(a_0, e_0, i_0)\, t + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \frac{B'^{(1)}_{jkl}(a_0, e_0, i_0)\ \sin\left(j\,\Omega_0 + k\,\omega_0 + l\,M_0\right)}{l\, n_0}$$

Secular term

6.4 Secular, periodic and mixed terms

- At 2nd order for the metric elements:

$$\dot{\alpha}_2 = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[A^{(2)}_{jkl}(a_0,e_0,i_0) + \frac{\partial A^{(1)}_{jkl}}{\partial a_0}\,a_1 + \frac{\partial A^{(1)}_{jkl}}{\partial e_0}\,e_1 + \frac{\partial A^{(1)}_{jkl}}{\partial i_0}\,i_1\right]\sin\left(j\,\Omega_0 + k\,\omega_0 + l\,M_0\right)$$
$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} A^{(1)}_{jkl}(a_0,e_0,i_0)\,(j\,\Omega_1 + k\,\omega_1 + l\,M_1)\,\cos\left(j\,\Omega_0 + k\,\omega_0 + l\,M_0\right)$$

$$\Rightarrow$$

$$\dot{\alpha}_2 = \sum_{j''=-\infty}^{\infty}\sum_{k''=-\infty}^{\infty}\sum_{l''=-\infty}^{\infty} A''_{j''k''l''}(a_0,e_0,i_0)\,\sin\left(j''\,\Omega_0 + k''\,\omega_0 + l''\,M_0\right)$$
$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} A'_{jkl}(a_0,e_0,i_0)\,t\,\cos\left(j\,\Omega_0 + k\,\omega_0 + l\,M_0\right)$$

6.4 Secular, periodic and mixed terms

- At 2nd order for the metric elements:

$$\dot{\alpha}_2 = \sum_{j''=-\infty}^{\infty} \sum_{k''=-\infty}^{\infty} \sum_{l''=-\infty}^{\infty} A''_{j''k''l''}(a_0, e_0, i_0) \sin(j''\,\Omega_0 + k''\,\omega_0 + l''\,M_0) + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} A'_{jkl}(a_0, e_0, i_0)\, t \cos(j\,\Omega_0 + k\,\omega_0 + l\,M_0)$$

$$\Rightarrow$$

$$\alpha_2 = -\sum_{j''=-\infty}^{\infty} \sum_{k''=-\infty}^{\infty} \sum_{l''=-\infty}^{\infty} \frac{A''_{j''k''l''}(a_0, e_0, i_0)}{l''\,n_0} \cos(j''\,\Omega_0 + k''\,\omega_0 + l''\,M_0) + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{A'_{jkl}(a_0, e_0, i_0)}{l\,n_0}\, t \sin(j\,\Omega_0 + k\,\omega_0 + l\,M_0) + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{A'_{jkl}(a_0, e_0, i_0)}{l^2\,n_0^2} \cos(j\,\Omega_0 + k\,\omega_0 + l\,M_0)$$

Mixed term

The increase with time of the mixed term restricts the validity of the development.

6.4 Secular, periodic and mixed terms

- How can we get rid of the mixed terms?
- Asymptotic development limited to *N* terms.

$$
\begin{aligned}
\alpha &= \alpha_0 + \sum_{n=1}^{N} \eta^n \, \alpha_n \\
\beta &= \beta_0 + \left(\sum_{n=1}^{N} \eta^n \, \overline{\beta_n} \right) t + \sum_{n=1}^{N} \eta^n \, \beta_n \\
&= \beta_0 + \eta \, \overline{\beta} \, t + \sum_{n=1}^{N} \eta^n \, \beta_n
\end{aligned}
$$

- The overlined quantities in the expressions of the angular elements are constants (angular velocities) that need to be determined.

6.4 Secular, periodic and mixed terms

- We introduce the arguments of the trigonometric functions:

$$\begin{aligned}
\gamma^0_{jkl} &= j\,\Omega_0 + k\,\omega_0 + l\,M_0 \\
\overline{\gamma}_{jkl} &= j\,\overline{\Omega} + k\,\overline{\omega} + l\,\overline{M} \\
\gamma^n_{jkl} &= j\,\Omega_n + k\,\omega_n + l\,M_n
\end{aligned}$$

- Leading to:

$$\begin{aligned}
\dot{\alpha}_0 + \eta\,\dot{\alpha}_1 + \eta^2\,\dot{\alpha}_2 \quad = \quad & \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\eta\,A^{(1)}_{jkl}(a_0,e_0,i_0) + \eta^2 \sum_{\alpha}\frac{\partial\,A^{(1)}_{jkl}}{\partial\alpha_0}\,\alpha_1 + \eta^2\,A^{(2)}_{jkl}(a_0,e_0,i_0)\right] \\
& \times \sin\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\,t + \eta\,\gamma^1_{jkl}\right)
\end{aligned}$$

$$\begin{aligned}
&\dot{\beta}_0 + \eta\,\overline{\beta_1} + \eta^2\,\overline{\beta_2} + \eta\,\dot{\beta}_1 + \eta^2\,\dot{\beta}_2 = n(a_0)\,\delta_{\beta M} + \frac{d\,n}{da_0}\left(\eta\,a_1 + \eta^2\,a_2\right)\delta_{\beta M} + \frac{1}{2}\frac{d^2\,n}{da_0^2}\left(\eta\,a_1 + \eta^2\,a_2\right)^2\delta_{\beta M} \\
&+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\eta\,B^{(1)}_{jkl}(a_0,e_0,i_0) + \eta^2\sum_{\alpha}\frac{\partial\,B^{(1)}_{jkl}}{\partial\alpha_0}\,\alpha_1 + \eta^2\,B^{(2)}_{jkl}(a_0,e_0,i_0)\right]\cos\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\,t + \eta\,\gamma^1_{jkl}\right)
\end{aligned}$$

6.4 Secular, periodic and mixed terms

- We express the trigonometric functions as

$$\sin\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\,t + \eta\,\gamma^1_{jkl}\right) = \sin\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\,t\right) + \cos\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\,t\right)\eta\,\gamma^1_{jkl}$$

$$\cos\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\,t + \eta\,\gamma^1_{jkl}\right) = \cos\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\,t\right) - \sin\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\,t\right)\eta\,\gamma^1_{jkl}$$

- At 0th order in η:

$$\begin{aligned} \dot{\alpha}_0 &= 0 \\ \dot{\beta}_0 &= n_0\,\delta_{\beta M} \end{aligned}$$

- At 1st order in η for the metric elements:

$$\begin{aligned} \dot{\alpha}_1 &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} A^{(1)}_{jkl}(a_0, e_0, i_0)\,\sin\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\,t\right) \\ \Rightarrow \alpha_1 &= -\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{A^{(1)}_{jkl}(a_0, e_0, i_0)}{n_0\,l + \eta\,\overline{\gamma}_{jkl}}\,\cos\left(\gamma^0_{jkl} + \eta\,\overline{\gamma}_{jkl}\,t\right) \end{aligned}$$

6.4 Secular, periodic and mixed terms

- At 1st order in η for the angular elements:

$$\overline{\beta_1} + \dot{\beta}_1 = \frac{dn}{da_0} a_1 \delta_{\beta M} + B^{(1)}_{000}(a_0, e_0, i_0) + \sum_{jkl}{}' B^{(1)}_{jkl}(a_0, e_0, i_0) \cos(\gamma^0_{jkl} + \eta \overline{\gamma}_{jkl} t)$$

with

$$\alpha_1 = -\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{A^{(1)}_{jkl}(a_0, e_0, i_0)}{n_0 l + \eta \overline{\gamma}_{jkl}} \cos(\gamma^0_{jkl} + \eta \overline{\gamma}_{jkl} t)$$

$\Rightarrow$

$$\overline{\beta_1} + \dot{\beta}_1 = B^{(1)}_{000}(a_0, e_0, i_0) + \sum_{jkl}{}' B'^{(1)}_{jkl}(a_0, e_0, i_0) \cos(\gamma^0_{jkl} + \eta \overline{\gamma}_{jkl} t)$$

$$\Rightarrow \overline{\beta_1} = B^{(1)}_{000}(a_0, e_0, i_0)$$

$$\beta_1 = \sum_{jkl}{}' \frac{B'^{(1)}_{jkl}(a_0, e_0, i_0)}{n_0 l + \eta \overline{\gamma}_{jkl}} \sin(\gamma^0_{jkl} + \eta \overline{\gamma}_{jkl} t)$$

6.4 Secular, periodic and mixed terms

- At 2nd order in η for the metric elements:

$$\dot{\alpha}_2 = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left[\sum_{\alpha}\frac{\partial A^{(1)}_{jkl}}{\partial\alpha_0}\alpha_1 + A^{(2)}_{jkl}(a_0,e_0,i_0)\right]\sin\left(\gamma^0_{jkl}+\eta\,\overline{\gamma}_{jkl}\,t\right)$$
$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}A^{(1)}_{jkl}(a_0,e_0,i_0)\,\cos\left(\gamma^0_{jkl}+\eta\,\overline{\gamma}_{jkl}\,t\right)\gamma^1_{jkl}$$

with

$$\alpha_1 = -\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{A^{(1)}_{jkl}(a_0,e_0,i_0)}{n_0\,l+\eta\,\overline{\gamma}_{jkl}}\cos\left(\gamma^0_{jkl}+\eta\,\overline{\gamma}_{jkl}\,t\right)$$

$\Rightarrow$

$$\dot{\alpha}_2 = \sum_{j''k''l''} A'^{(2)}_{j''k''l''}\sin\left(\gamma^0_{j''k''l''}+\eta\,\overline{\gamma}_{j''k''l''}\,t\right)$$

$$\Rightarrow \alpha_2 = -\sum_{j''k''l''}\frac{A'^{(2)}_{j''k''l''}}{n_0\,l''+\eta\,\overline{\gamma}_{j''k''l''}}\cos\left(\gamma^0_{j''k''l''}+\eta\,\overline{\gamma}_{j''k''l''}\,t\right)$$

6.4 Secular, periodic and mixed terms

- At 2nd order in η for the angular elements:

$$\overline{\beta_2} + \dot{\beta}_2 = \frac{d\,n}{da_0}\, a_2\, \delta_{\beta M} + \frac{1}{2}\frac{d^2\, n}{da_0^2}\, a_1^2\, \delta_{\beta M} + B^{(2)}_{000}(a_0, e_0, i_0) + \sum_{\alpha} \frac{\partial\, B^{(1)}_{000}}{\partial \alpha_0}\, \alpha_1$$
$$+ {\sum_{jkl}}' \left[\sum_{\alpha} \frac{\partial\, B^{(1)}_{jkl}}{\partial \alpha_0}\, \alpha_1 + B^{(2)}_{jkl}(a_0, e_0, i_0) \right] \cos\left(\gamma^0_{jkl} + \eta\, \overline{\gamma}_{jkl}\, t\right)$$
$$- {\sum_{jkl}}' B^{(1)}_{jkl}(a_0, e_0, i_0)\, \sin\left(\gamma^0_{jkl} + \eta\, \overline{\gamma}_{jkl}\, t\right) \gamma^1_{jkl}$$

since

$$\alpha_1 = -\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{A^{(1)}_{jkl}(a_0, e_0, i_0)}{n_0\, l + \eta\, \overline{\gamma}_{jkl}} \cos\left(\gamma^0_{jkl} + \eta\, \overline{\gamma}_{jkl}\, t\right)$$

&

$$\alpha_2 = -\sum_{j''k''l''} \frac{A'^{(2)}_{j''k''l''}}{n_0\, l'' + \eta\, \overline{\gamma}_{j''k''l''}} \cos\left(\gamma^0_{j''k''l''} + \eta\, \overline{\gamma}_{j''k''l''}\, t\right)$$

6.4 Secular, periodic and mixed terms

- We finally obtain:

$$\overline{\beta_2} + \dot{\beta}_2 = B'^{(2)}_{000} + \sum'_{j''k''l''} B'^{(2)}_{j''k''l''} \cos\left(\gamma^0_{j''k''l''} + \eta\,\overline{\gamma}_{j''k''l''}\,t\right)$$

$$\Rightarrow \overline{\beta_2} = B'^{(2)}_{000}(a_0, e_0, i_0)$$

$$\Rightarrow \beta_2 = \sum_{j''k''l''} \frac{B'^{(2)}_{j''k''l''}}{n_0\,l'' + \eta\,\overline{\gamma}_{j''k''l''}} \sin\left(\gamma^0_{j''k''l''} + \eta\,\overline{\gamma}_{j''k''l''}\,t\right)$$

- There are no mixed terms left and the behaviour of the development over long time intervals is much better (although it remains limited in time).

6.4 Secular, periodic and mixed terms

- There are various periodic terms that appear in the solution.
- We distinguish terms corresponding to short periods (frequency $l\, n_0$ with $l \neq 0$) and terms corresponding to long periods (frequencies proportional to η with $l = 0$).
- The long-period terms of order η^n have actual amplitudes proportional to η^{n-1}.

6.5 Perturbations due to J_2

- Potential associated with J_2: $$U' = J_2\,\mu \frac{R_e^2}{a^3}\,\frac{a^3}{r^3}\,(\frac{3}{2}\,\sin^2\theta - \frac{1}{2})$$

  with $$\sin\theta = \sin i\,\sin(\omega+\phi)$$

  $\Rightarrow$

$$\begin{aligned} U' &= J_2\,\mu \frac{R_e^2}{a^3}\,\frac{a^3}{r^3}\,(\frac{3}{2}\,\sin^2 i\,\sin^2(\omega+\phi) - \frac{1}{2}) \\ &= J_2\,\mu \frac{R_e^2}{a^3}\,\frac{a^3}{r^3}\,\left(\frac{3}{4}\,\sin^2 i - \frac{1}{2} - \frac{3}{4}\,\sin^2 i\,\cos 2\omega\,\cos 2\phi + \frac{3}{4}\,\sin^2 i\,\sin 2\omega\,\sin 2\phi\right) \end{aligned}$$

The potential depends on three variable quantities:

$$\left(\frac{a}{r}\right)^3 \qquad \left(\frac{a}{r}\right)^3 \cos 2\phi \qquad \left(\frac{a}{r}\right)^3 \sin 2\phi$$

Idea: develop these functions in Fourier series of *M*, expanding the coefficients into series of powers of *e*.

6.5 Perturbations due to J_2

- The constant terms of $\left(\frac{a}{r}\right)^3 \cos(p\,\phi)$ are $\frac{1}{(1-e^2)^{3/2}}\,(2\,\delta_{p0} + e\,\delta_{p1})$

- At 3rd order in *e*, we have that

$$\left(\frac{a}{r}\right)^3 = (1-e\,\cos E)^{-3} = 1 + 3\,e\,\cos E + 6\,e^2\,\cos^2 E + 10\,e^3\,\cos^3 E + \mathcal{O}(e^4)$$
$$= 1 + 3\,e^2 + (3\,e + \frac{15\,e^3}{2})\,\cos E + 3\,e^2\,\cos(2\,E) + \frac{5\,e^3}{2}\,\cos(3\,E) + \mathcal{O}(e^4)$$

$$\left(\frac{a}{r}\right)^3 \cos 2\,\phi = \left(\frac{a}{r}\right)^3 (2\,\cos^2\phi - 1) = \left(\frac{a}{r}\right)^3 \left(\frac{2\,(\cos E - e)^2}{(1-e\,\cos E)^2} - 1\right)$$
$$= 2\left(\frac{a}{r}\right)^5 (\cos E - e)^2 - \left(\frac{a}{r}\right)^3 = 2\,(1-e\,\cos E)^{-5}(\cos E - e)^2 - \left(\frac{a}{r}\right)^3$$
$$= 2\,(1 + 5\,e\,\cos E + 15\,e^2\,\cos^2 E + 35\,e^3\,\cos^3 E)(\cos^2 E - 2\,e\,\cos E + e^2)$$
$$-1 - 3\,e\,\cos E - 6\,e^2\,\cos^2 E - 10\,e^3\,\cos^3 E + \mathcal{O}(e^4)$$
$$= \frac{e^2}{4} + \left(\frac{e}{2} + \frac{5\,e^3}{4}\right)\cos E + (1 + 2\,e^2)\,\cos(2\,E) + \left(\frac{5\,e}{2} + \frac{35\,e^3}{8}\right)\cos(3\,E)$$
$$+\frac{15\,e^2}{4}\,\cos(4\,E) + \frac{35\,e^3}{8}\,\cos(5\,E) + \mathcal{O}(e^4)$$

6.5 Perturbations due to J_2

- At 3rd order in *e*, we note that

$$\begin{aligned}
\left(\frac{a}{r}\right)^3 \sin 2\phi &= 2\left(\frac{a}{r}\right)^3 \sin\phi\,\cos\phi = 2\left(\frac{a}{r}\right)^3 \frac{\sqrt{1-e^2}\,\sin E}{1-e\,\cos E}\,\frac{\cos E - e}{1-e\,\cos E} \\
&= 2\sqrt{1-e^2}\,\sin E\,(\cos E - e)\,(1-e\,\cos E)^{-5} \\
&= \sin E\,(2-e^2)\,(1+5\,e\,\cos E + 15\,e^2\,\cos^2 E + 35\,e^3\,\cos^3 E)(\cos E - e) + \mathcal{O}(e^4) \\
&= \left(\frac{e}{2}+e^3\right)\sin E + (1+2\,e^2)\,\sin(2\,E) + \left(\frac{5\,e}{2}+\frac{35\,e^3}{8}\right)\sin(3\,E) \\
&\quad + \frac{15\,e^2}{4}\,\sin(4\,E) + \frac{35\,e^3}{8}\,\sin(5\,E) + \mathcal{O}(e^4)
\end{aligned}$$

- On the other hand,

$$\begin{aligned}
\cos(p\,E) &= \delta_{p0} - \frac{e}{2}\,\delta_{p1} + \sum_{k=1}^{+\infty} \frac{p}{k}\left(J_{k-p}(k\,e) - J_{k+p}(k\,e)\right)\,\cos(k\,M) \\
\sin(p\,E) &= \sum_{k=1}^{+\infty} \frac{p}{k}\left(J_{k-p}(k\,e) + J_{k+p}(k\,e)\right)\,\sin(k\,M)
\end{aligned}$$

with $J_s(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+s)!}\left(\frac{x}{2}\right)^{2m+s}$

6.5 Perturbations due to J_2

- We thus find that…

$$\cos E = \frac{-e}{2} + \left(1 - \frac{3e^2}{8}\right)\cos M + \frac{e}{2}\cos(2M) + \frac{3e^2}{8}\cos(3M) + \mathcal{O}(e^3)$$

$$\sin E = \left(1 - \frac{e^2}{8}\right)\sin M + \frac{e}{2}\sin(2M) + \frac{3e^2}{8}\sin(3M) + \mathcal{O}(e^3)$$

$$\cos(2E) = \left(-e + \frac{e^3}{12}\right)\cos M + (1-e^2)\cos(2M) + \left(e - \frac{9e^3}{8}\right)\cos(3M) + e^2\cos(4M) + \frac{25e^3}{24}\cos(5M) + \mathcal{O}(e^4)$$

$$\sin(2E) = \left(-e + \frac{e^3}{6}\right)\sin M + (1-e^2)\sin(2M) + \left(e - \frac{9e^3}{8}\right)\sin(3M) + e^2\sin(4M) + \frac{25e^3}{24}\sin(5M) + \mathcal{O}(e^4)$$

$$\cos(3E) = \frac{3e^2}{8}\cos M - \frac{3e}{2}\cos(2M) + \left(1 - \frac{9e^2}{4}\right)\cos(3M) + \frac{3e}{2}\cos(4M) + \frac{15e^2}{8}\cos(5M) + \mathcal{O}(e^3)$$

$$\sin(3E) = \frac{3e^2}{8}\sin M - \frac{3e}{2}\sin(2M) + \left(1 - \frac{9e^2}{4}\right)\sin(3M) + \frac{3e}{2}\sin(4M) + \frac{15e^2}{8}\sin(5M) + \mathcal{O}(e^3)$$

6.5 Perturbations due to J_2

- …and:

$$\begin{aligned}
\cos(4E) &= -2e\cos(3M) + \cos(4M) + 2e\cos(5M) + \mathcal{O}(e^2)\\
\sin(4E) &= -2e\sin(3M) + \sin(4M) + 2e\sin(5M) + \mathcal{O}(e^2)\\
\cos(5E) &= \cos(5M) + \mathcal{O}(e)\\
\sin(5E) &= \sin(5M) + \mathcal{O}(e)
\end{aligned}$$

- This leads to

$$\left(\frac{a}{r}\right)^3 = \frac{1}{(1-e^2)^{3/2}} + \left(3e + \frac{27e^3}{8}\right)\cos M + \frac{9e^2}{2}\cos(2M) + \frac{53e^3}{8}\cos(3M) + \mathcal{O}(e^4)$$

$$\begin{aligned}
\left(\frac{a}{r}\right)^3\cos(2\phi) &= \left(-\frac{e}{2} + \frac{e^3}{12}\right)\cos M + \left(1 - \frac{5e^2}{2}\right)\cos(2M) + \left(\frac{7e}{2} - \frac{123e^3}{16}\right)\cos(3M)\\
&\quad + \frac{17e^2}{2}\cos(4M) + \frac{845e^3}{48}\cos(5M) + \mathcal{O}(e^4)\\
\left(\frac{a}{r}\right)^3\sin(2\phi) &= \left(-\frac{e}{2} + \frac{e^3}{24}\right)\sin M + \left(1 - \frac{5e^2}{2}\right)\sin(2M) + \left(\frac{7e}{2} - \frac{123e^3}{16}\right)\sin(3M)\\
&\quad + \frac{17e^2}{2}\sin(4M) + \frac{845e^3}{48}\sin(5M) + \mathcal{O}(e^4)
\end{aligned}$$

6.5 Perturbations due to J_2

- We thus find for *U'*

$$
\begin{aligned}
U' \quad = \quad & J_2\,\mu \frac{R_e^2}{a^3}\left(\frac{3}{4}\,\sin^2 i - \frac{1}{2}\right)\frac{1}{(1-e^2)^{3/2}} \\
& + J_2\,\mu \frac{R_e^2}{a^3}\left\{\left(\frac{3}{4}\,\sin^2 i - \frac{1}{2}\right)\left[\left(3\,e + \frac{27\,e^3}{8}\right)\cos M + \frac{9\,e^2}{2}\cos(2\,M) + \frac{53\,e^3}{8}\cos(3\,M)\right]\right. \\
& - \frac{3}{4}\,\sin^{i}\left[\frac{e^3}{48}\cos(2\,\omega - M) + \left(-\frac{e}{2} + \frac{e^3}{16}\right)\cos(2\,\omega + M) + \left(1 - \frac{5\,e^2}{2}\right)\cos(2\,\omega + 2\,M)\right. \\
& \left.\left. + \left(\frac{7\,e}{2} - \frac{123\,e^3}{16}\right)\cos(2\,\omega + 3\,M) + \frac{17\,e^2}{2}\cos(2\,\omega + 4\,M) + \frac{845\,e^3}{48}\cos(2\,\omega + 5\,M)\right]\right\} \\
& + \mathcal{O}(e^4)
\end{aligned}
$$

$$
\begin{bmatrix} \frac{d\,a}{dt} \\ \frac{d\,M}{dt} \\ \frac{d\,e}{dt} \\ \frac{d\,\omega}{dt} \\ \frac{d\,i}{dt} \\ \frac{d\,\Omega}{dt} \end{bmatrix}
=
\begin{bmatrix} 0 \\ n(t) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
+ \frac{1}{n\,a^2}
\begin{bmatrix}
0 & -2\,a & 0 & 0 & 0 & 0 \\
2\,a & 0 & \frac{1-e^2}{e} & 0 & 0 & 0 \\
0 & -\frac{1-e^2}{e} & 0 & \frac{\sqrt{1-e^2}}{e} & 0 & 0 \\
0 & 0 & -\frac{\sqrt{1-e^2}}{e} & 0 & \frac{\cos i}{\sqrt{1-e^2}\,\sin i} & 0 \\
0 & 0 & 0 & -\frac{\cos i}{\sqrt{1-e^2}\,\sin i} & 0 & \frac{1}{\sqrt{1-e^2}\,\sin i} \\
0 & 0 & 0 & 0 & \frac{-1}{\sqrt{1-e^2}\,\sin i} & 0
\end{bmatrix}
\begin{bmatrix} \frac{\partial\,U'}{\partial a} \\ \frac{\partial\,U'}{\partial M} \\ \frac{\partial\,U'}{\partial e} \\ \frac{\partial\,U'}{\partial \omega} \\ \frac{\partial\,U'}{\partial i} \\ \frac{\partial\,U'}{\partial \Omega} \end{bmatrix}
$$

6.5 Perturbations due to J_2

- The Lagrange equations hence lead to

$$\begin{aligned}
\frac{d\,a}{dt} &= 2\,\frac{n\,J_2\,R_e^2}{a}\left[\left(\frac{3}{4}\sin^2 i - \frac{1}{2}\right) 3\,e\,\sin M - \frac{3}{4}\sin^2 i\,\left(\frac{-e}{2}\sin(2\,\omega + M) + 2\,\sin(2\,\omega + 2\,M)\right.\right.\\
&\quad \left.\left. + \frac{21\,e}{2}\sin(2\,\omega + 3\,M)\right)\right] + \mathcal{O}(e^2)\\
\frac{d\,e}{dt} &= \frac{n\,J_2\,R_e^2}{a^2}\left[\left(\frac{3}{4}\sin^2 i - \frac{1}{2}\right)(3\,\sin M + 9\,e\,\sin(2\,M)) - \frac{3}{4}\sin^2 i\,\left(\frac{1}{2}\sin(2\,\omega + M)\right.\right.\\
&\quad \left.\left. - e\,\sin(2\,\omega + 2\,M) + \frac{7}{2}\sin(2\,\omega + 3\,M) + 17\,e\,\sin(2\,\omega + 4\,M)\right)\right] + \mathcal{O}(e^2)\\
\frac{d\,i}{dt} &= \frac{3\,n\,J_2\,R_e^2}{4\,a^2}\sin i\,\cos i\,[e\,\sin(2\,\omega + M) - 2\,\sin(2\,\omega + 2\,M) - 7\,e\,\sin(2\,\omega + 3\,M)] + \mathcal{O}(e^2)
\end{aligned}$$

$$
\begin{aligned}
\frac{d\,\Omega}{dt} &= -\frac{3\,n\,J_2\,R_e^2\,\cos i}{2\,a^2}\left[\frac{1}{(1-e^2)^2} + 3\,e\,\cos M + \frac{e}{2}\,\cos\left(2\,\omega + M\right) - \cos\left(2\,\omega + 2\,M\right)\right. \\
&\quad \left. -\frac{7\,e}{2}\,\cos\left(2\,\omega + 3\,M\right)\right] + \mathcal{O}(e^2) \\
\frac{d\,\omega}{dt} &= \frac{3\,n\,J_2\,R_e^2}{a^2\,(1-e^2)^2}\left(1 - \frac{5}{4}\,\sin^2 i\right) + \frac{n\,J_2\,R_e^2}{a^2}\left\{\left[\left(\frac{1}{2} - \frac{3}{4}\,\sin^2 i\right)\left(\frac{3}{e} + \frac{69\,e}{8}\right) + \frac{9\,e}{2}\,\cos^2 i\right]\cos M\right. \\
&\quad + \left(\frac{1}{2} - \frac{3}{4}\,\sin^2 i\right)\left(9\,\cos\left(2\,M\right) + \frac{159\,e}{8}\,\cos\left(3\,M\right)\right) + \frac{3\,e}{64}\,\sin^2 i\,\cos\left(2\,\omega - M\right) \\
&\quad + \left[\frac{3}{4}\,\sin^2 i\left(\frac{-1}{2\,e} + \frac{7\,e}{16}\right) + \frac{3\,e}{4}\,\cos^2 i\right]\cos\left(2\,\omega + M\right) - \frac{3}{2}\left(1 + \frac{3}{2}\,\sin^2 i\right)\cos\left(2\,\omega + 2\,M\right) \\
&\quad + \left[\frac{3}{4}\,\sin^2 i\left(\frac{7}{2\,e} - \frac{397\,e}{16}\right) - \frac{21\,e}{4}\,\cos^2 i\right]\cos\left(2\,\omega + 3\,M\right) \\
&\quad \left. + \frac{3}{4}\,\sin^2 i\left(17\,\cos\left(2\,\omega + 4\,M\right) + \frac{845\,e}{16}\,\cos\left(2\,\omega + 5\,M\right)\right)\right\} + \mathcal{O}(e^2) \\
\frac{d\,M}{dt} &= n + \frac{3\,n\,J_2\,R_e^2}{a^2\,(1-e^2)^{3/2}}\left(\frac{1}{2} - \frac{3}{4}\,\sin^2 i\right) + \frac{n\,J_2\,R_e^2}{a^2}\left\{\left(\frac{1}{2} - \frac{3}{4}\,\sin^2 i\right)\left[\left(\frac{-3}{e} + \frac{87\,e}{8}\right)\cos M\right.\right. \\
&\quad \left. -9\,\cos\left(2\,M\right) - \frac{159\,e}{8}\,\cos\left(3\,M\right)\right] + \frac{3}{4}\,\sin^2 i\left[\frac{-e}{16}\,\cos\left(2\,\omega - M\right) + \left(\frac{1}{2\,e} - \frac{59\,e}{16}\right)\cos\left(2\,\omega + M\right)\right. \\
&\quad + 11\,\cos\left(2\,\omega + 2\,M\right) + \left(\frac{-7}{2\,e} + \frac{761\,e}{16}\right)\cos\left(2\,\omega + 3\,M\right) - 17\,\cos\left(2\,\omega + 4\,M\right) \\
&\quad \left.\left. -\frac{845\,e}{16}\,\cos\left(2\,\omega + 5\,M\right)\right]\right\} + \mathcal{O}(e^2)
\end{aligned}
$$

6.5 Perturbations due to J_2

- Let us apply an asymptotic development:

$$\begin{aligned}\alpha &= \alpha_0 + J_2\,\alpha_1 + (J_2)^2\,\alpha_2 + \ldots \\ \beta &= \beta_0 + (J_2\,\overline{\beta_1} + (J_2)^2\,\overline{\beta_2})\,t + J_2\,\beta_1 + (J_2)^2\,\beta_2 + \ldots \\ &= \beta_0 + J_2\,\overline{\beta}\,t + J_2\,\beta_1 + (J_2)^2\,\beta_2 + \ldots\end{aligned}$$

- To 0th order in J_2:

$$\begin{aligned}a_0 &= Cte & \Omega_0 &= Cte \\ e_0 &= Cte & \omega_0 &= Cte \\ i_0 &= Cte & M_0 &= m_0 + n_0\,t\end{aligned}$$

- Secular terms of the angular elements at 1st order in J_2:

$$\begin{aligned}J_2\,\overline{\Omega_1} &= -\frac{3\,n_0\,J_2\,R_e^2\,\cos i_0}{2\,a_0^2\,(1-e_0^2)^2} \\ J_2\,\overline{\omega_1} &= \frac{3\,n_0\,J_2\,R_e^2}{a_0^2\,(1-e_0^2)^2}\left(1-\frac{5}{4}\,\sin^2 i_0\right) \\ J_2\,\overline{M_1} &= \frac{3\,n_0\,J_2\,R_e^2}{a_0^2\,(1-e_0^2)^{3/2}}\left(\frac{1}{2}-\frac{3}{4}\,\sin^2 i_0\right)\end{aligned}$$

6.5 Perturbations due to J_2

- Periodic terms at 1st order in J_2:

$$J_2 \frac{d\,a_1}{dt} = \frac{n_0\, J_2\, R_e^2}{a_0} \left[\left(\frac{9}{2} \sin^2 i_0 - 3 \right) e_0 \sin\left(M_0 + J_2\, \overline{M}\, t\right) \right.$$
$$+ \frac{3}{2} \sin^2 i_0 \left(\frac{e_0}{2} \sin\left(2\omega_0 + M_0 + J_2\,(2\overline{\varpi} + \overline{M})\,t\right) - 2 \sin\left(2\,\omega_0 + 2\,M_0 + 2\,J_2\,(\overline{\varpi} + \overline{M})\,t\right) \right.$$
$$\left.\left. - \frac{21\, e_0}{2} \sin\left(2\,\omega_0 + 3\,M_0 + J_2\,(2\,\overline{\varpi} + 3\,\overline{M})\,t\right) \right) \right]$$

$$\ldots$$

$$J_2 \frac{d\,M_1}{dt} = \frac{-3\,n_0}{2\,a_0} J_2\, a_1 + \frac{n_0\, J_2\, R_e^2}{a_0^2} \left[\left[\left(\frac{1}{2} - \frac{3}{4} \sin^2 i_0 \right) \left(\frac{-3}{e_0} \cos\left(M_0 + J_2\, \overline{M}\, t\right) \right.\right.\right.$$
$$\left.\left. -9 \cos\left(2\,M_0 + 2\;J_2 \overline{M}\,t\right) \right] + \frac{3}{4} \sin^2 i_0 \left[\frac{1}{2\,e_0} \cos\left(2\,\omega_0 + M_0 + J_2\,(2\,\overline{\varpi} + \overline{M})\,t\right) \right.$$
$$+11 \cos\left(2\,\omega_0 + 2\,M_0 + 2\,J_2\,(\overline{\varpi} + \overline{M})\,t\right) - \frac{7}{2\,e_0} \cos\left(2\,\omega_0 + 3\,M_0 + J_2\,(2\,\overline{\varpi} + 3\,\overline{M})\,t\right)$$
$$\left.\left. -17 \cos\left(2\,\omega_0 + 4\,M_0 + 2\,J_2\,(\overline{\varpi} + 2\,\overline{M})\,t\right) \right] \right\}$$

6.5 Perturbations due to J_2

- Periodic terms at 1st order in J_2:

$$J_2 \frac{d\,a_1}{dt} = \frac{n_0\,J_2\,R_e^2}{a_0}\Bigg[\left(\frac{9}{2}\sin^2 i_0 - 3\right) e_0 \sin\left(M_0 + J_2\,\overline{M}\,t\right) + \frac{3}{2}\sin^2 i_0 \left(\frac{e_0}{2}\sin\left(2\,\omega_0 + M_0 + J_2\left(2\,\overline{\varpi} + \overline{M}\right)t\right) - 2\sin\left(2\,\omega_0 + 2\,M_0 + 2\,J_2\left(\overline{\varpi} + \overline{M}\right)t\right) - \frac{21\,e_0}{2}\sin\left(2\,\omega_0 + 3\,M_0 + J_2\left(2\,\overline{\varpi} + 3\,\overline{M}\right)t\right)\right)\Bigg]$$

$\Rightarrow$

$$J_2\,a_1 = -\frac{n_0\,J_2\,R_e^2}{a_0}\Bigg[\left(\frac{9}{2}\sin^2 i_0 - 3\right) e_0 \frac{\cos\left(M_0 + J_2\,\overline{M}\,t\right)}{n_0 + J_2\,\overline{M}} + \frac{3}{2}\sin^2 i_0 \left(\frac{e_0}{2}\frac{\cos\left(2\,\omega_0 + M_0 + J_2\left(2\,\overline{\varpi} + \overline{M}\right)t\right)}{n_0 + J_2\left(\overline{M} + 2\,\overline{\varpi}\right)} - \frac{\cos\left(2\,\omega_0 + 2\,M_0 + 2\,J_2\left(\overline{\varpi} + \overline{M}\right)t\right)}{n_0 + J_2\left(\overline{M} + \overline{\varpi}\right)} - \frac{21\,e_0}{2}\frac{\cos\left(2\,\omega_0 + 3\,M_0 + J_2\left(2\,\overline{\varpi} + 3\,\overline{M}\right)t\right)}{3\,n_0 + J_2\left(3\,\overline{M} + 2\,\overline{\varpi}\right)}\right)\Bigg]$$

with $\frac{J_2}{n_0+\overline{\beta}\,J_2} = \frac{J_2}{n_0} + \mathcal{O}((J_2)^2)$

6.5 Perturbations due to J_2

- Periodic terms at 1st order in J_2:

$$J_2 \frac{d\,a_1}{dt} = \frac{n_0\,J_2\,R_e^2}{a_0}\left[\left(\frac{9}{2}\sin^2 i_0 - 3\right) e_0 \sin\left(M_0 + J_2\,\overline{M}\,t\right) + \frac{3}{2}\sin^2 i_0 \left(\frac{e_0}{2}\sin\left(2\,\omega_0 + M_0 + J_2\left(2\,\overline{\omega} + \overline{M}\right)t\right) - 2\sin\left(2\,\omega_0 + 2\,M_0 + 2\,J_2\left(\overline{\omega} + \overline{M}\right)t\right) - \frac{21\,e_0}{2}\sin\left(2\,\omega_0 + 3\,M_0 + J_2\left(2\,\overline{\omega} + 3\,\overline{M}\right)t\right)\right)\right]$$

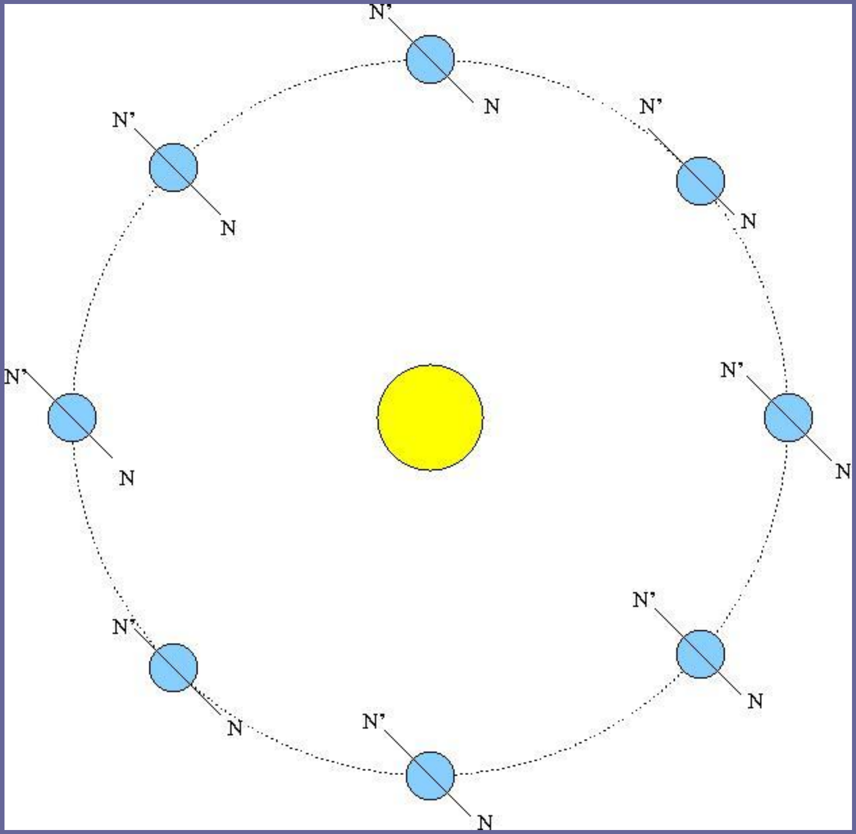
$\Rightarrow$

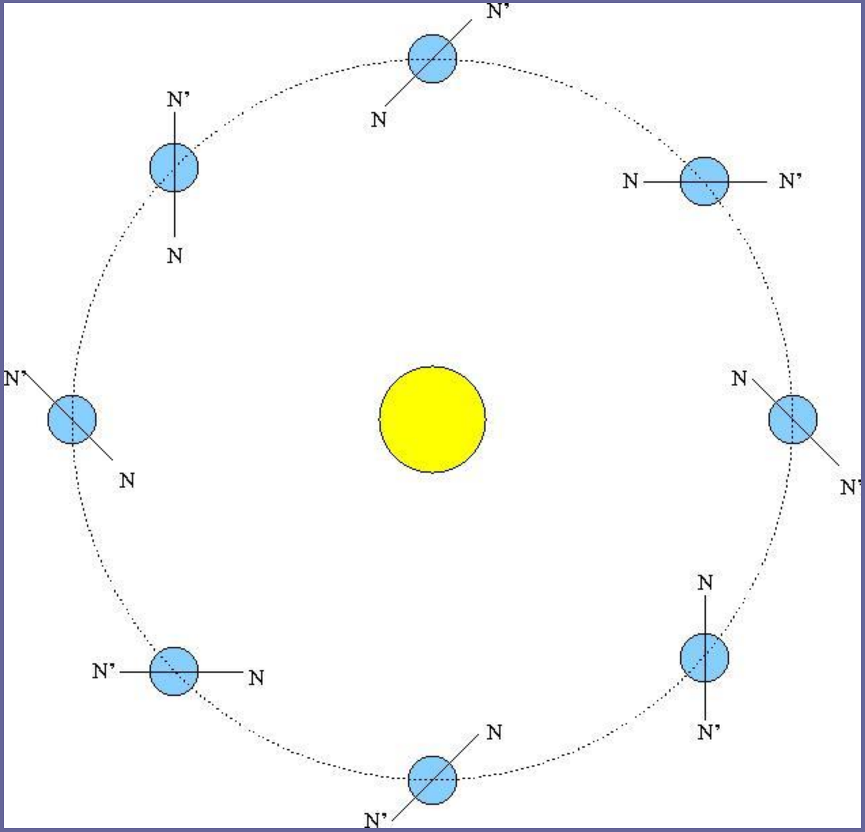
$$J_2\,a_1 = -\frac{J_2\,R_e^2}{a_0}\left[\left(\frac{9}{2}\sin^2 i_0 - 3\right) e_0 \cos\left(M_0 + J_2\,\overline{M}\,t\right) + \frac{3}{2}\sin^2 i_0 \left(\frac{e_0}{2}\cos\left(2\,\omega_0 + M_0 + J_2\left(2\,\overline{\omega} + \overline{M}\right)t\right) - \cos\left(2\,\omega_0 + 2\,M_0 + 2\,J_2\left(\overline{\omega} + \overline{M}\right)t\right) - \frac{7\,e_0}{2}\cos\left(2\,\omega_0 + 3\,M_0 + J_2\left(2\,\overline{\omega} + 3\,\overline{M}\right)t\right)\right)\right]$$

6.5 Perturbations due to J_2

- Application of the 1st order secular terms in J_2: the Sun-synchronous orbit:

$$J_2\,\overline{\Omega_1} = -\frac{3\,n_0\,J_2\,R_e^2\,\cos i_0}{2\,a_0^2\,(1-e_0^2)^2}$$ = 360°/year



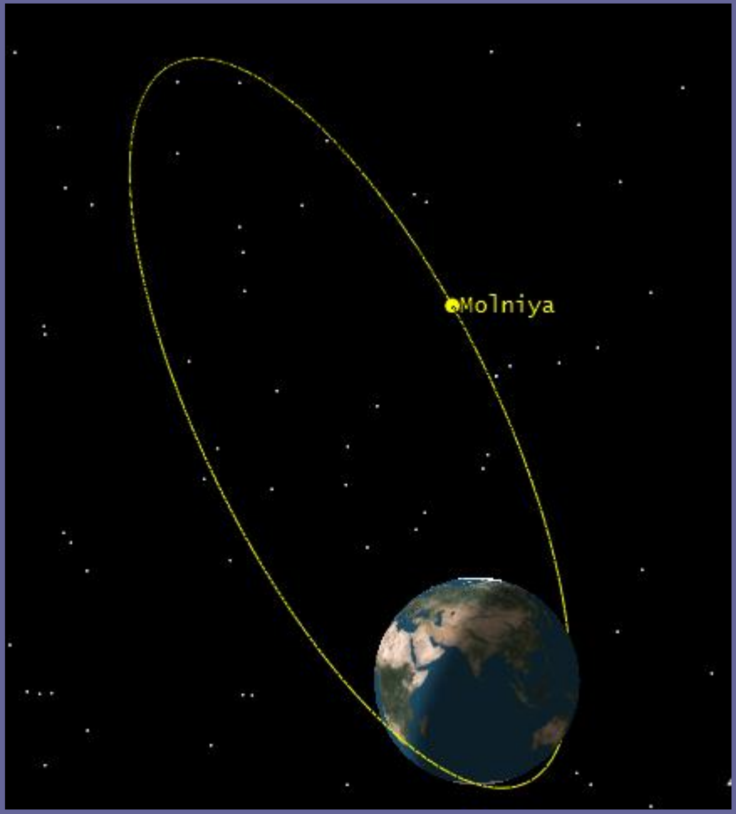


6.5 Perturbations due to J_2

- Application of the 1st order secular terms in J_2: the Tundra and Molniya orbits

$$J_2 \overline{\omega_1} = \frac{3\, n_0\, J_2\, R_e^2}{a_0^2\,(1-e_0^2)^2} \left(1 - \frac{5}{4}\sin^2 i_0\right)$$ $= 0$ if $i_0 = 63.43°$

- The longitude of the apocentre remains constant as a function of time. Highly eccentric orbits with periods of 12h (Molniya) or 24h (Tundra).
- The satellite moves very slowly at apogee allowing an extended visibility.

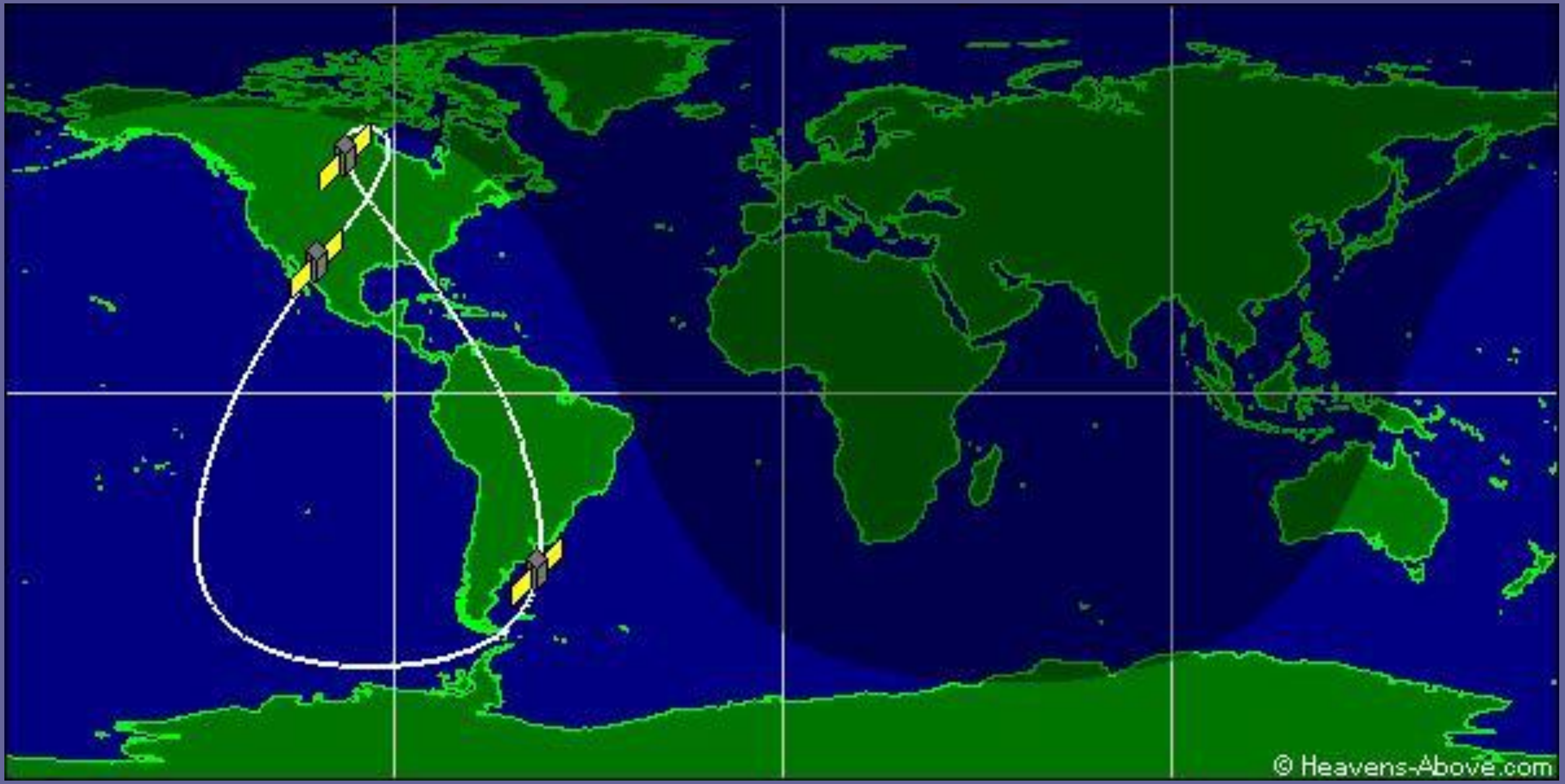


6.5 Perturbations due to J_2

- Molniya:

6.5 Perturbations due to J_2

- Tundra:



6.5 Perturbations due to J_2

- Other zonal terms: J_n of order J_2^2
- Same kind of perturbations (secular, long- and short-term periodicities).
- Sectoral and tesseral terms depend on λ:

$$\lambda = \alpha(t) - t_*$$

$$t_* = \nu_\oplus (t - t_0)$$

- Diurnal perturbations due to the Earth’s rotation.

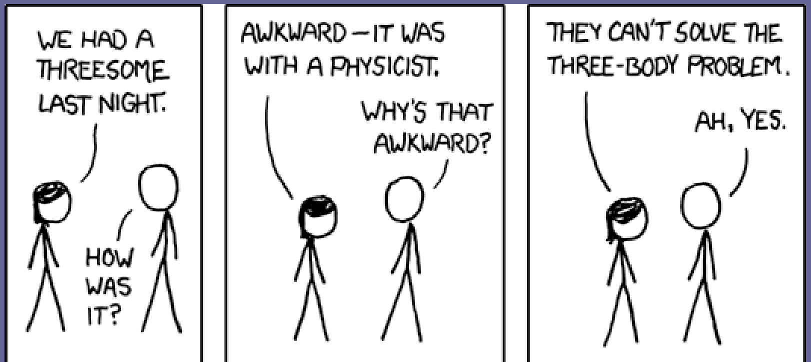
Chapter VII: The *N* body problem

- Consider a system consisting of N point-like or spherical masses isolated in space (without external forces).
- Newton's equations (inertial frame of reference):

$$m_k \frac{d^2 \vec{OP_k}}{dt^2} = \sum_{i=0}^{k-1} \frac{G\, m_k\, m_i}{|\vec{P_kP_i}|^3}\, \vec{P_kP_i} + \sum_{i=k+1}^{n} \frac{G\, m_k\, m_i}{|\vec{P_kP_i}|^3}\, \vec{P_kP_i}$$

for $k=0,\ldots,n=N-1$

- These equations depend on 6 *N* constants of integration.
- BUT: only 10 classical integrals can be formulated.
- There is thus no general analytic solution for the *N* body problem.



7.1 Integrals of the equation of motion

- Absence of external forces $\Rightarrow$ conservation of the momentum:

$$\sum_{k=0}^{n} m_k \, \frac{d^2 \, \vec{OP_k}}{dt^2} = \vec{0}$$

- Therefore the centre of mass moves in uniform velocity straight line motion:

$$\Rightarrow \left(\sum_{k=0}^{n} m_k \right) \frac{d^2 \, \vec{OC}}{dt^2} = \vec{0}$$

$$\vec{OC}(t) = \vec{OC}(t_0) + \vec{V_C} \, t$$

$\Rightarrow$ 6 classical integrals (components of the vectors that describe the initial position and velocity of the centre of mass)

7.1 Integrals of the equation of motion

- Absence of external forces ⇒ conservation of the total angular momentum of the system:

$$
\begin{aligned}
\sum_{k=0}^{n} m_k \, O\vec{P}_k \wedge \frac{d^2\, O\vec{P}_k}{dt^2} &= \sum_{k=0}^{n} \sum_{i \neq k} O\vec{P}_k \wedge \frac{G\, m_i\, m_k}{|P_k\vec{P}_i|^3}\, P_k\vec{P}_i \\
&= \sum_{k=0}^{n} \sum_{i \neq k} (O\vec{P}_k + P_k\vec{P}_i) \wedge \frac{G\, m_i\, m_k}{|P_k\vec{P}_i|^3}\, P_k\vec{P}_i \\
&= -\sum_{k=0}^{n} \sum_{i \neq k} O\vec{P}_i \wedge \frac{G\, m_i\, m_k}{|P_k\vec{P}_i|^3}\, P_i\vec{P}_k = \vec{0}
\end{aligned}
$$

$$
\Rightarrow \frac{d}{dt}\left(\sum_{k=0}^{n} O\vec{P}_k \wedge m_k\, \frac{d\, O\vec{P}_k}{dt}\right) = \vec{0}
$$

$$
\sum_{k=0}^{n} O\vec{P}_k \wedge m_k\, \frac{d\, O\vec{P}_k}{dt} = C\vec{t}e
$$

7.1 Integrals of the equation of motion

- One can express the conservation of the angular momentum with respect to the centre of mass:

$$\sum_{k=0}^{n} C\vec{P}_k \wedge m_k \frac{d\,C\vec{P}_k}{dt} = \vec{h}$$

$\Rightarrow$ 3 classical integrals (components of the total angular momentum)

7.1 Integrals of the equation of motion

- Absence of external forces $\Rightarrow$ conservation of total energy:

$$
\begin{aligned}
\frac{dT}{dt} &= \sum_{k=0}^{n} m_k \frac{d\,\vec{OP_k}}{dt} \cdot \frac{d^2\,\vec{OP_k}}{dt^2} \\
&= \sum_{k=0}^{n} \sum_{i \neq k} \frac{d\,\vec{OP_k}}{dt} \cdot \frac{G\, m_i\, m_k}{|\vec{P_k P_i}|^3}\, \vec{P_k P_i} \\
&= \sum_{k=0}^{n-1} \sum_{i=k+1}^{n} \left(\frac{d\,\vec{OP_k}}{dt} - \frac{d\,\vec{OP_i}}{dt}\right) \cdot \frac{G\, m_i\, m_k}{|\vec{P_k P_i}|^3}\, \vec{P_k P_i} \\
&= -\sum_{k=0}^{n-1} \sum_{i=k+1}^{n} \frac{d\,\vec{P_k P_i}}{dt} \cdot \frac{G\, m_i\, m_k}{|\vec{P_k P_i}|^3}\, \vec{P_k P_i} \\
&= \frac{d}{dt}\left(\sum_{k=0}^{n-1} \sum_{i=k+1}^{n} \frac{G\, m_i\, m_k}{|\vec{P_k P_i}|} \right)
\end{aligned}
$$

7.1 Integrals of the equation of motion

- Absence of external forces $\Rightarrow$ conservation of total energy:

$$\frac{dT}{dt} = \frac{d}{dt}\left(\sum_{k=0}^{n-1}\sum_{i=k+1}^{n}\frac{G\,m_i\,m_k}{|\vec{P_k P_i}|}\right)$$

$$\frac{1}{2}\sum_{k=0}^{n} m_k \left|\frac{d\,\vec{OP_k}}{dt}\right|^2 - \left(\sum_{k=0}^{n-1}\sum_{i=k+1}^{n}\frac{G\,m_i\,m_k}{|\vec{P_k P_i}|}\right) = \mathcal{E}$$

$$U = -\left(\sum_{k=0}^{n-1}\sum_{i=k+1}^{n}\frac{G\,m_i\,m_k}{|\vec{P_k P_i}|}\right) = -\frac{1}{2}\sum_{k=0}^{n-1}\sum_{i\neq k}\frac{G\,m_i\,m_k}{|\vec{P_k P_i}|}$$

$\Rightarrow$ 1 classical integral

- A total of 10 classical integrals allowing to check the stability of a numerical solution.

7.2 The relative motion

- … with respect to the centre of mass (origin of an inertial frame of reference):

$$\vec{u_k} = C\vec{P_k}$$

$$\Rightarrow \vec{u_0} = -\sum_{k=1}^{n} \frac{m_k}{m_0} \vec{u_k}$$

- Newton's equation

$$m_k \frac{d^2\, \vec{OP_k}}{dt^2} = \sum_{i=0}^{k-1} \frac{G\, m_k\, m_i}{|\vec{P_k P_i}|^3} \vec{P_k P_i} + \sum_{i=k+1}^{n} \frac{G\, m_k\, m_i}{|\vec{P_k P_i}|^3} \vec{P_k P_i}$$

writes:

$$\frac{d^2\, \vec{u_k}}{dt^2} = G\, m_0 \frac{\vec{u_0} - \vec{u_k}}{|\vec{u_0} - \vec{u_k}|^3} + G \sum_{i=1, i\neq k}^{n} m_i \frac{\vec{u_i} - \vec{u_k}}{|\vec{u_i} - \vec{u_k}|^3}$$

$$m_k \frac{d^2\, \vec{u_k}}{dt^2} = -\nabla_k U$$ with $$U = -\sum_{j=0}^{n-1} \sum_{i=j+1}^{n} \frac{G\, m_i\, m_j}{[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]^{1/2}}$$

7.2 The relative motion

- … with respect to the most massive mass P_0 (not the origin of an inertial frame of reference): $\vec{r_k} = \vec{P_0 P_k} = \vec{u_k} - \vec{u_0}$
  $\Rightarrow \vec{r_0} = \vec{0}$
- Newton's equation
  $$m_k \frac{d^2\, \vec{OP_k}}{dt^2} = \sum_{i=0}^{k-1} \frac{G\, m_k\, m_i}{|\vec{P_k P_i}|^3} \vec{P_k P_i} + \sum_{i=k+1}^{n} \frac{G\, m_k\, m_i}{|\vec{P_k P_i}|^3} \vec{P_k P_i}$$
  writes:

$$\frac{d^2\, \vec{r_k}}{dt^2} = \sum_{i=0, i\neq k}^{n} G\, m_i \frac{\vec{r_i} - \vec{r_k}}{|\vec{r_i} - \vec{r_k}|^3} - \sum_{i=1}^{n} G\, m_i \frac{\vec{r_i}}{|\vec{r_i}|^3}$$
$$= -G\,(m_0 + m_k) \frac{\vec{r_k}}{|\vec{r_k}|^3} + \sum_{i=1, i\neq k}^{n} G\, m_i \left(\frac{\vec{r_i} - \vec{r_k}}{|\vec{r_i} - \vec{r_k}|^3} - \frac{\vec{r_i}}{|\vec{r_i}|^3} \right)$$

$$\frac{d^2\, \vec{r_k}}{dt^2} = -\frac{G\,(m_0 + m_k)\, \vec{r_k}}{|\vec{r_k}|^3} + \vec{\nabla}_k\, V_k$$

with $V_k = \sum_{i=1, i\neq k}^{n} G\, m_i \left(\frac{1}{|\vec{r_i} - \vec{r_k}|} - \frac{\vec{r_i} \cdot \vec{r_k}}{|\vec{r_i}|^3} \right)$

7.3 The 3-body problem

- Consider $N = 3$:

$$\frac{d^2 \vec{r_k}}{dt^2} = \sum_{i=0, i \neq k}^{n} G\, m_i \frac{\vec{r_i} - \vec{r_k}}{|\vec{r_i} - \vec{r_k}|^3} - \sum_{i=1}^{n} G\, m_i \frac{\vec{r_i}}{|\vec{r_i}|^3}$$

$$= -G\,(m_0 + m_k)\frac{\vec{r_k}}{|\vec{r_k}|^3} + \sum_{i=1, i \neq k}^{n} G\, m_i \left(\frac{\vec{r_i} - \vec{r_k}}{|\vec{r_i} - \vec{r_k}|^3} - \frac{\vec{r_i}}{|\vec{r_i}|^3} \right)$$

$\Rightarrow$

$$\frac{d^2 \vec{r_1}}{dt^2} = -G\,(m_0 + m_1)\frac{\vec{r_1}}{|\vec{r_1}|^3} + G\, m_2 \left(\frac{\vec{r_2} - \vec{r_1}}{|\vec{r_2} - \vec{r_1}|^3} - \frac{\vec{r_2}}{|\vec{r_2}|^3} \right)$$

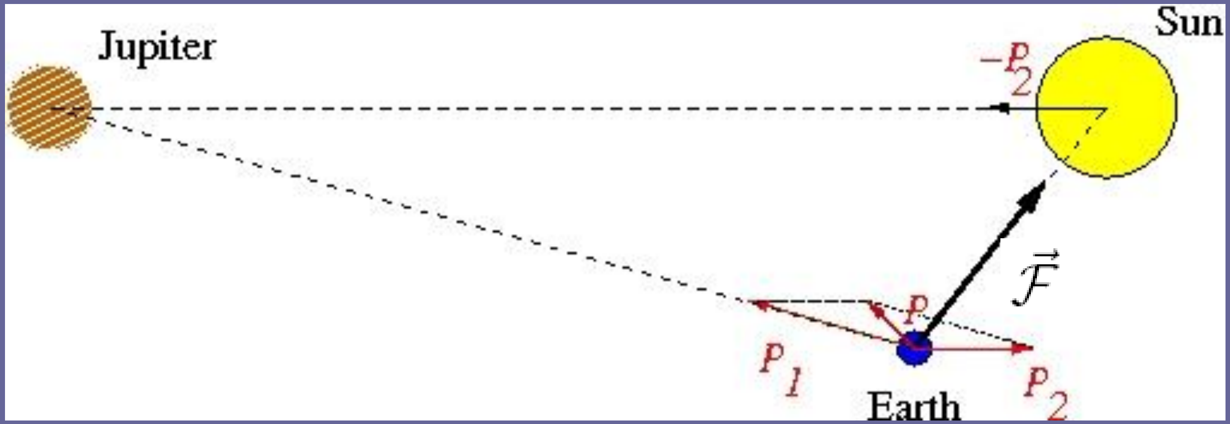
$$\frac{d^2 \vec{r_2}}{dt^2} = -G\,(m_0 + m_2)\frac{\vec{r_2}}{|\vec{r_2}|^3} + G\, m_1 \left(\frac{\vec{r_1} - \vec{r_2}}{|\vec{r_1} - \vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3} \right)$$

7.3 The 3-body problem

- Example: the Earth's motion around the Sun, accounting for the presence of Jupiter:

$$\frac{d^2 \vec{r_{♁}}}{dt^2} + G\,(m_\odot + m_{♁})\,\frac{\vec{r_{♁}}}{|\vec{r_{♁}}|^3} = G\,m_{♃}\left(\frac{\vec{r_{♃}} - \vec{r_{♁}}}{|\vec{r_{♃}} - \vec{r_{♁}}|^3} - \frac{\vec{r_{♃}}}{|\vec{r_{♃}}|^3}\right) = \vec{\mathcal{P}}$$

$$\vec{\mathcal{F}} = -G\,(m_\odot + m_{♁})\,\frac{\vec{r_{♁}}}{|\vec{r_{♁}}|^3}$$



7.3 The 3-body problem

- The Earth's motion around the Sun, accounting for the presence of Jupiter; order of magnitude of the force due to Jupiter:

$$\frac{|\vec{\mathcal{P}}|}{|\vec{\mathcal{F}}|} = \frac{m_{♃}}{m_{\odot} + m_{♁}} \left(\frac{r_{♁}}{r_{♃}}\right)^3$$

$$\Rightarrow \quad \boxed{\frac{|\vec{\mathcal{P}}|}{|\vec{\mathcal{F}}|} \simeq 10^{-5}}$$

- Same order of magnitude for the force due the Moon on a satellite in geostationary orbit.

$\Rightarrow$ In this case, one can treat the force of the 3rd body as a perturbation.

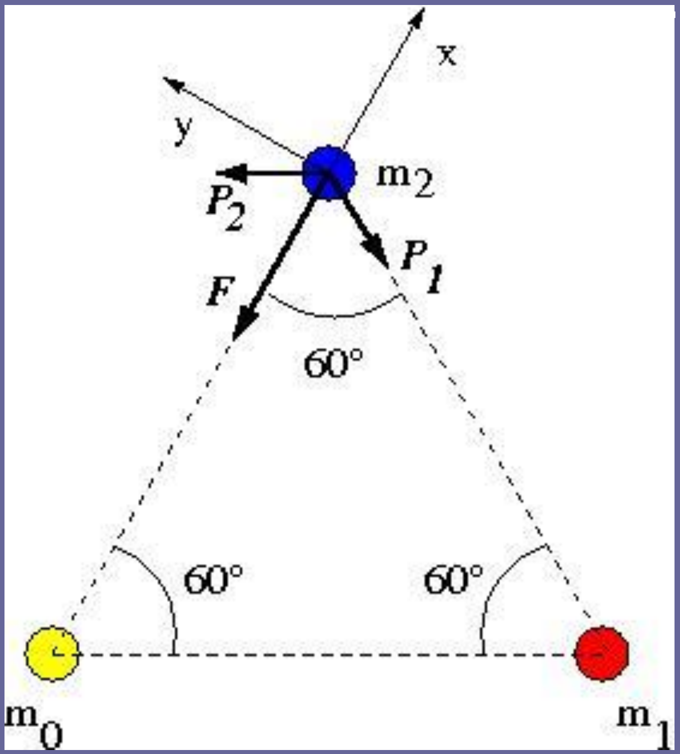
This approximation does not hold in all situations (e.g. triple stellar systems).

7.3.1 The Lagrange solutions

- Consider the 3-body problem with the masses m_0, m_1 and m_2 having arbitrary values.

➢ There exists a family of 5 analytical solutions (Lagrange solutions) where the problem can be formulated as the combination of two equivalent 2-body problems.

1. The masses are located at the an equilateral triangle (2 symmetric configurations with respect to the line m_0m_1).



7.3.1 The Lagrange solutions

Masses at the summits of an equilateral triangle:

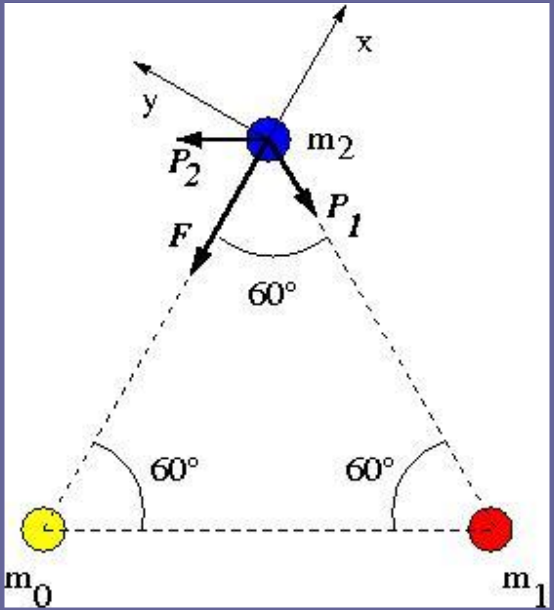
$$\frac{d^2 \vec{r_2}}{dt^2} = -G\,(m_0+m_2)\,\frac{\vec{r_2}}{|\vec{r_2}|^3} + G\,m_1\,\left(\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3}\right)$$

$$\Rightarrow \quad \ddot{x}_2 = -G\,(m_0+m_2)\,\frac{1}{r_2^2} - \frac{1}{2}\,G\,m_1\,\left(\frac{1}{r_2^2}+\frac{1}{r_2^2}\right) = -\frac{G\,(m_0+m_1+m_2)}{r_2^2}$$

$$\ddot{y}_2 = -\frac{G\,m_1}{r_2^2}\,\left(\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right) = 0$$

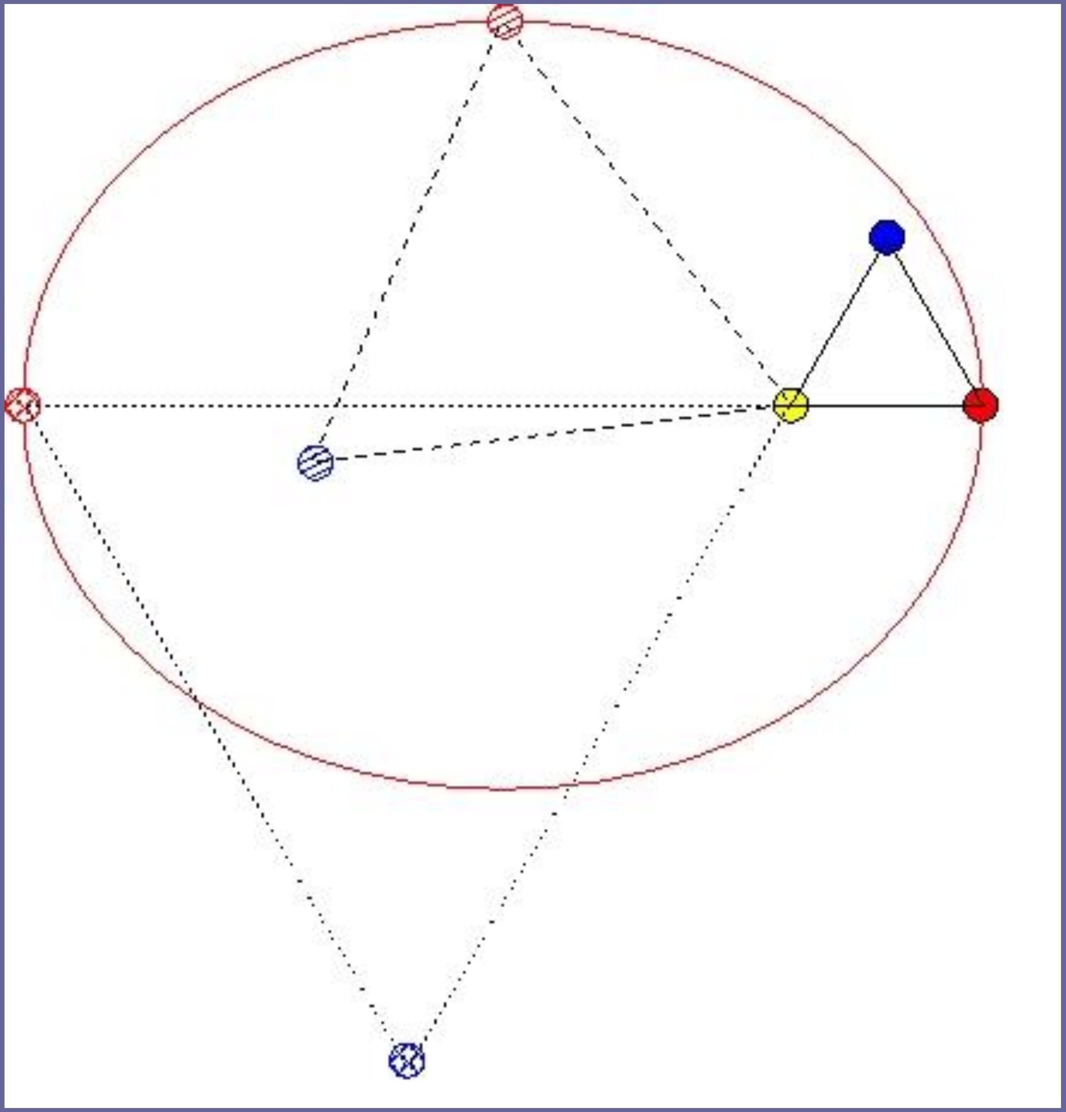
$$\Rightarrow \quad \frac{d^2 \vec{r_1}}{dt^2} = -G\,(m_0+m_1+m_2)\,\frac{\vec{r_1}}{|\vec{r_1}|^3}$$

$$\frac{d^2 \vec{r_2}}{dt^2} = -G\,(m_0+m_1+m_2)\,\frac{\vec{r_2}}{|\vec{r_2}|^3}$$

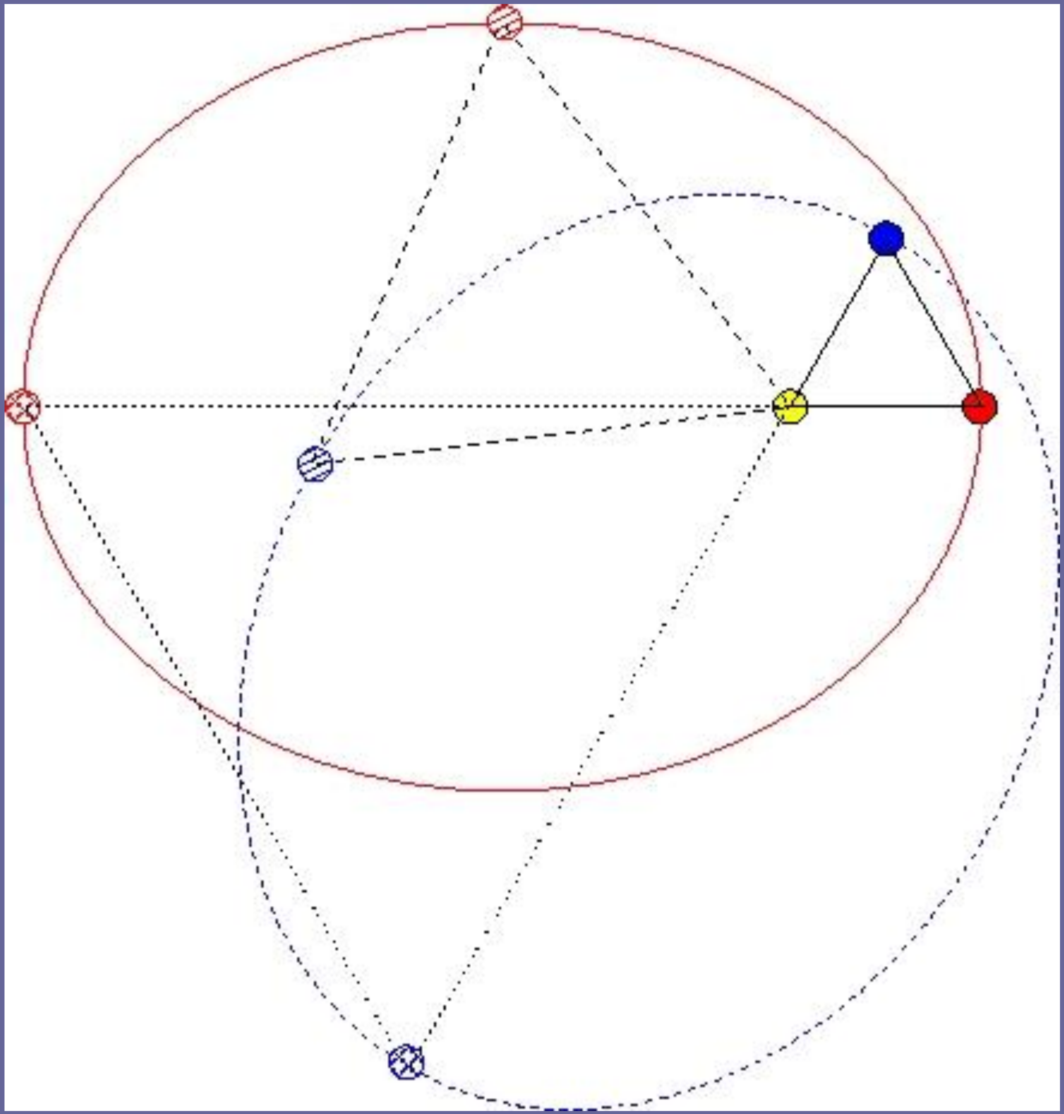


Solution if $|\dot{\vec{r_1}}| = |\dot{\vec{r_2}}|$ & $\vec{r_1} \wedge \dot{\vec{r_1}} = \vec{r_2} \wedge \dot{\vec{r_2}}$ at t = 0: identical conical sections rotated by 60°.

7.3.1 The Lagrange solutions



7.3.1 The Lagrange solutions



7.3.1 The Lagrange solutions

2. The masses are aligned with P_0P_1: $\vec{r_2} = \xi\,\vec{r_1}$

$$\begin{aligned}
\frac{d^2\,\vec{r_1}}{dt^2} &= -G\,(m_0+m_1)\,\frac{\vec{r_1}}{|\vec{r_1}|^3} + G\,m_2\left(\frac{\vec{r_2}-\vec{r_1}}{|\vec{r_2}-\vec{r_1}|^3} - \frac{\vec{r_2}}{|\vec{r_2}|^3}\right)\\
\frac{d^2\,\vec{r_2}}{dt^2} &= -G\,(m_0+m_2)\,\frac{\vec{r_2}}{|\vec{r_2}|^3} + G\,m_1\left(\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3}\right)
\end{aligned}$$

$$\Rightarrow\quad \begin{aligned}
\frac{d^2\,\vec{r_1}}{dt^2} &= -G\,(m_0+m_1)\,\frac{\vec{r_1}}{r_1^3} + G\,m_2\left(\frac{\xi-1}{|\xi-1|^3}\,\frac{\vec{r_1}}{r_1^3} - \frac{\xi}{|\xi|^3}\,\frac{\vec{r_1}}{r_1^3}\right)\\
\frac{d^2\,\vec{r_2}}{dt^2} &= -G\,(m_0+m_2)\,\frac{\xi}{|\xi|^3}\,\frac{\vec{r_1}}{r_1^3} + G\,m_1\left(-\frac{\xi-1}{|\xi-1|^3}\,\frac{\vec{r_1}}{r_1^3} - \frac{\vec{r_1}}{r_1^3}\right)\\
&= \xi\left[-G\,(m_0+m_1)\,\frac{\vec{r_1}}{r_1^3} + G\,m_2\left(\frac{\xi-1}{|\xi-1|^3}\,\frac{\vec{r_1}}{r_1^3} - \frac{\xi}{|\xi|^3}\,\frac{\vec{r_1}}{r_1^3}\right)\right]
\end{aligned}$$

7.3.1 The Lagrange solutions

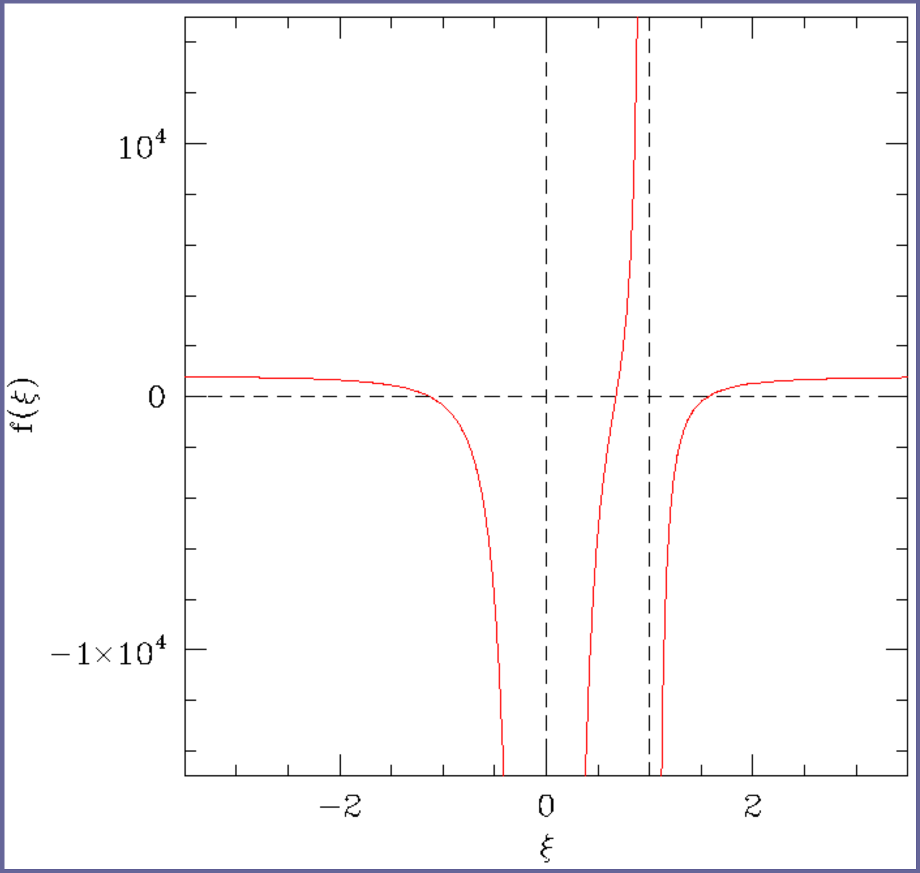
Masses aligned:

$$f(\xi) = m_0 + m_1 + m_2 \left(\frac{\xi}{|\xi|^3} + \frac{1-\xi}{|1-\xi|^3} \right) - \frac{1}{|\xi|^3} \left(m_0 + m_2 + m_1 \frac{|\xi|^3}{\xi} \left(1 - \frac{1-\xi}{|1-\xi|^3} \right) \right) = 0$$

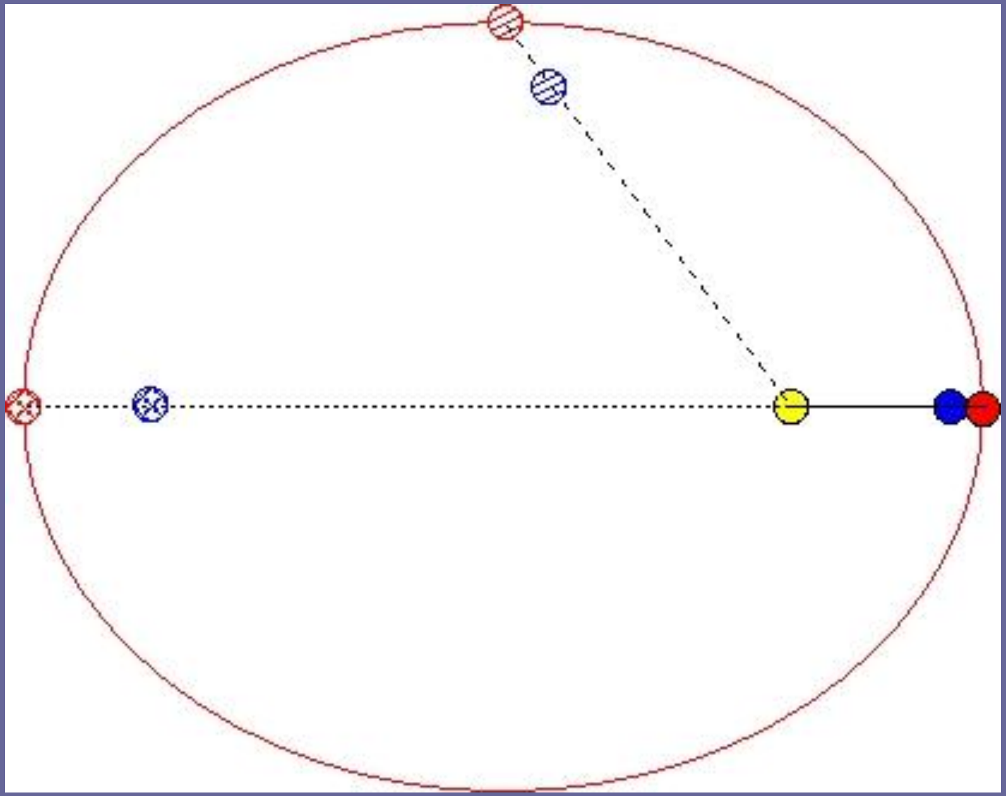
$\Rightarrow$ $\exists$ 3 zeros for this equation

1. one solution for $\xi < 0$
2. one solution for $0 < \xi < 1$
3. one solution for $\xi > 1$

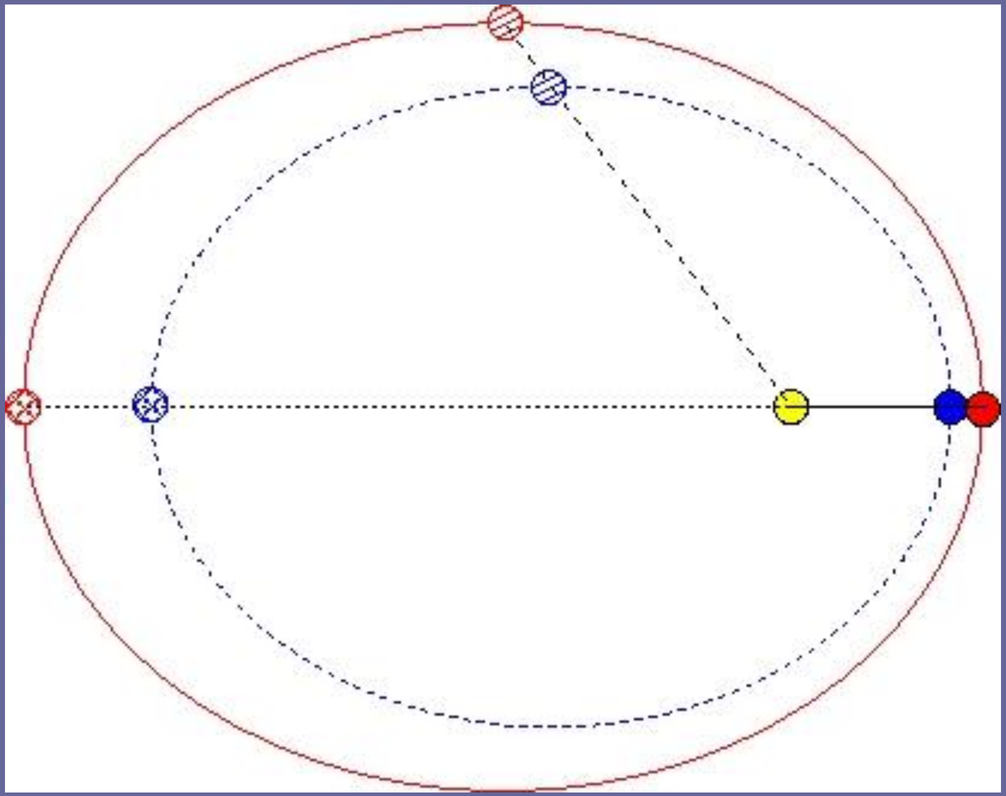
Solution if $\xi^2 \vec{r_1} \wedge \dot{\vec{r_1}} = \vec{r_2} \wedge \dot{\vec{r_2}}$ at t = 0: homothetic conical sections.



7.3.1 The Lagrange solutions



7.3.1 The Lagrange solutions



7.3.2 The restricted circular 3-body problem

- Suppose that $m_2 << \min(m_0, m_1)$ and that m_1 moves on a circular orbit around m_0 with $n_1 = \sqrt{\frac{G\,(m_0+m_1)}{a_1^3}}$

- We neglect m_2

$$\begin{aligned} \frac{d^2\,\vec{r_2}}{dt^2} &= -G\,m_0\,\frac{\vec{r_2}}{|\vec{r_2}|^3} + G\,m_1\left(\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3}\right) \\ &= \vec{\nabla}\left(\frac{G\,m_0}{r_2} + \frac{G\,m_1}{|\vec{r_1}-\vec{r_2}|} - m_1\,G\,\frac{\vec{r_1}\cdot\vec{r_2}}{a_1^3}\right) \end{aligned}$$

- In the frame of reference rotating with P_0P_1, angular velocity $\vec{\Omega} = n_1\,\vec{e_z}$

$$\frac{d^2\,\vec{r_2}}{dt^2} = \frac{\delta^2\,\vec{r_2}}{\delta t^2} + 2\,\vec{\Omega}\wedge\frac{\delta\,\vec{r_2}}{\delta t} + \vec{\Omega}\wedge(\vec{\Omega}\wedge\vec{r_2})$$

$$\Rightarrow \begin{aligned} \frac{\delta^2\,\vec{r_2}}{\delta t^2} + 2\,n_1\,\vec{e_z}\wedge\frac{\delta\,\vec{r_2}}{\delta t} + n_1^2\,\vec{e_z}\wedge(\vec{e_z}\wedge\vec{r_2}) &= -G\,m_0\,\frac{\vec{r_2}}{|\vec{r_2}|^3} + G\,m_1\left(\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3}\right) \\ &= \vec{\nabla}\left(\frac{G\,m_0}{r_2} + \frac{G\,m_1}{|\vec{r_1}-\vec{r_2}|} - m_1\,G\,\frac{\vec{r_1}\cdot\vec{r_2}}{a_1^3}\right) \end{aligned}$$

7.3.2 The restricted circular 3-body problem

- We project the equation on the axes:

$$\begin{aligned}\frac{\delta^2 \vec{r_2}}{\delta t^2} + 2\,n_1\,\vec{e_z}\wedge\frac{\delta\vec{r_2}}{\delta t} + n_1^2\,\vec{e_z}\wedge(\vec{e_z}\wedge\vec{r_2}) &= -G\,m_0\,\frac{\vec{r_2}}{|\vec{r_2}|^3} + G\,m_1\left(\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3}\right)\\ &= \vec{\nabla}\left(\frac{G\,m_0}{r_2} + \frac{G\,m_1}{|\vec{r_1}-\vec{r_2}|} - m_1\,G\,\frac{\vec{r_1}\cdot\vec{r_2}}{a_1^3}\right)\end{aligned}$$

$$\Rightarrow\quad \begin{aligned}\ddot{x} - 2\,n_1\,\dot{y} &= \frac{\partial}{\partial x}\left(\frac{G\,m_0}{r_2} + \frac{G\,m_1}{|\vec{r_1}-\vec{r_2}|} - m_1\,G\,\frac{\vec{r_1}\cdot\vec{r_2}}{a_1^3}\right) + n_1^2\,x\\ \ddot{y} + 2\,n_1\,\dot{x} &= \frac{\partial}{\partial y}\left(\frac{G\,m_0}{r_2} + \frac{G\,m_1}{|\vec{r_1}-\vec{r_2}|} - m_1\,G\,\frac{\vec{r_1}\cdot\vec{r_2}}{a_1^3}\right) + n_1^2\,y\\ \ddot{z} &= \frac{\partial}{\partial z}\left(\frac{G\,m_0}{r_2} + \frac{G\,m_1}{|\vec{r_1}-\vec{r_2}|} - m_1\,G\,\frac{\vec{r_1}\cdot\vec{r_2}}{a_1^3}\right)\end{aligned}$$

- Roche potential:

$$\Phi = \frac{G\,m_0}{\sqrt{x^2+y^2+z^2}} + \frac{G\,m_1}{\sqrt{(a_1-x)^2+y^2+z^2}} + \frac{n_1^2}{2}\left[\left(x - \frac{m_1}{m_0+m_1}\,a_1\right)^2 + y^2\right]$$

7.3.2 The restricted circular 3-body problem

- Jacobi integral: $\frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)=\Phi-C_J$
- Symmetrical expression of the Roche potential:

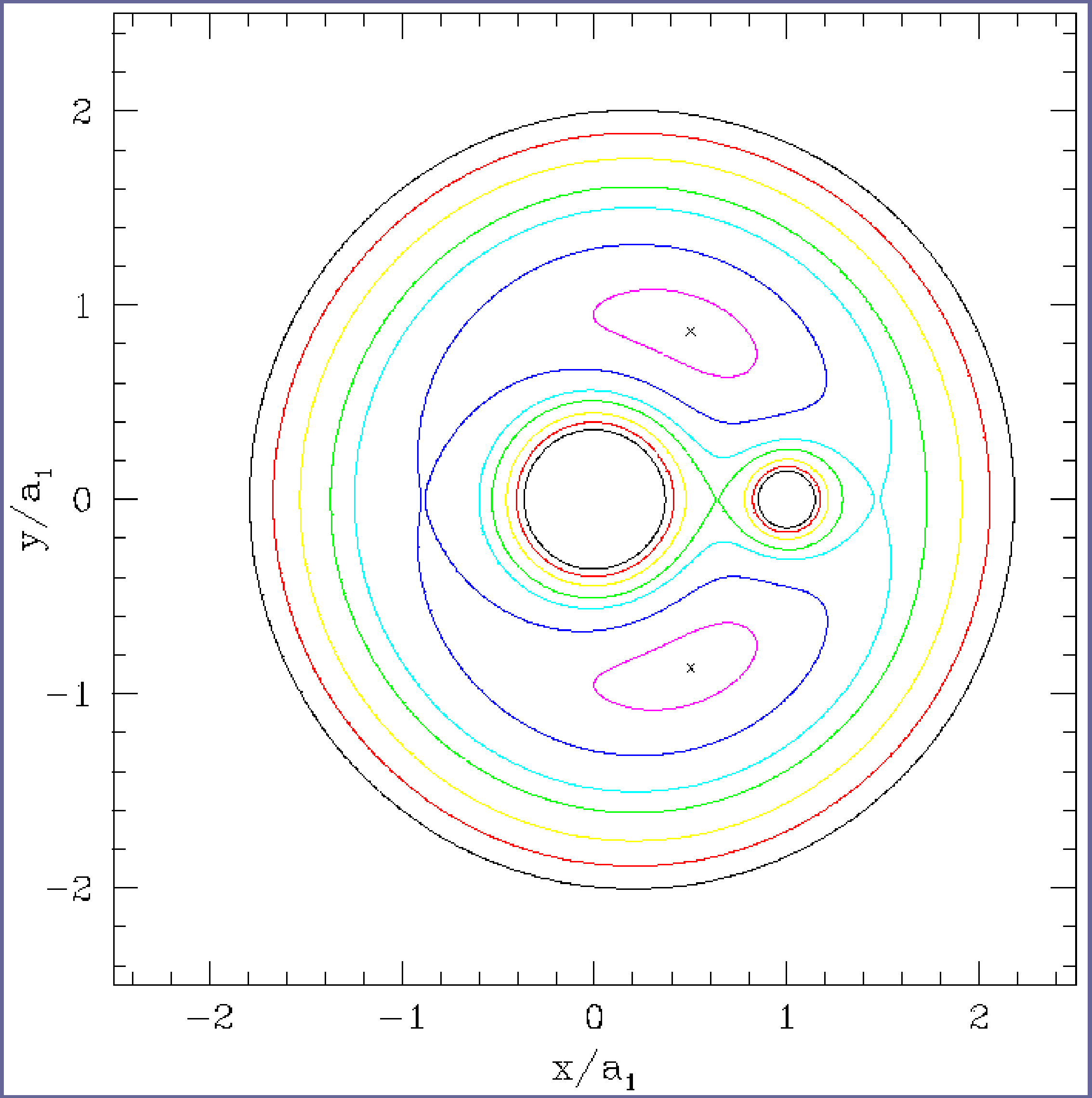
$$\Phi = n_1^2\, a_1^3 \left\{\left[\frac{m_0}{m_0+m_1}\left(\frac{1}{\rho_0}+\frac{\rho_0^2}{2\,a_1^3}\right)+\frac{m_1}{m_0+m_1}\left(\frac{1}{\rho_1}+\frac{\rho_1^2}{2\,a_1^3}\right)\right]-\frac{1}{2\,a_1^3}\left(z^2+\frac{m_0\,m_1}{(m_0+m_1)^2}\,a_1^2\right)\right\}$$

- Equipotential surfaces contained in a cylinder of equation

$$n_1^2\left[\left(x-\frac{m_1}{m_0+m_1}\,a_1\right)^2+y^2\right]=2\,C_J$$

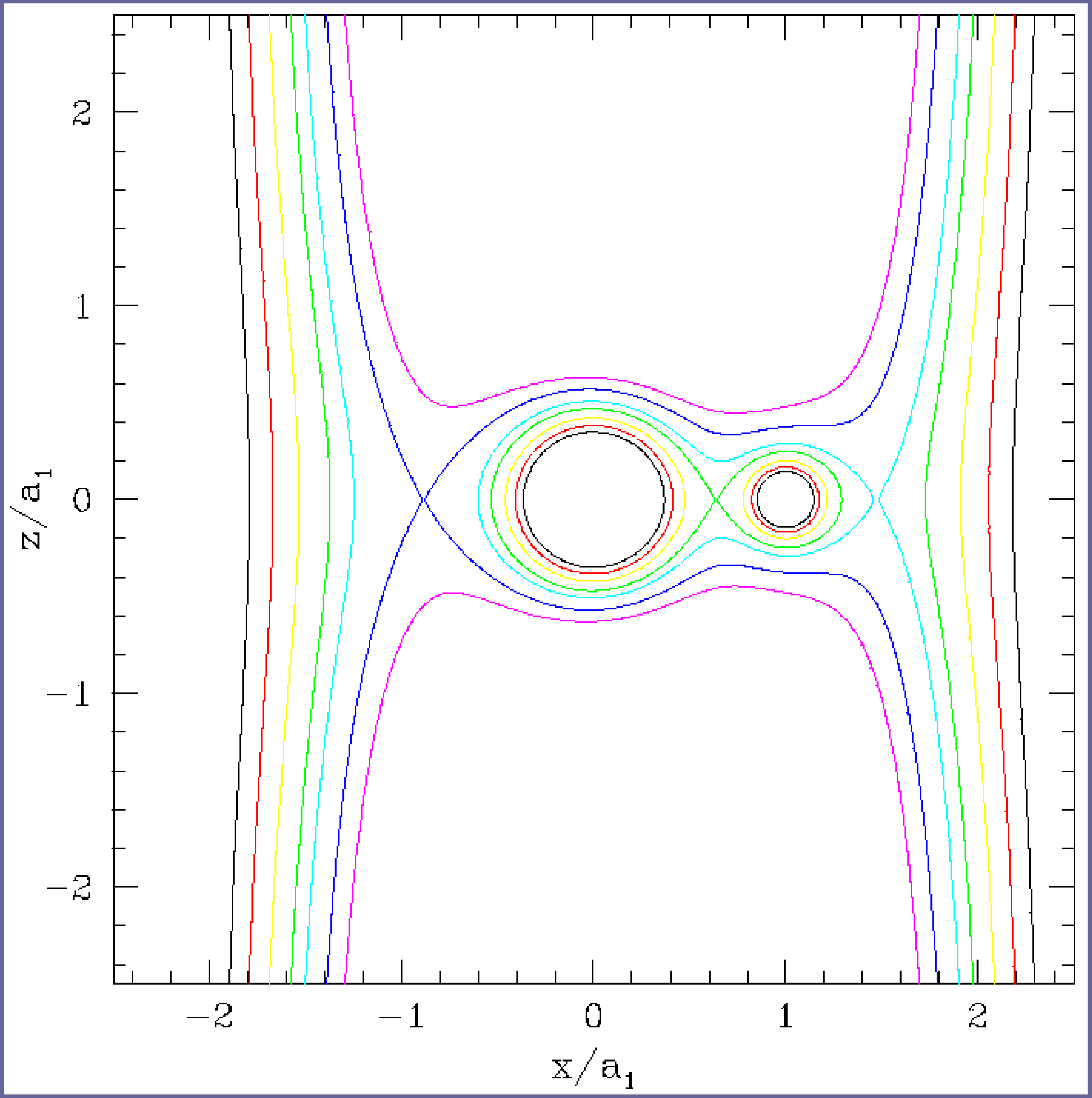
7.3.2 The restricted circular 3-body problem

- Section of the Roche potential in the *x y* plane:



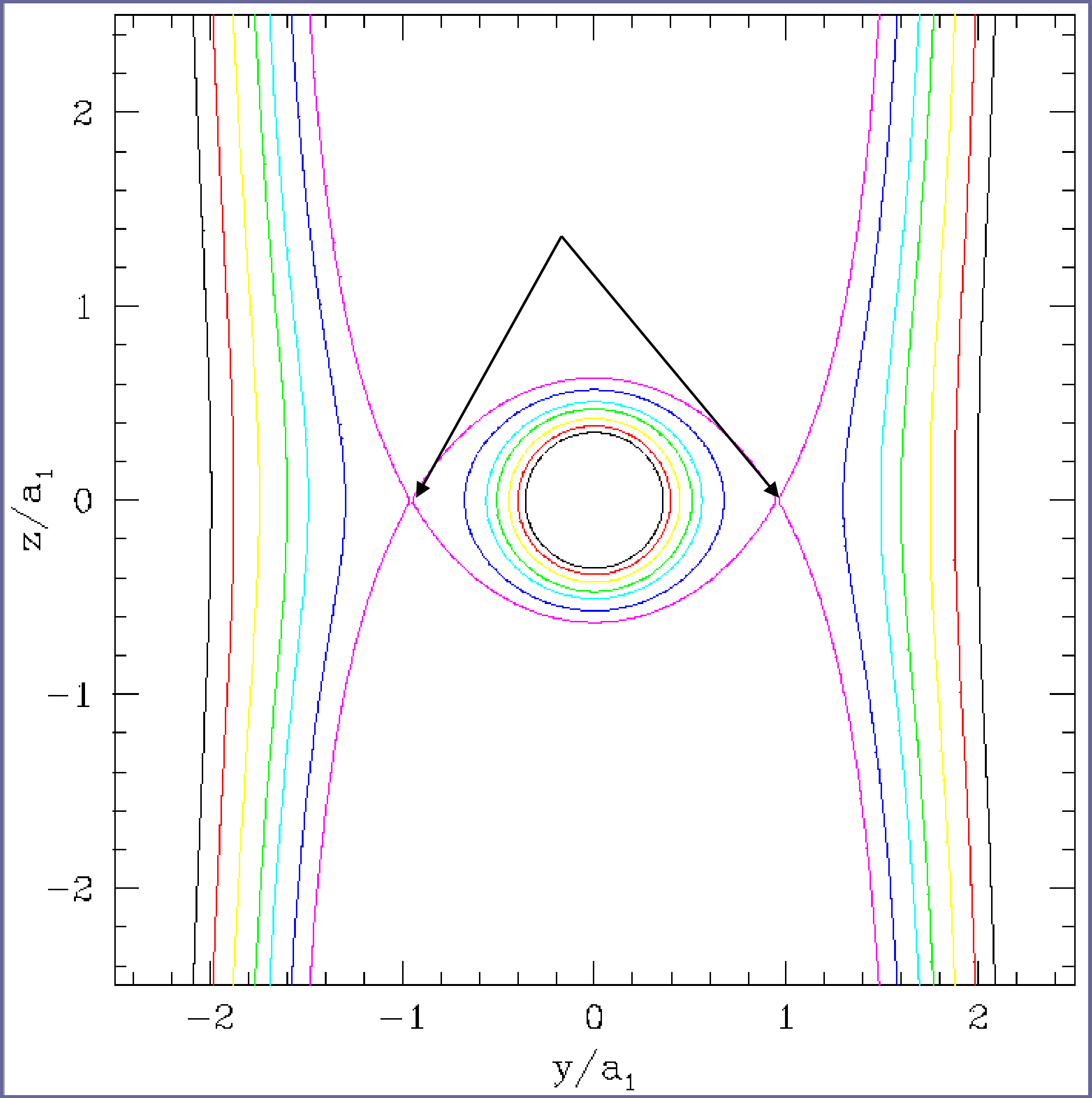
7.3.2 The restricted circular 3-body problem

- Section of the Roche potential in the *x z* plane:



7.3.2 The restricted circular 3-body problem

- Section of the Roche potential in the *y z* plane:



7.3.2 The restricted circular 3-body problem

- Lagrangian points = relative equilibrium positions:

$$\vec{\nabla}\,\Phi = \vec{0} \Rightarrow \begin{cases} \frac{\partial\,\Phi}{\partial x} = -\frac{G\,m_0\,x}{[x^2+y^2+z^2]^{3/2}} - \frac{G\,m_1\,(x-a_1)}{[(a_1-x)^2+y^2+z^2]^{3/2}} + n_1^2\left(x - \frac{m_1}{m_0+m_1}\,a_1\right) = 0 \\ \frac{\partial\,\Phi}{\partial y} = -\frac{G\,m_0\,y}{[x^2+y^2+z^2]^{3/2}} - \frac{G\,m_1\,y}{[(a_1-x)^2+y^2+z^2]^{3/2}} + n_1^2\,y = 0 \\ \frac{\partial\,\Phi}{\partial z} = -\frac{G\,m_0\,z}{[x^2+y^2+z^2]^{3/2}} - \frac{G\,m_1\,z}{[(a_1-x)^2+y^2+z^2]^{3/2}} = 0 \end{cases}$$

- All solutions of these equations yield *z = 0*.
- Different possibilities depending on the value of *y*:

1. *y = 0*

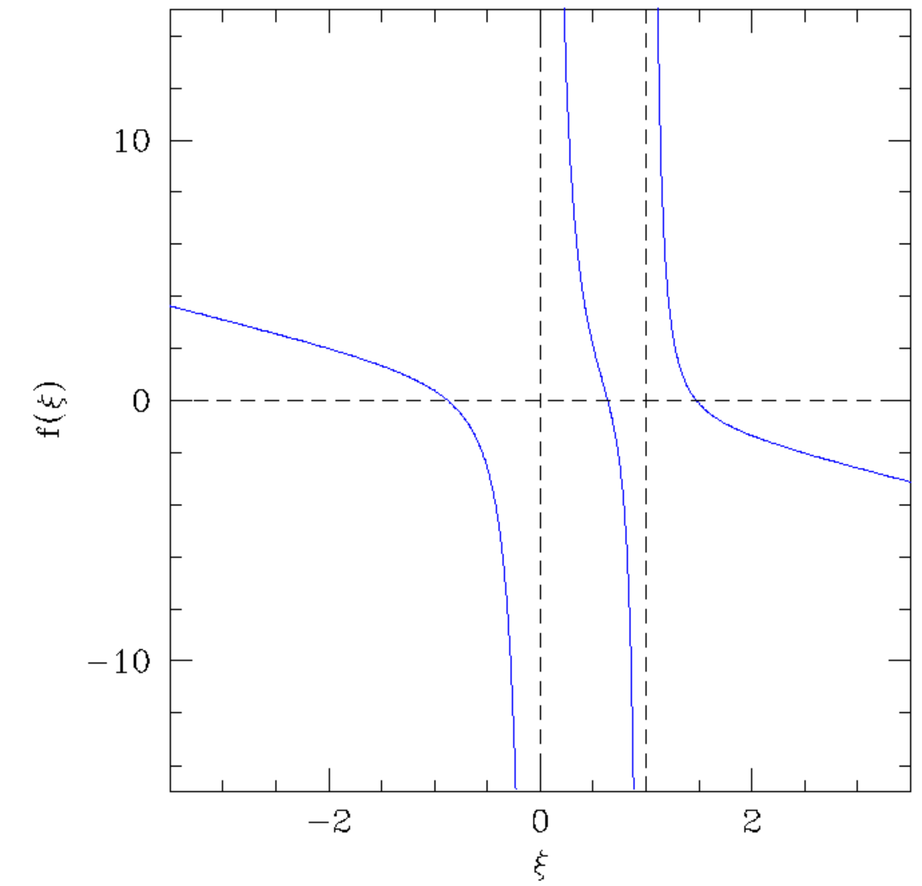
$$\frac{-G\,m_0\,x}{|x|^3} - \frac{G\,m_1\,(x-a_1)}{|x-a_1|^3} + n_1^2\left(x - \frac{m_1}{m_0+m_1}\,a_1\right) = 0$$

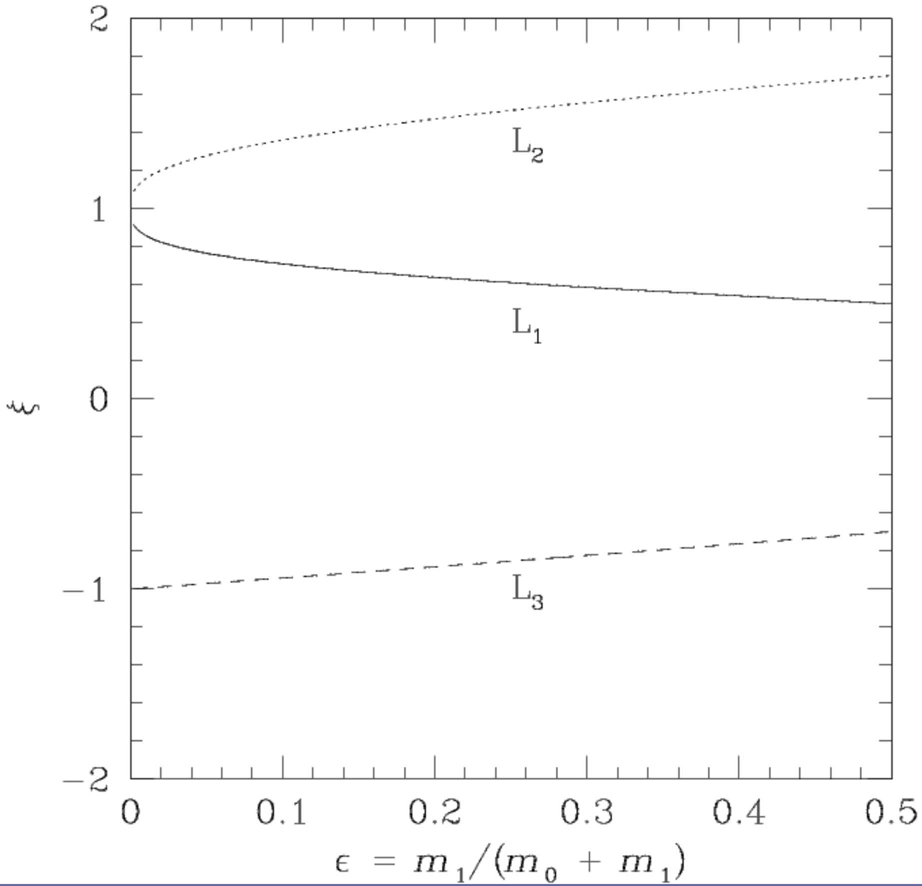
$x = \xi\,a_1 \Rightarrow$

$$f(\xi) = \frac{m_0}{m_0+m_1}\,\frac{\xi}{|\xi|^3} + \frac{m_1}{m_0+m_1}\,\frac{\xi-1}{|\xi-1|^3} - \left(\xi - \frac{m_1}{m_0+m_1}\right) = 0$$

7.3.2 The restricted circular 3-body problem

$$f(\xi) = \frac{m_0}{m_0 + m_1}\frac{\xi}{|\xi|^3} + \frac{m_1}{m_0 + m_1}\frac{\xi - 1}{|\xi - 1|^3} - \left(\xi - \frac{m_1}{m_0 + m_1}\right) = 0$$





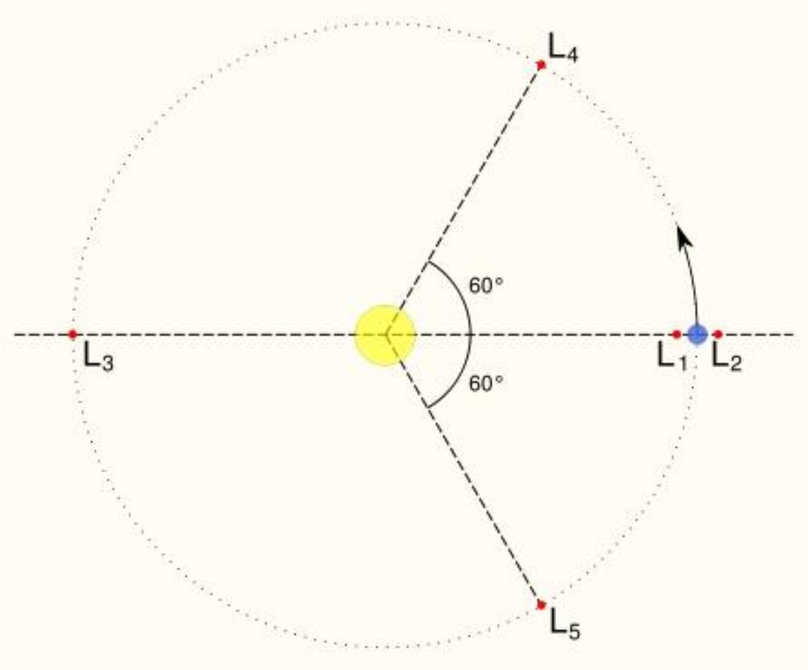
7.3.2 The restricted circular 3-body problem

2. $y \neq 0$

$$-\frac{G\,m_0}{[x^2+y^2]^{3/2}} - \frac{G\,m_1}{[(a_1-x)^2+y^2]^{3/2}} + n_1^2 = 0$$

$$\Rightarrow \quad \begin{aligned} (x-a_1)^2+y^2 &= a_1^2 \\ x^2+y^2 &= a_1^2 \end{aligned}$$

$$\Rightarrow \quad x = a_1/2 \quad \& \quad y = \pm\frac{\sqrt{3}}{2}\,a_1$$



7.3.2 The restricted circular 3-body problem

- Stability of the Lagrangian points: linear analysis

$$
\begin{aligned}
x &= x_0 + \delta x \\
y &= y_0 + \delta y \\
z &= \delta z
\end{aligned}
$$

$$
\begin{aligned}
\ddot{x} - 2\,n_1\,\dot{y} &= \frac{\partial}{\partial x}\left(\frac{G\,m_0}{r_2} + \frac{G\,m_1}{|\vec{r_1} - \vec{r_2}|} - m_1\,G\,\frac{\vec{r_1}\cdot\vec{r_2}}{a_1^3}\right) + n_1^2\,x \\
\ddot{y} + 2\,n_1\,\dot{x} &= \frac{\partial}{\partial y}\left(\frac{G\,m_0}{r_2} + \frac{G\,m_1}{|\vec{r_1} - \vec{r_2}|} - m_1\,G\,\frac{\vec{r_1}\cdot\vec{r_2}}{a_1^3}\right) + n_1^2\,y \\
\ddot{z} &= \frac{\partial}{\partial z}\left(\frac{G\,m_0}{r_2} + \frac{G\,m_1}{|\vec{r_1} - \vec{r_2}|} - m_1\,G\,\frac{\vec{r_1}\cdot\vec{r_2}}{a_1^3}\right)
\end{aligned}
$$

$\Rightarrow$

$$
\begin{aligned}
\delta\ddot{x} &= \frac{\partial^2\,\Phi}{\partial x^2}\,\delta x + \frac{\partial^2\,\Phi}{\partial x\,\partial y}\,\delta y + \frac{\partial^2\,\Phi}{\partial x\,\partial z}\,\delta z + 2\,n_1\,\delta\dot{y} \\
\delta\ddot{y} &= \frac{\partial^2\,\Phi}{\partial x\,\partial y}\,\delta x + \frac{\partial^2\,\Phi}{\partial y^2}\,\delta y + \frac{\partial^2\,\Phi}{\partial y\,\partial z}\,\delta z - 2\,n_1\,\delta\dot{x} \\
\delta\ddot{z} &= \frac{\partial^2\,\Phi}{\partial x\,\partial z}\,\delta x + \frac{\partial^2\,\Phi}{\partial y\,\partial z}\,\delta y + \frac{\partial^2\,\Phi}{\partial z^2}\,\delta z
\end{aligned}
$$

7.3.2 The restricted circular 3-body problem

- Stability of the Lagrangian points: linear analysis

$$\frac{\partial^2 \Phi}{\partial y\, \partial z} = \frac{\partial^2 \Phi}{\partial x\, \partial z} = 0 \qquad \frac{\partial^2 \Phi}{\partial z^2} = -\frac{G\, m_0}{[x^2+y^2]^{3/2}} - \frac{G\, m_1}{[(a_1-x)^2+y^2]^{3/2}} = -\lambda^2 < 0$$

$$\begin{aligned}
\delta\ddot{x} &= \frac{\partial^2 \Phi}{\partial x^2}\,\delta x + \frac{\partial^2 \Phi}{\partial x\, \partial y}\,\delta y + \frac{\partial^2 \Phi}{\partial x\, \partial z}\,\delta z + 2\, n_1\, \delta\dot{y} \\
\delta\ddot{y} &= \frac{\partial^2 \Phi}{\partial x\, \partial y}\,\delta x + \frac{\partial^2 \Phi}{\partial y^2}\,\delta y + \frac{\partial^2 \Phi}{\partial y\, \partial z}\,\delta z - 2\, n_1\, \delta\dot{x} \\
\delta\ddot{z} &= \frac{\partial^2 \Phi}{\partial x\, \partial z}\,\delta x + \frac{\partial^2 \Phi}{\partial y\, \partial z}\,\delta y + \frac{\partial^2 \Phi}{\partial z^2}\,\delta z
\end{aligned}$$

$$\Rightarrow \quad \delta\ddot{z} + \lambda^2\, \delta z = 0$$

$$\Rightarrow \quad \delta z = Z_1 \cos(\lambda\, t) + Z_2 \sin(\lambda\, t)$$

- Stable along z.

7.3.2 The restricted circular 3-body problem

- Stability of the Lagrangian points: linear analysis

$$\begin{aligned}
\delta\ddot{x} &= \frac{\partial^2\Phi}{\partial x^2}\,\delta x + \frac{\partial^2\Phi}{\partial x\,\partial y}\,\delta y + \frac{\partial^2\Phi}{\partial x\,\partial z}\,\delta z + 2\,n_1\,\delta\dot{y} \\
\delta\ddot{y} &= \frac{\partial^2\Phi}{\partial x\,\partial y}\,\delta x + \frac{\partial^2\Phi}{\partial y^2}\,\delta y + \frac{\partial^2\Phi}{\partial y\,\partial z}\,\delta z - 2\,n_1\,\delta\dot{x} \\
\delta\ddot{z} &= \frac{\partial^2\Phi}{\partial x\,\partial z}\,\delta x + \frac{\partial^2\Phi}{\partial y\,\partial z}\,\delta y + \frac{\partial^2\Phi}{\partial z^2}\,\delta z
\end{aligned}$$

$$u = \frac{\partial^2\Phi}{\partial x^2},\; v = \frac{\partial^2\Phi}{\partial x\,\partial y} \quad \& \quad w = \frac{\partial^2\Phi}{\partial y^2}$$

- Solutions of the kind *δx=X exp(αt)* and *δy=Y exp(αt)*

$$\Rightarrow \begin{pmatrix} \alpha^2 - u & -v - 2\,n_1\,\alpha \\ -v + 2\,n_1\,\alpha & \alpha^2 - w \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow (\alpha^2 - u)\,(\alpha^2 - w) - (v + 2\,n_1\,\alpha)\,(v - 2\,n_1\,\alpha) = 0$$

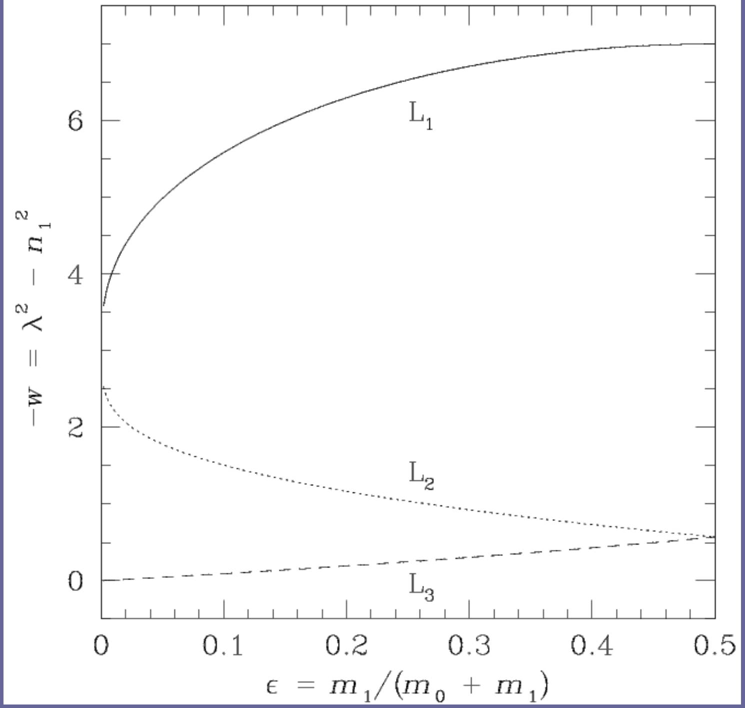
7.3.2 The restricted circular 3-body problem

- Stability of the Lagrangian points: linear analysis

$$(\alpha^2 - u)(\alpha^2 - w) - (v + 2 n_1 \alpha)(v - 2 n_1 \alpha) = 0$$

$$\Rightarrow \quad \alpha^4 + (4 n_1^2 - u - w)\,\alpha^2 + (u\,w - v^2) = 0$$

- L_1, L_2, L_3 points:



$$u\,w - v^2 < 0$$

$\Rightarrow$ unstable equilibrium

7.3.2 The restricted circular 3-body problem

- Stability of the Lagrangian points: linear analysis

$$(\alpha^2 - u)(\alpha^2 - w) - (v + 2\,n_1\,\alpha)(v - 2\,n_1\,\alpha) = 0$$

$\Rightarrow$ $$\alpha^4 + (4\,n_1^2 - u - w)\,\alpha^2 + (u\,w - v^2) = 0$$

- L_4 and L_5:

$u = \frac{3}{4}\,n_1^2,\ v = \frac{3\sqrt{3}}{4}\,(m_0 - m_1)\,\frac{G}{a_1^3}$ & $w = \frac{9}{4}\,n_1^2.$

$\Rightarrow$ $$\alpha^4 + n_1^2\,\alpha^2 + \frac{27}{4}\,n_1^4\,\frac{m_1\,m_0}{(m_1 + m_0)^2} = 0$$

7.3.2 The restricted circular 3-body problem

- The L_4 and L_5 points: $$\alpha^4 + n_1^2 \alpha^2 + \frac{27}{4} n_1^4 \frac{m_1 m_0}{(m_1 + m_0)^2} = 0$$

$$\Rightarrow \quad \Delta = n_1^4 - 27 n_1^4 \frac{m_1 m_0}{(m_1 + m_0)^2} \quad \& \quad \alpha^2 = \frac{-n_1^2 \pm \sqrt{\Delta}}{2}$$

- Solutions for α are purely imaginary provided that

$$0 \leq \Delta < n_1^4$$

- L_4 and L_5 are stable if $q > (25 + \sqrt{621})/2 = 24.96$

7.3.2 The restricted circular 3-body problem

- Orbits about the L_4 and L_5 points. Let us assume that

$$q > (25 + \sqrt{621})/2 = 24.96$$

and

$$\varepsilon = \frac{1}{1+q} << (25.96)^{-1} = 0.0385$$

$$\alpha^4 + n_1^2\,\alpha^2 + \frac{27}{4}\,n_1^4\,\frac{m_1\,m_0}{(m_1+m_0)^2} = 0 \Rightarrow \alpha^4 + n_1^2\,\alpha^2 + \frac{27}{4}\,n_1^4\,\varepsilon\,(1-\varepsilon) = 0$$

$$\alpha^2 \simeq \frac{-n_1^2 \pm n_1^2\,\sqrt{1-27\,\varepsilon}}{2} \simeq \frac{n_1^2}{2}\left[-1 \pm \left(1 - \frac{27\,\varepsilon}{2}\right)\right]$$

- The solutions are long-period oscillations and oscillations with frequency close to n_1

$$\Rightarrow \quad \begin{aligned} \alpha &\simeq \pm\frac{3\sqrt{3\,\varepsilon}}{2}\,n_1\,j \\ \alpha &\simeq \pm\sqrt{1-\frac{27\,\varepsilon}{4}}\,n_1\,j \simeq \left(1-\frac{27\,\varepsilon}{8}\right)\,n_1\,j \end{aligned}$$

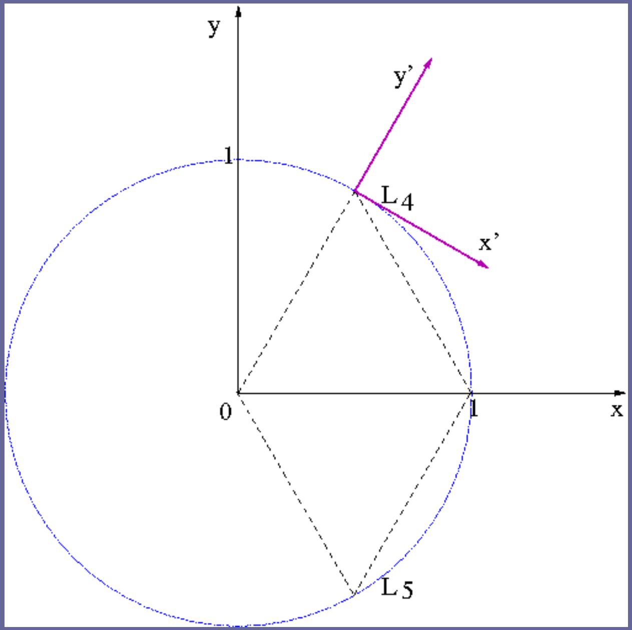
7.3.2 The restricted circular 3-body problem

- The amplitudes are solution of

$$\begin{pmatrix} \alpha^2 - u & -v - 2\,n_1\,\alpha \\ -v + 2\,n_1\,\alpha & \alpha^2 - w \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- We rotate our axes by –π/6:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix}$$



- So that:

$$\begin{pmatrix} \alpha^2 - \frac{3\,n_1^2}{4} & -\frac{3\sqrt{3}}{4}\,n_1^2\,(1-2\,\varepsilon) - 2\,n_1\,\alpha \\ -\frac{3\sqrt{3}}{4}\,n_1^2\,(1-2\,\varepsilon) + 2\,n_1\,\alpha & \alpha^2 - \frac{9\,n_1^2}{4} \end{pmatrix} \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$

$$\left(\frac{\sqrt{3}}{2}\,\alpha^2 - \frac{3\sqrt{3}}{4}\,\varepsilon\,n_1^2 + n_1\,\alpha\right) X' + \left(\frac{\alpha^2}{2} - \frac{3\,n_1^2}{2} + \frac{9\,n_1^2}{4}\,\varepsilon - \sqrt{3}\,n_1\,\alpha\right) Y' = 0$$

7.3.2 The restricted circular 3-body problem

$$\left(\frac{\sqrt{3}}{2}\,\alpha^2 - \frac{3\,\sqrt{3}}{4}\,\varepsilon\, n_1^2 + n_1\,\alpha\right)\,X' + \left(\frac{\alpha^2}{2} - \frac{3\,n_1^2}{2} + \frac{9\,n_1^2}{4}\,\varepsilon - \sqrt{3}\,n_1\,\alpha\right)Y' = 0$$

- For the low-frequency terms, $\alpha \;\simeq\; \pm\frac{3\,\sqrt{3\,\varepsilon}}{2}\,n_1\,j$ we find:

$$Y' \simeq \pm\sqrt{3\,\varepsilon}\,j\,X'$$

$$\Rightarrow \quad \delta x' = X'_{lp}\,\cos\left[\frac{3\,\sqrt{3\,\varepsilon}}{2}\,n_1\,t - \Phi_{lp}\right]$$

$$\delta y' = -\sqrt{3\,\varepsilon}\,X'_{lp}\,\sin\left[\frac{3\,\sqrt{3\,\varepsilon}}{2}\,n_1\,t - \Phi_{lp}\right]$$

- This libration mode has much larger amplitude along the tangential direction than along the radial direction.

7.3.2 The restricted circular 3-body problem

$$\left(\frac{\sqrt{3}}{2}\,\alpha^2 - \frac{3\sqrt{3}}{4}\,\varepsilon\, n_1^2 + n_1\,\alpha\right) X' + \left(\frac{\alpha^2}{2} - \frac{3\,n_1^2}{2} + \frac{9\,n_1^2}{4}\,\varepsilon - \sqrt{3}\,n_1\,\alpha\right) Y' = 0$$

- For the high-frequency terms, $\alpha \simeq \pm\sqrt{1-\frac{27\,\varepsilon}{4}}\,n_1\,j \simeq \left(1-\frac{27\,\varepsilon}{8}\right) n_1\,j$ we find: $Y' \simeq \pm\frac{j\,X'}{2}$

$\Rightarrow$

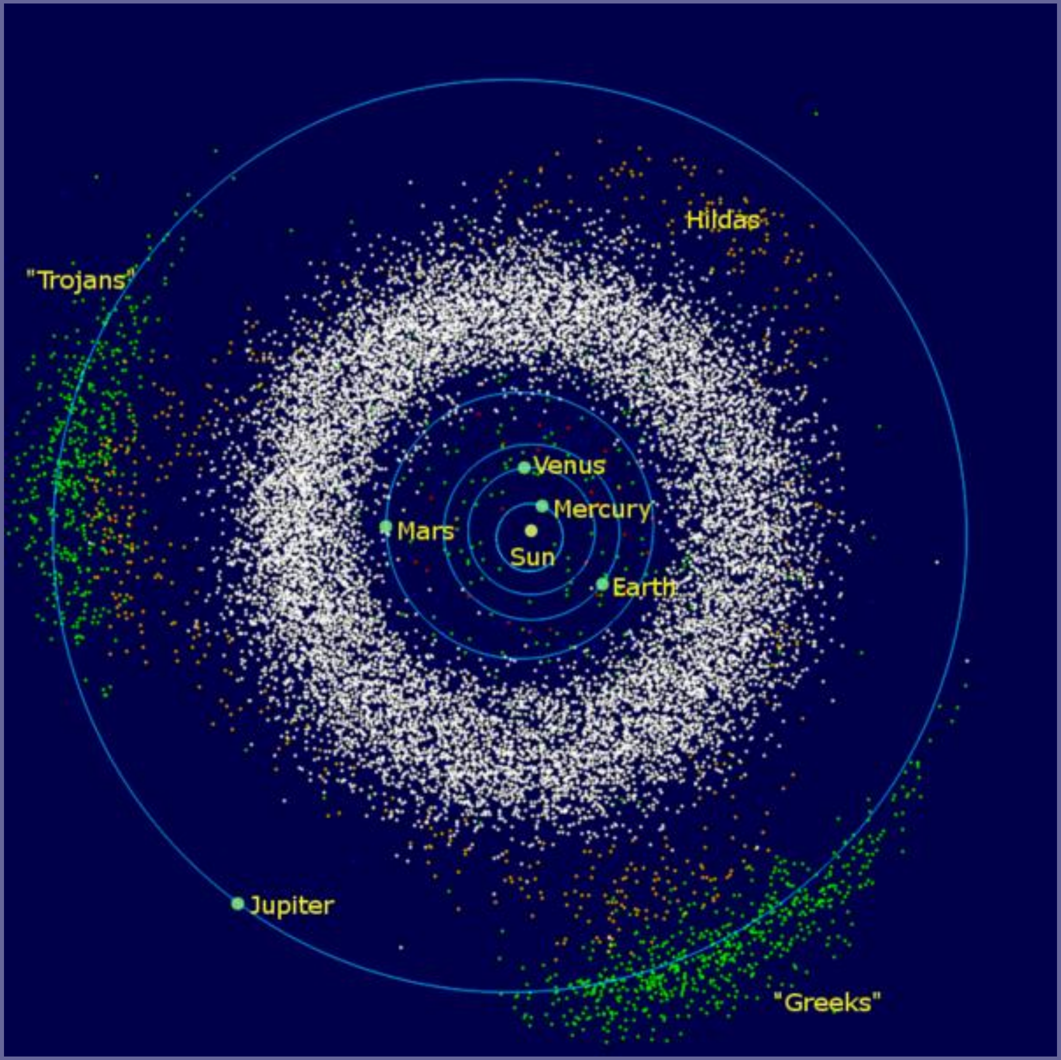
$$\delta x' = X'_{sp}\,\cos\left[\left(1-\frac{27\,\varepsilon}{8}\right) n_1\,t - \Phi_{sp}\right]$$

$$\delta y' = -\frac{X'_{sp}}{2}\,\sin\left[\left(1-\frac{27\,\varepsilon}{8}\right) n_1\,t - \Phi_{sp}\right]$$

- This oscillation mode corresponds to an elliptical motion.

7.3.2 The restricted circular 3-body problem

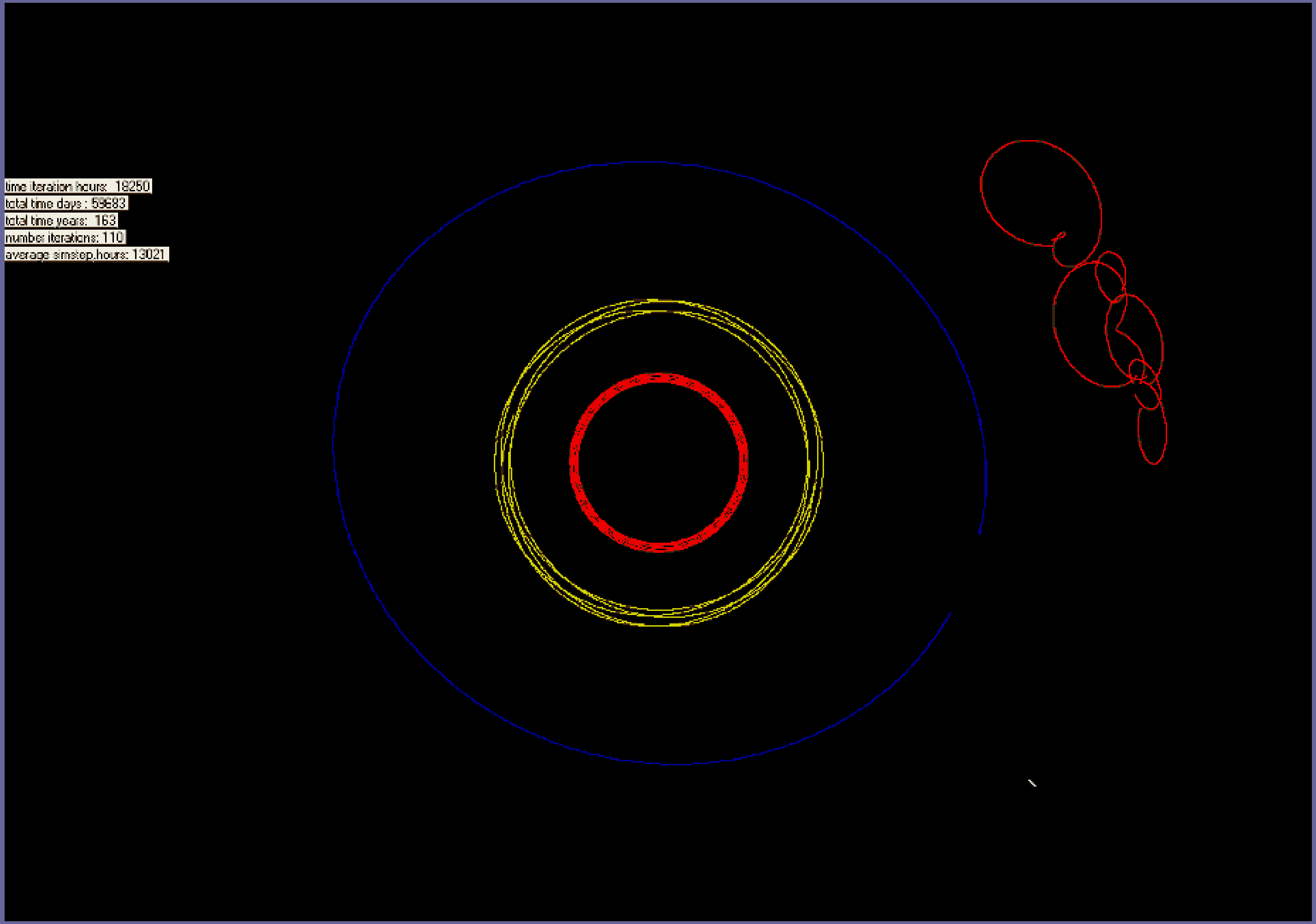
- The « Trojans » and the « Greeks »:



7.3.2 The restricted circular 3-body problem

7.3.2 The restricted circular 3-body problem

- Libration of Neptune’s Trojans under the influence of the other planets:



7.3.2 The restricted circular 3-body problem

- Are there stable orbits around the libration points L_1, L_2, L_3?

$$\begin{aligned} \delta\ddot{x} - 2\,n_1\,\delta\dot{y} - (2\,\lambda^2 + n_1^2)\,\delta x &= 0 \\ \delta\ddot{y} + 2\,n_1\,\delta\dot{x} + (\lambda^2 - n_1^2)\,\delta y &= 0 \\ \delta\ddot{z} + \lambda^2\,\delta z &= 0 \end{aligned}$$

- We have seen that $\delta z = Z\,\cos(\lambda\,t + \phi_z)$
- For solutions of the kind $\delta x = X_c\,exp(\alpha t)$ and $\delta y = Y_c\,exp(\alpha t)$, one can find initial conditions such that the argument of the exponential is a pure imaginary number: $\alpha = j\,\omega$
- $\delta x = \mathcal{R}[X_c\,exp(j\omega t)]$ and $\delta y = \mathcal{R}[Y_c\,exp(j\omega t)]$

$$\Rightarrow \begin{pmatrix} -(\omega^2 + 2\,\lambda^2 + n_1^2) & -2\,n_1\,\omega\,j \\ 2\,n_1\,\omega\,j & -(\omega^2 + n_1^2 - \lambda^2) \end{pmatrix} \begin{pmatrix} X_c \\ Y_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

7.3.2 The restricted circular 3-body problem

$$\begin{pmatrix} -(\omega^2 + 2\lambda^2 + n_1^2) & -2\,n_1\,\omega\,j \\ 2\,n_1\,\omega\,j & -(\omega^2 + n_1^2 - \lambda^2) \end{pmatrix} \begin{pmatrix} X_c \\ Y_c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \omega^4 + (\lambda^2 - 2\,n_1^2)\,\omega^2 + (n_1^4 + n_1^2\,\lambda^2 - 2\,\lambda^4) = 0$$

- A positive solution for ω² is given by $\omega^2 = \dfrac{\lambda\sqrt{9\,\lambda^2 - 8\,n_1^2} - \lambda^2 + 2\,n_1^2}{2}$
- This yields a relation between the complex amplitudes of δx and δy: $X_c = \dfrac{-(\omega^2 + n_1^2 - \lambda^2)\,j\,Y_c}{2\,n_1\,\omega}$
- Finally, we obtain: $\delta x = X\,\sin(\omega\,t + \phi_0)$ $\delta y = Y\,\cos(\omega\,t + \phi_0)$

& $X = \frac{\omega^2 + n_1^2 - \lambda^2}{2\,n_1\,\omega} Y$

	n_1 (rad/day)	λ^2/n_1^2	ω/n_1	X/Y
L_1 (Sun - Earth)	0.01720	4.060	2.086	0.310
L_2 (Sun - Earth)	0.01720	3.941	2.057	0.314
L_1 (Earth - Moon)	0.22997	5.147	2.334	0.279
L_2 (Earth - Moon)	0.22997	3.190	1.863	0.349

7.3.2 The restricted circular 3-body problem

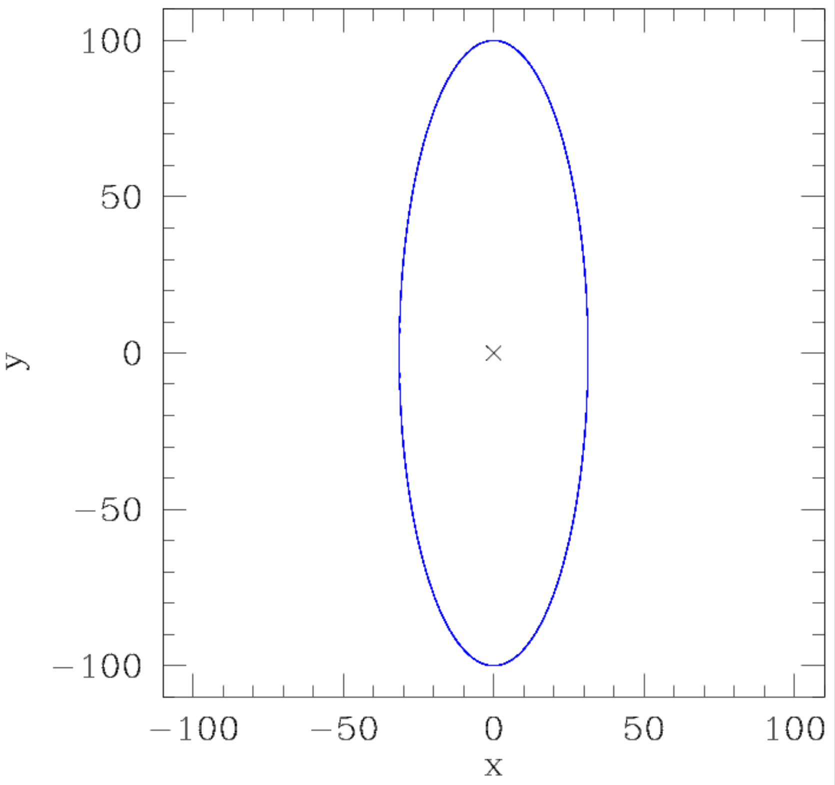
- Lissajous orbit:

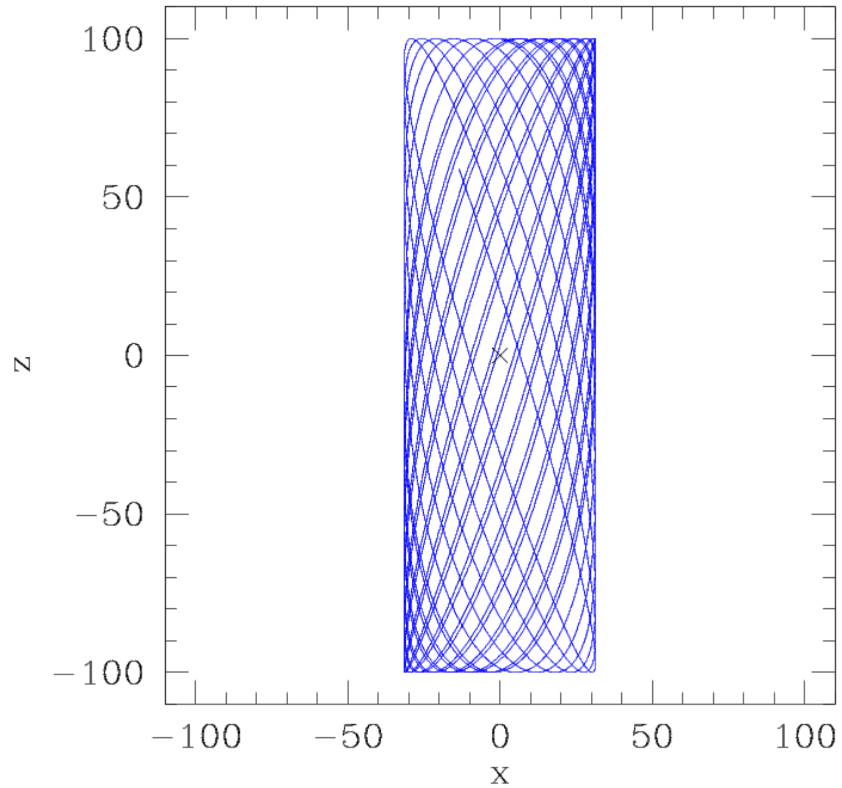
$$\delta x = X \sin(\omega t + \phi_0)$$

$$\delta y = Y \cos(\omega t + \phi_0)$$

$$\delta z = Z \cos(\lambda t + \phi_z)$$

$$\omega^2 = \frac{\lambda\sqrt{9\lambda^2 - 8n_1^2} - \lambda^2 + 2n_1^2}{2}$$





7.3.2 The restricted circular 3-body problem

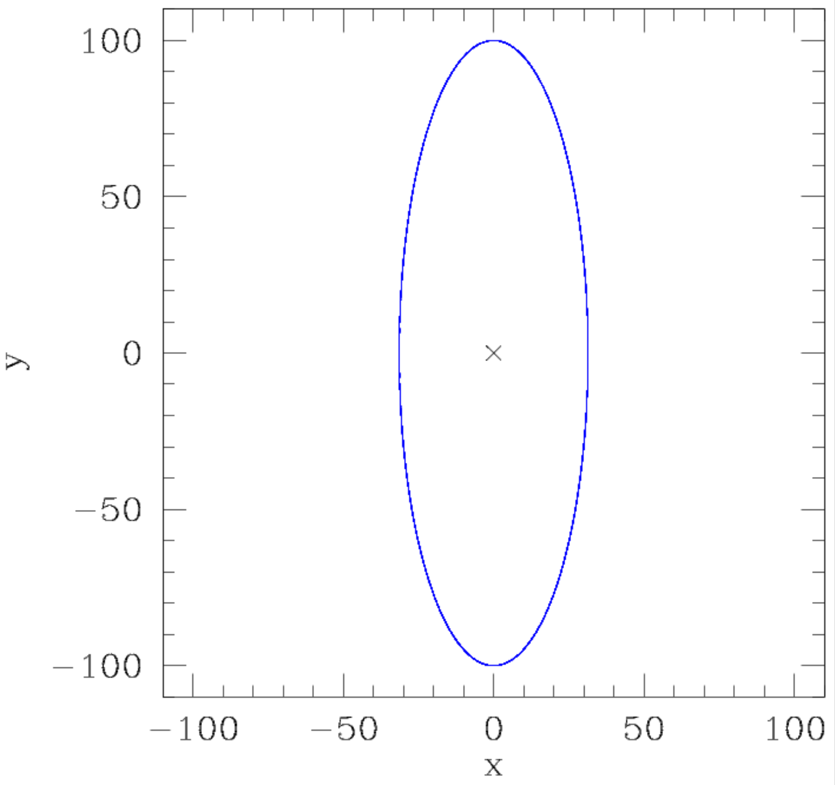
- Lissajous orbit:

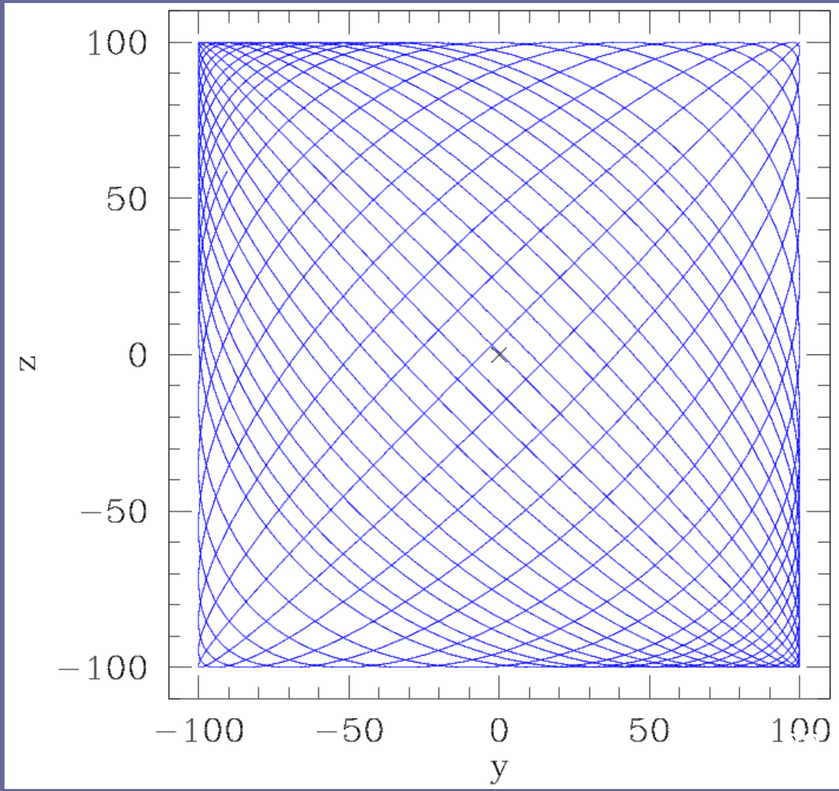
$$\delta x = X \sin(\omega t + \phi_0)$$

$$\delta y = Y \cos(\omega t + \phi_0)$$

$$\delta z = Z \cos(\lambda t + \phi_z)$$

$$\omega^2 = \frac{\lambda \sqrt{9\lambda^2 - 8 n_1^2} - \lambda^2 + 2 n_1^2}{2}$$





7.3.3 The sphere of influence

- Three-body problem: the Sun (P_0), planet (P_1) and spaceship (P_2):

$$\frac{d^2\vec{r_2}}{dt^2} = -G\,(m_0+m_2)\,\frac{\vec{r_2}}{|\vec{r_2}|^3} + G\,m_1\left(\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3}\right)$$

- When can this problem be treated in the heliocentric frame of reference?

$$R = \frac{m_1}{m_0+m_2}\,\frac{\left|\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3}\right|}{\frac{1}{|\vec{r_2}|^2}} < \eta$$

7.3.3 The sphere of influence

- Let φ be the angle between $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_0}$

$$\alpha = \frac{|\vec{r_2}-\vec{r_1}|}{|\vec{r_1}|}$$

$$\Rightarrow \left|\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3}\right| = \frac{1}{|\vec{r_1}-\vec{r_2}|^2}\left(1+\alpha^4 - 2\,\alpha^2\,\cos\phi\right)^{1/2}$$

$$|\vec{r_2}|^2 = |\vec{r_1}|^2\left(1+\alpha^2 - 2\,\alpha\,\cos\phi\right)$$

$$R = \frac{m_1}{m_0+m_2}\,\frac{\left|\frac{\vec{r_1}-\vec{r_2}}{|\vec{r_1}-\vec{r_2}|^3} - \frac{\vec{r_1}}{|\vec{r_1}|^3}\right|}{\frac{1}{|\vec{r_2}|^2}} < \eta$$

$$\Rightarrow R = \frac{m_1}{m_0+m_2}\,\frac{1}{\alpha^2}\left(1+\alpha^2 - 2\,\alpha\,\cos\phi\right)\left(1+\alpha^4 - 2\,\alpha^2\,\cos\phi\right)^{1/2}$$

$$\Rightarrow \frac{m_1}{m_0+m_2}\,\frac{1}{\alpha^2} \simeq \frac{m_1}{m_0}\,\frac{1}{\alpha^2} < \eta$$

7.3.3 The sphere of influence

- Three-body problem: the Sun (P_0), planet (P_1) and spaceship (P_2):

$$\frac{d^2\left(\vec{r_2}-\vec{r_1}\right)}{dt^2} = -G\left(m_1+m_2\right)\frac{\vec{r_2}-\vec{r_1}}{|\vec{r_2}-\vec{r_1}|^3} + G\,m_0\left(\frac{\vec{r_1}}{|\vec{r_1}|^3}-\frac{\vec{r_2}}{|\vec{r_2}|^3}\right)$$

- When can this problem be treated in the planetocentric frame of reference?

$$R' = \frac{m_0}{m_1+m_2}\,\frac{\left|\frac{\vec{r_1}}{|\vec{r_1}|^3}-\frac{\vec{r_2}}{|\vec{r_2}|^3}\right|}{\frac{1}{|\vec{r_2}-\vec{r_1}|^2}} < \eta$$

7.3.3 The sphere of influence

- Let φ be the angle between $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_0}$

$$\alpha = \frac{|\vec{r_2}-\vec{r_1}|}{|\vec{r_1}|}$$

$$\Rightarrow \quad |\vec{r_2}|^2 = |\vec{r_1}|^2\,(1+\alpha^2 - 2\,\alpha\,\cos\phi)$$

$$\left|\frac{\vec{r_1}}{|\vec{r_1}|^3} - \frac{\vec{r_2}}{|\vec{r_2}|^3}\right| = \frac{\alpha}{|\vec{r_1}|^2}\,(1+3\,\cos^2\phi + \mathcal{O}(\alpha))^{1/2}$$

$$R' = \frac{m_0}{m_1+m_2}\,\frac{\left|\frac{\vec{r_1}}{|\vec{r_1}|^3} - \frac{\vec{r_2}}{|\vec{r_2}|^3}\right|}{\frac{1}{|\vec{r_2}-\vec{r_1}|^2}} < \eta$$

$$\Rightarrow \quad R' \simeq \frac{m_0}{m_1+m_2}\,\alpha^3\sqrt{1+3\,\cos^2\phi} \geq \frac{m_0\,\alpha^3}{m_1+m_2}$$

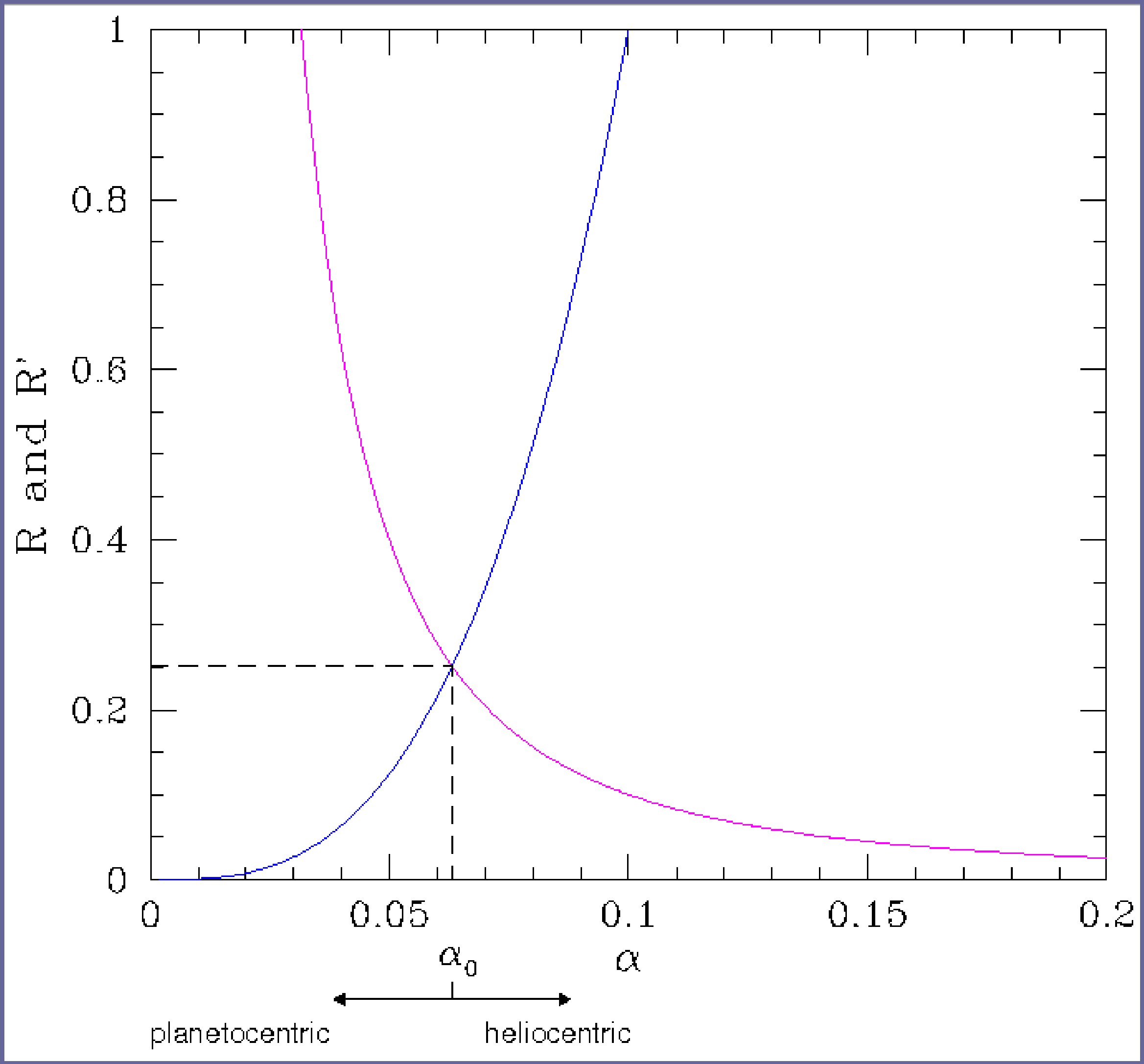
$$\Rightarrow \quad \frac{m_0}{m_1+m_2}\,\alpha^3 < \eta$$

7.3.3 The sphere of influence

- The two quantities are of the same order when

$$\frac{m_1}{m_0}\frac{1}{\alpha^2} = \frac{m_0}{m_1}\alpha^3$$

$$\Rightarrow \quad \alpha_0 = \left(\frac{m_1}{m_0}\right)^{2/5}$$



$$\alpha = \frac{|\vec{r_2} - \vec{r_1}|}{|\vec{r_1}|}$$

7.3.3 The sphere of influence

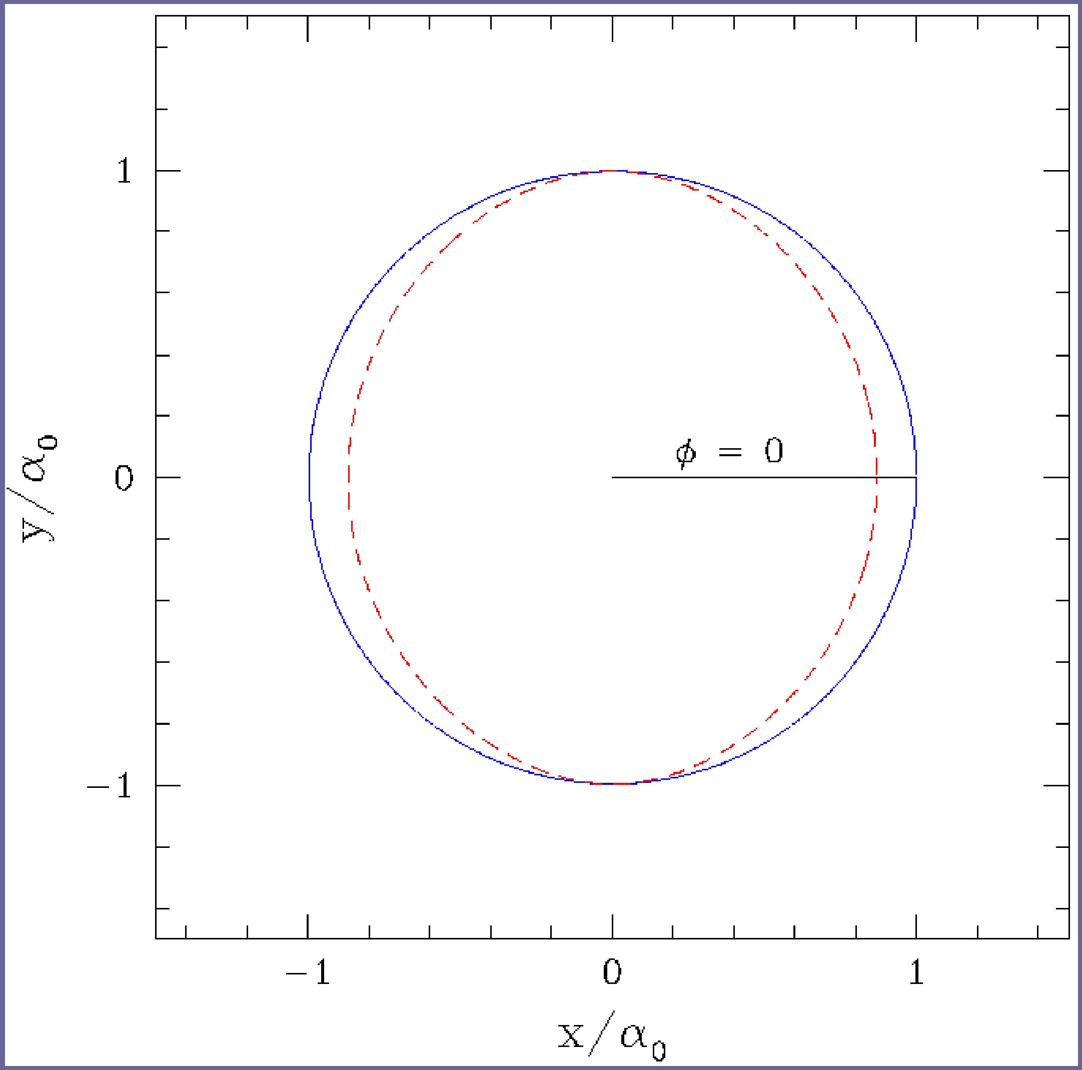
Planet	m_1/m_0	α_0	R_0	a_1 (AU)	$\alpha_0\, a_1$ (10^6 km)
Mercury	$1.66\, 10^{-7}$	0.0019	0.044	0.387	0.112
Venus	$2.45\, 10^{-6}$	0.0057	0.075	0.723	0.616
Earth	$3.00\, 10^{-6}$	0.0062	0.079	1.00	0.925
Mars	$3.23\, 10^{-7}$	0.0025	0.050	1.52	0.577
Jupiter	$9.55\, 10^{-4}$	0.0620	0.249	5.20	48.2
Saturn	$2.86\, 10^{-4}$	0.0382	0.196	9.54	54.6
Uranus	$4.37\, 10^{-5}$	0.0180	0.134	19.2	51.8
Neptune	$5.18\, 10^{-5}$	0.0193	0.138	30.1	86.8

$$\alpha_0 = \left(\frac{m_1}{m_0}\right)^{2/5}$$

7.3.3 The sphere of influence

- The more accurate solution includes a factor $(1 + 3\cos^2\phi)^{1/2}$

$$\alpha_0 = (1 + 3\cos^2\phi)^{-1/10}\left(\frac{m_1}{m_0}\right)^{2/5}$$



7.3.4 The Tisserand invariant

- Circular restricted 3-body problem with $m_0 >> m_1 >> m_2$
- Position vector of m_2 in reference frame centred on center of mass of m_0 and m_1

$$\vec{R_2} = \vec{r_2} - \frac{m_1}{m_0 + m_1}\, a_1\, \vec{e_x}$$

$$\vec{\Omega} = \sqrt{\frac{G\,(m_0+m_1)}{a_1^3}}\; \vec{e_z} \quad \Rightarrow \quad \frac{\delta\,\vec{r_2}}{\delta t} = \dot{\vec{R_2}} - \vec{\Omega} \wedge \vec{R_2}$$

- Thus, $$|\frac{\delta\,\vec{r_2}}{\delta t}|^2 = |\dot{\vec{R_2}}|^2 + |\vec{\Omega} \wedge \vec{R_2}|^2 - 2\,\vec{\Omega} \cdot \vec{h}$$

- The Roche potential yields

$$\begin{aligned} C_J &= \Phi - \frac{1}{2}|\frac{\delta\,\vec{r_2}}{\delta t}|^2 \\ &= \frac{G\,m_0}{\rho_0} + \frac{G\,m_1}{\rho_1} - \frac{1}{2}|\dot{\vec{R_2}}|^2 + \vec{\Omega} \cdot \vec{h} \\ &\simeq \vec{\Omega} \cdot \vec{h} - \varepsilon \end{aligned}$$

&

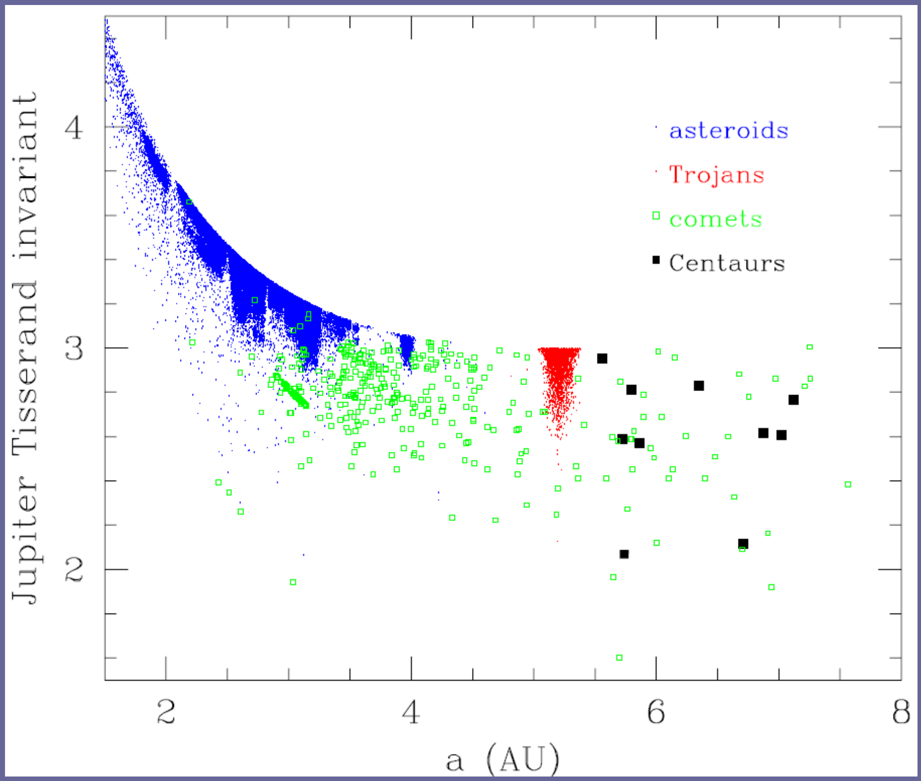
$$\varepsilon = \frac{1}{2}|\dot{\vec{R_2}}|^2 - \frac{G\,m_0}{\rho_0}$$

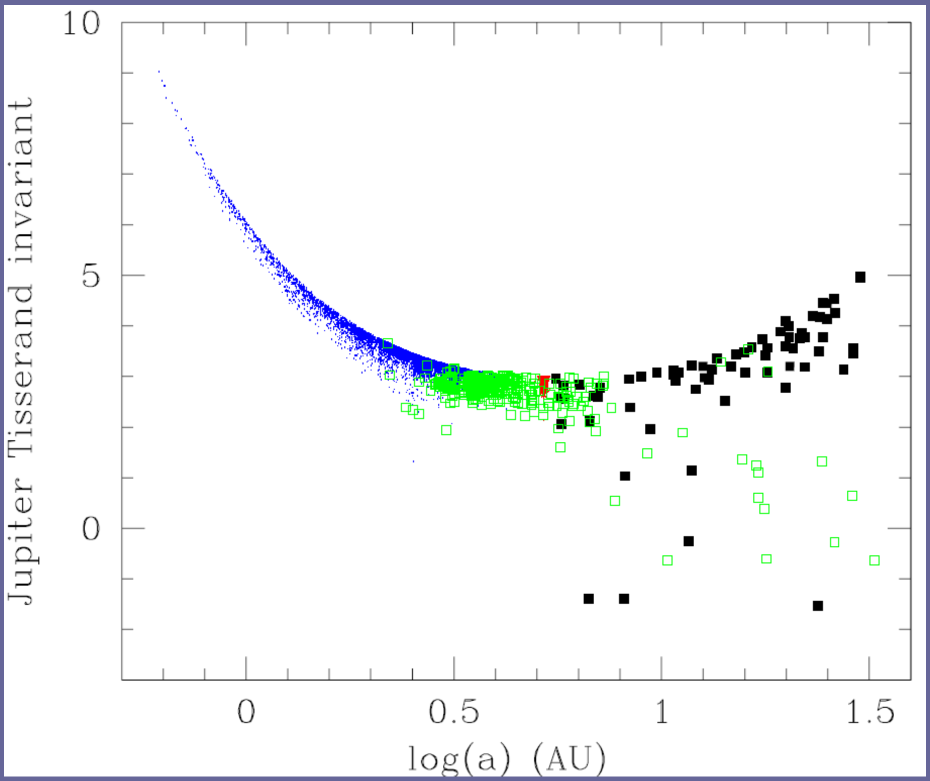
$$\Rightarrow \quad \vec{\Omega} \cdot \vec{h} - \varepsilon \simeq Cte$$

7.3.4 The Tisserand invariant

- Tisserand invariant: $\vec{\Omega}\cdot\vec{h}-\varepsilon \simeq Cte$

- E.g. Tisserand invariant for elliptical orbits in the Sun-Jupiter system: $$\frac{a_{\text{♃}}}{a}+2\sqrt{\frac{a\left(1-e^{2}\right)}{a_{\text{♃}}}}\cos i \simeq \frac{a_{\text{♃}}}{a'}+2\sqrt{\frac{a'\left(1-e'^{2}\right)}{a_{\text{♃}}}}\cos i' \simeq Cte$$





7.4 The motions in a planetary system

- The more massive body (the Sun) is much more massive than all the planets ($m_0 >> m_k$)
- Eccentricities and inclinations are small, avoiding planets to come too close to each-other.
- Newton's equation:

$$\frac{d^2 \vec{r_k}}{dt^2} = -\frac{G\,(m_0 + m_k)\,\vec{r_k}}{|\vec{r_k}|^3} + \sum_{n=1,n\neq k}^{N} \vec{\nabla}_k\, U_{k,n}$$

$$U_{k,n} = G\, m_n \left(\frac{1}{|\vec{r_n} - \vec{r_k}|} - \frac{\vec{r_n} \cdot \vec{r_k}}{|\vec{r_n}|^3} \right)$$

The potentials are different for the different planets.

7.4 The motions in a planetary system

- We consider the problem of two planets (1 and 2):

$$U_{k,n} = G\,m_n \left(\frac{1}{|\vec{r_n}-\vec{r_k}|} - \frac{\vec{r_n}\cdot\vec{r_k}}{|\vec{r_n}|^3}\right)$$

$$\Delta_{1,2} = |\vec{r_1} - \vec{r_2}|$$

$$\Rightarrow$$

$$U_{1,2} = G\,m_2 \left(\frac{1}{\Delta_{1,2}} - \frac{r_1\,\cos\gamma}{r_2^2}\right)$$

$$U_{2,1} = G\,m_1 \left(\frac{1}{\Delta_{1,2}} - \frac{r_2\,\cos\gamma}{r_1^2}\right)$$

- We introduce the following parameters: $\rho = \frac{r_1}{r_2} < 1$ $\alpha = \frac{a_1}{a_2} < 1$

$$\rho = \alpha\,\frac{r_1}{a_1}\,\frac{a_2}{r_2}$$

$$\Rightarrow$$

$$\begin{aligned}\frac{1}{\Delta_{1,2}} &= (r_1^2 + r_2^2 - 2\,r_1\,r_2\,\cos\gamma)^{-1/2} \\ &= \frac{1}{r_2}\,(1 - 2\,\rho\,\cos\gamma + \rho^2)^{-1/2} \\ &= \frac{1}{r_2}\sum_{n=0}^{\infty} \rho^n\,P_n(\cos\gamma)\end{aligned}$$

$$\frac{1}{\sqrt{1 - 2\,x\,t + t^2}} = \sum_{n=0}^{+\infty} P_n(x)\,t^n$$

7.4 The motions in a planetary system

- Developing to 2nd order in α:

$$\frac{1}{\Delta_{1,2}} = \frac{1}{a_2}\left(\frac{a_2}{r_2} + \alpha\frac{r_1}{a_1}\left(\frac{a_2}{r_2}\right)^2\cos\gamma + \alpha^2\left(\frac{r_1}{a_1}\right)^2\left(\frac{a_2}{r_2}\right)^3\left(\frac{3}{2}\cos^2\gamma - \frac{1}{2}\right) + \mathcal{O}(\alpha^3)\right)$$

Planet	a (AU)	e	i (°)	Ω (°)	ϖ (°)	L_0 (°)	n (arcsec day^{-1})
Mercury	0.3871	0.2056	7.00	48.33	77.46	252.25	14732.42
Venus	0.7233	0.0068	3.39	76.68	131.56	181.98	5767.67
Earth	1.0000	0.0167	–	–	102.94	100.47	3548.19
Mars	1.5237	0.0934	1.85	49.56	336.06	355.43	1886.52
Jupiter	5.2028	0.0485	1.30	100.46	14.33	34.35	299.128
Saturn	9.5388	0.0555	2.49	113.66	93.06	50.08	120.455
Uranus	19.182	0.0463	0.77	74.01	173.00	314.05	42.231
Neptune	30.058	0.0090	1.77	131.78	48.12	304.39	21.534
Pluto	39.44	0.2485	17.33	110.7	224.6	237.7	14.3

The majority of planets have roughly circular orbits with low inclinations with respect to the ecliptic.

7.4 The motions in a planetary system

- Problem of the definition of the longitude of the line of nodes and on the longitude of pericentre...
- We introduce more regular elements (for small *e* and *i*):

$$z = e\ \exp\left(j\,\varpi\right)$$

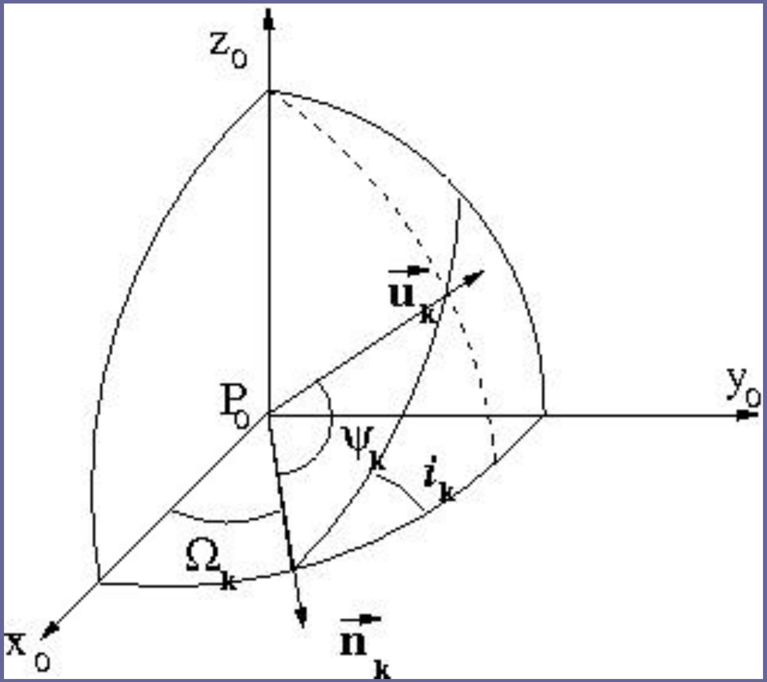
$$X = e\ \exp\left(j\,M\right) = e\ \exp\left[j\,(L - \varpi)\right]$$

$$\zeta = \sin\left(i/2\right)\ \exp\left(j\,\Omega\right)$$

$$Y = \sin\left(i/2\right)\ \exp\left[j\,(L - \Omega)\right]$$

$$\varpi = \Omega + \omega \qquad L = M + \varpi$$

$$l = L + (\phi - M) \qquad \psi_1 = l_1 - \Omega_1$$



7.4 The motions in a planetary system

- We express the potential as a function of the elements of the osculating orbit (here we restrict ourselves to e^2):

$$\frac{a}{r} = 1 + (e - \frac{e^3}{8} + \frac{e^5}{192}) \cos M + (e^2 - \frac{e^4}{3}) \cos(2M) + (\frac{9e^3}{8} - \frac{81e^5}{128}) \cos(3M) + \frac{4e^4}{3} \cos(4M) + \frac{625e^5}{384} \cos(5M) + \mathcal{O}(e^6)$$

$$\frac{r}{a} = 1 + \frac{e^2}{2} - (e - \frac{3e^3}{8} + \frac{5e^5}{192}) \cos M - (\frac{e^2}{2} - \frac{e^4}{3}) \cos(2M) - (\frac{3e^3}{8} - \frac{45e^5}{128}) \cos(3M) - \frac{e^4}{3} \cos(4M) - \frac{125e^5}{384} \cos(5M) + \mathcal{O}(e^6)$$

$$\frac{a}{r} = 1 + \frac{1}{2}(X + \overline{X}) + \frac{1}{2}(X^2 + \overline{X}^2) + \mathcal{O}(e^3)$$

$$\frac{r}{a} = 1 - \frac{1}{2}(X + \overline{X}) - \frac{1}{4}(X^2 - 2X\overline{X} + \overline{X}^2) + \mathcal{O}(e^3)$$

$$\theta = \exp[j(\phi - M)] = 1 + (X - \overline{X}) + \frac{1}{8}(9X^2 - 8X\overline{X} - \overline{X}^2) + \mathcal{O}(e^3)$$

7.4 The motions in a planetary system

- We express the potential as a function of the elements of the osculating orbit (here we restrict ourselves to order *d* in *e*):

$$\begin{aligned}
\frac{a}{r} &= 1 + \frac{1}{2}\,(X + \overline{X}) + \frac{1}{2}\,(X^2 + \overline{X}^2) + \mathcal{O}(e^3) \\
\frac{r}{a} &= 1 - \frac{1}{2}\,(X + \overline{X}) - \frac{1}{4}\,(X^2 - 2\,X\,\overline{X} + \overline{X}^2) + \mathcal{O}(e^3) \\
\theta = \exp\left[j\,(\phi - M)\right] &= 1 + (X - \overline{X}) + \frac{1}{8}\,(9\,X^2 - 8\,X\,\overline{X} - \overline{X}^2) + \mathcal{O}(e^3)
\end{aligned}$$

$$\Rightarrow \quad \left(\frac{r}{a}\right)^n \exp\left[j\,m\,(\phi - M)\right] = \sum_{0 \le p + p' \le d} X^p \overline{X}^{p'} C^{n,m}_{p,p'} + \mathcal{O}(e^{d+1})$$

7.4 The motions in a planetary system

- The development of the potential contains terms of the kind

$$\frac{r_1^n}{r_2^{n+1}} = \frac{\alpha^n}{a_2} \left(\frac{r_1}{a_1}\right)^n \left(\frac{a_2}{r_2}\right)^{n+1}$$

$$\Rightarrow \quad \frac{r_1^n}{r_2^{n+1}} = \frac{\alpha^n}{a_2} \sum_{0 \le p_1 + p_1' + p_2 + p_2' \le d} C_{p_1,p_1'}^{n,0} \, C_{p_2,p_2'}^{-n-1,0} X_1^{p_1} \, \overline{X_1}^{p_1'} \, X_2^{p_2} \, \overline{X_2}^{p_2'}$$

$$= \frac{\alpha^n}{a_2} \sum_{0 \le p_1 + p_1' + p_2 + p_2' \le d} C_{p_1,p_1'}^{n,0} \, C_{p_2,p_2'}^{-n-1,0} e_1^{(p_1+p_1')} \, e_2^{(p_2+p_2')} \times$$

$$\exp\left\{j\left[(p_1 - p_1')(L_1 - \varpi_1) + (p_2 - p_2')(L_2 - \varpi_2)\right]\right\}$$

since $X = e \exp(j\,M) = e \exp[j\,(L - \varpi)]$

7.4 The motions in a planetary system

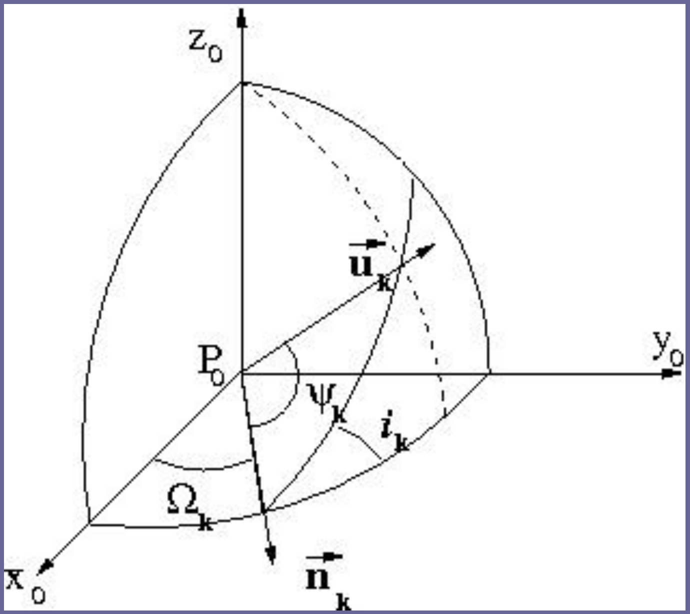
- We still need to express $\cos\gamma = \vec{u_1} \cdot \vec{u_2}$

$$\vec{u_1} = (\cos\psi_1\ \cos\Omega_1 - \sin\psi_1\ \sin\Omega_1\ \cos i_1, \cos\psi_1\ \sin\Omega_1 + \sin\psi_1\ \cos\Omega_1\ \cos i_1, \sin\psi_1\ \sin i_1)$$

$$\psi_1 = l_1 - \Omega_1$$

$$\varpi = \Omega + \omega$$

$$l = L + (\phi - M)$$



$$\Rightarrow \cos\gamma = \Re\left\{\cos^2\frac{i_1}{2}\ \cos^2\frac{i_2}{2}\ \exp\left[j\,(l_1 - l_2)\right] + \sin^2\frac{i_1}{2}\ \sin^2\frac{i_2}{2}\ \exp\left[j\,(l_1 - l_2 - 2\,\Omega_1 + 2\,\Omega_2)\right]\right.$$
$$+\sin^2\frac{i_1}{2}\ \cos^2\frac{i_2}{2}\ \exp\left[j\,(l_1 + l_2 - 2\,\Omega_1)\right] + \cos^2\frac{i_1}{2}\ \sin^2\frac{i_2}{2}\ \exp\left[j\,(l_1 + l_2 - 2\,\Omega_2)\right]$$
$$\left. + \frac{1}{2}\ \sin i_1\ \sin i_2\ \left(\exp\left[j\,(l_1 - l_2 - \Omega_1 + \Omega_2)\right] - \exp\left[j\,(l_1 + l_2 - \Omega_1 - \Omega_2)\right]\right)\right\}$$

7.4 The motions in a planetary system

- We introduce the resulting expressions of the potential in the Lagrange equations.
- We obtain a system of differential equations for the orbital elements.
- These equations contain terms $\exp\left[j\left(p\,L_1 + (p+q)\,L_2\right)\right]$
- The integration of these terms can lead to resonances

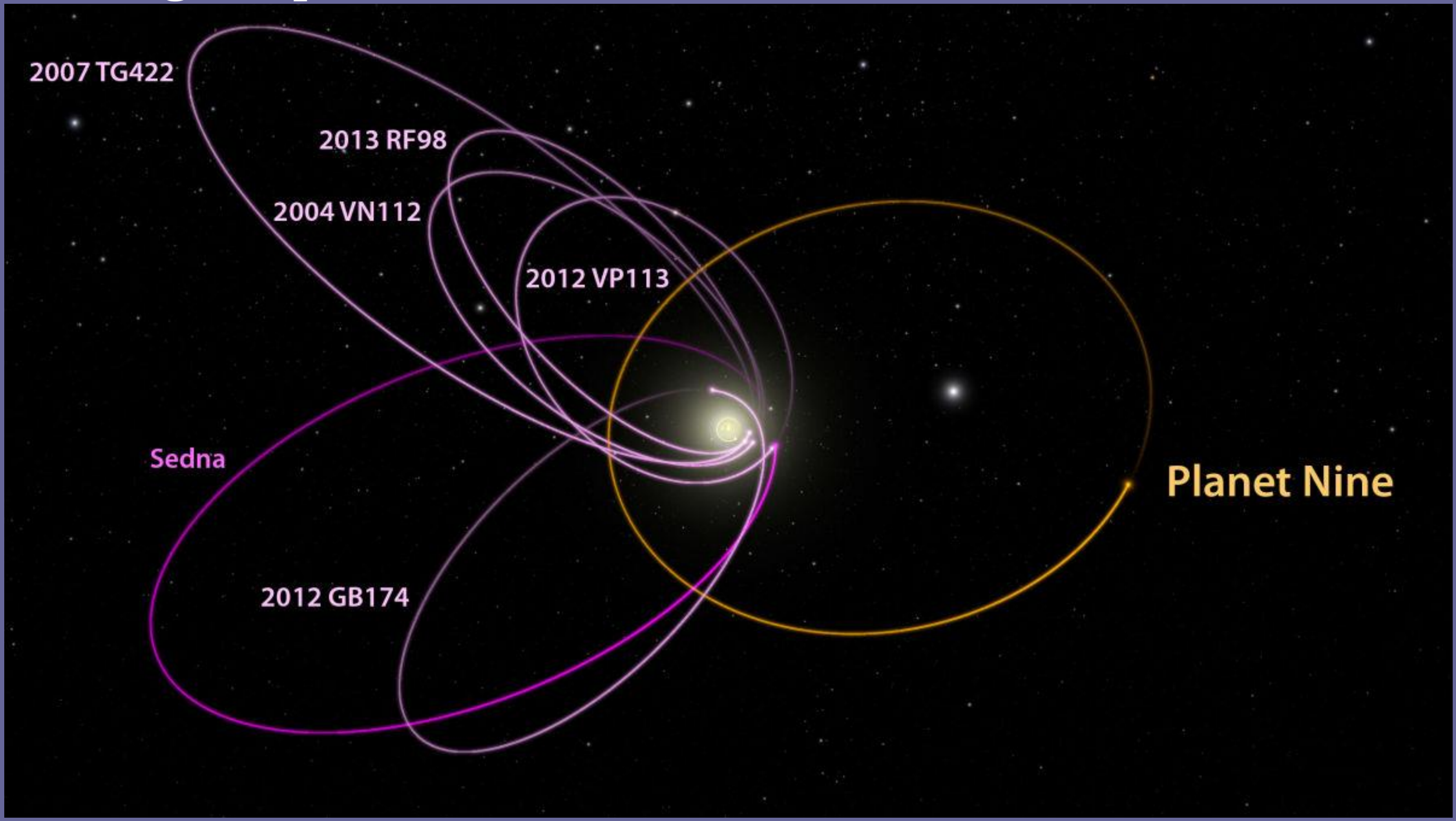
$$2\,n_{♃} - 5\,n_{♄} = -4.02\text{''}\ \mathrm{day}^{-1}$$

$$n_{♅} - 2\,n_{♆}$$

Planet	n (arcsec day^{-1})
Jupiter	299.128
Saturn	120.455
Uranus	42.231
Neptune	21.534

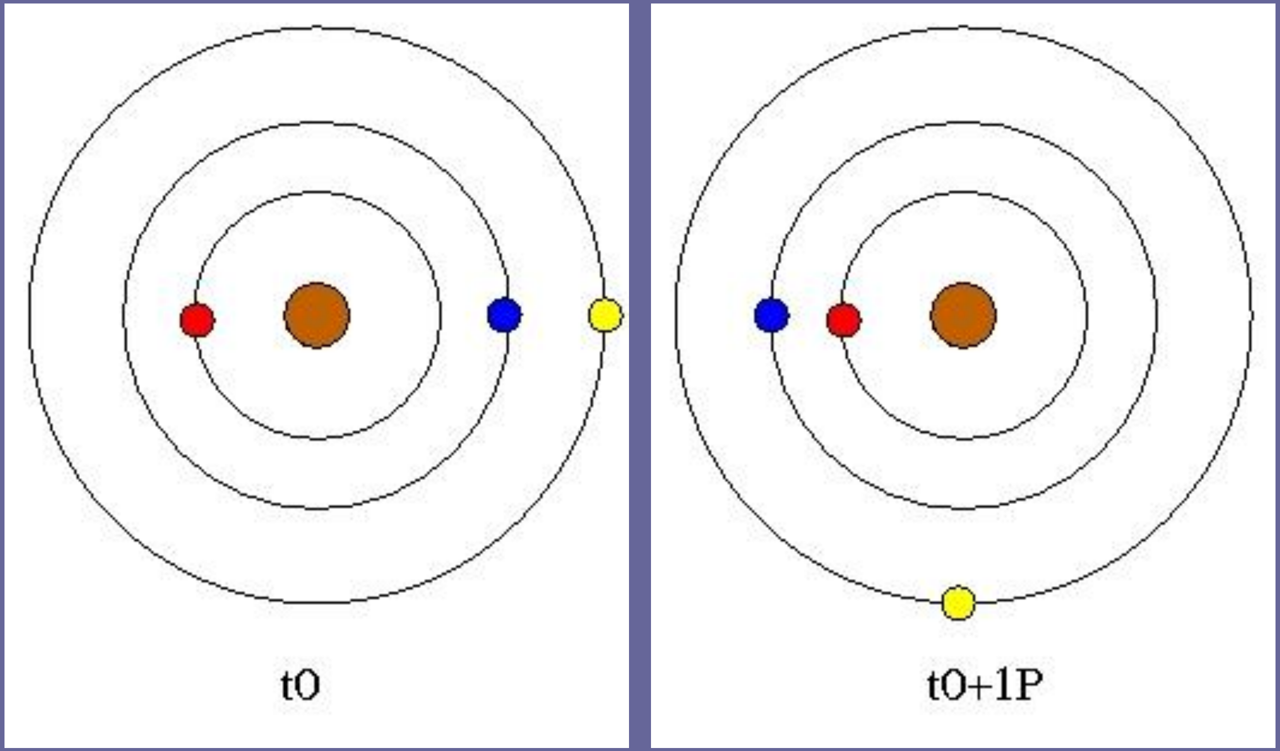
7.4 The motions in a planetary system

- The haunt for Planet Nine or is there anybody out there?
- Grouping in perihelion position of distant Kuiper Belt objects suggests gravitational influence of a massive planet with an anti-aligned perihelion.



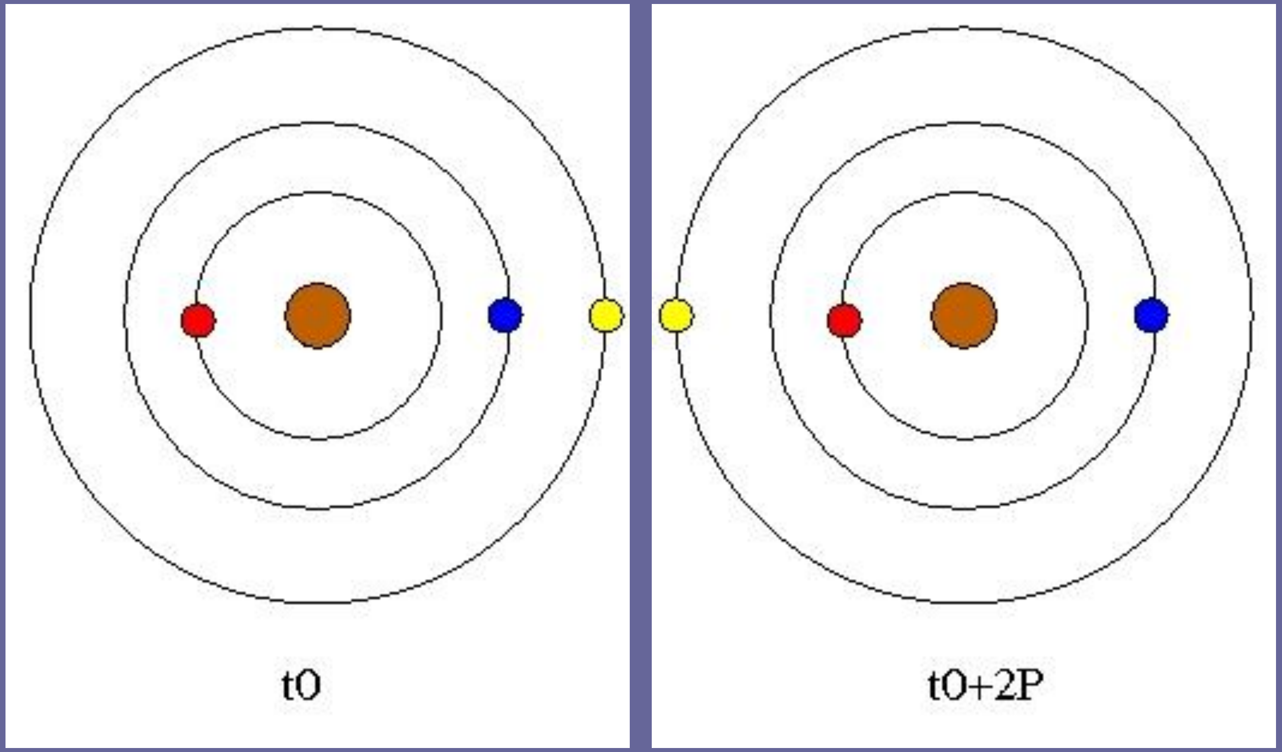
7.5 The Laplace resonance

- Resonance between the orbital periods of 3 or more bodies (simple integer number ratio).
- Most spectacular example: the Galilean moons Io, Europa and Ganymede: ratios 1:2:4.



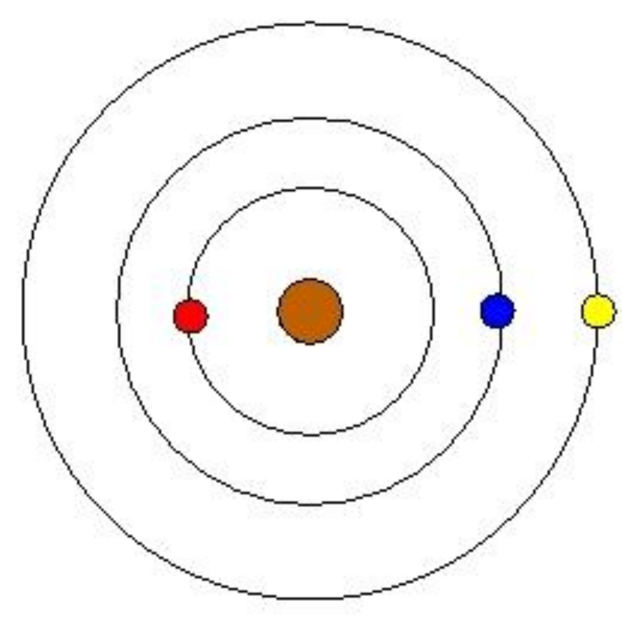
7.5 The Laplace resonance

- Resonance between the orbital periods of 3 or more bodies (simple integer number ratio).
- Most spectacular example: the Galilean moons Io, Europa and Ganymede: ratios 1:2:4.

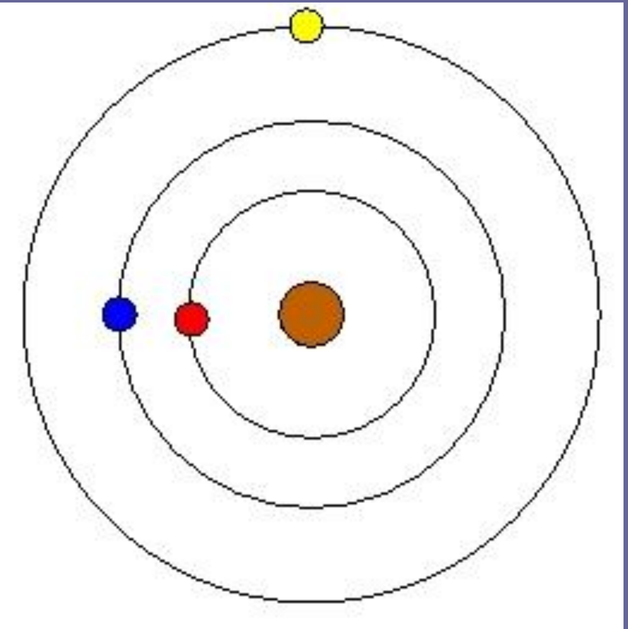


7.5 The Laplace resonance

- Resonance between the orbital periods of 3 or more bodies (simple integer number ratio).
- Most spectacular example: the Galilean moons Io, Europa and Ganymede: ratios 1:2:4.



t0 +4P



t0+3P

7.5 The Laplace resonance

- The orbital periods are not exactly in the 1:2:4 ratios.

$$2\,n_{\rm Europa} - n_{\rm Io} = -0.74° \text{ day}^{-1}$$

- The Io – Europa conjunction is tied to the perijove of Io and the apojove of Europa.

$$\omega_{\rm Io} = (2\,n_{\rm Europa} - n_{\rm Io})\,t - \theta_1$$

$$\omega_{\rm Europa} = (2\,n_{\rm Europa} - n_{\rm Io})\,t - \theta_2$$

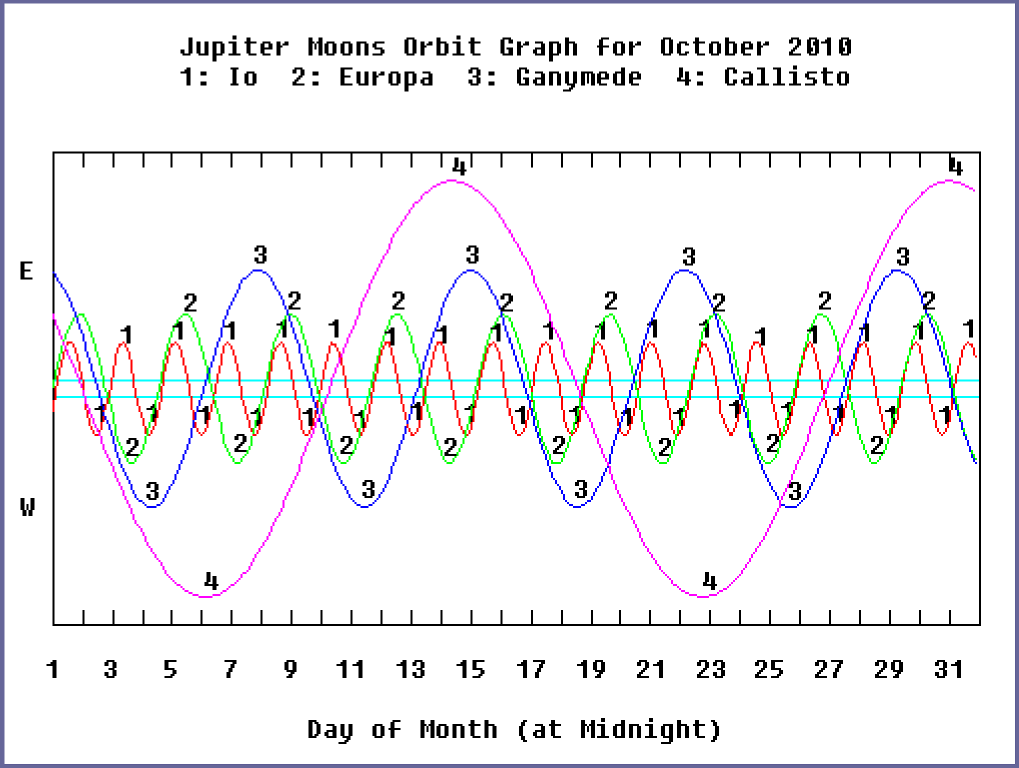
- θ_1 et θ_2 oscillate around 0° and 180°.

Moon	*a* (km)	*P* (days)
Io	421 700	1.769
Europa	671 034	3.551
Ganymede	1 070 412	7.154
Callisto	1 882 709	16.689

- The Europa-Ganymede conjunction is tied to the perijove of Europa: $\omega_{\rm Europa} = (2\,n_{\rm Ganymede} - n_{\rm Europa})\,t - \theta_3$
- θ_3 oscillates around 0°.

7.5 The Laplace resonance

- In practice, the Laplace resonance implies a simple commensurability between the rates of motion of the Io - Europa and Europa - Ganymede conjunctions: $\dot{\omega}_{\rm Io} = \dot{\omega}_{\rm Europa}$
- The Laplace resonance prevents triple conjunctions of the three moons.

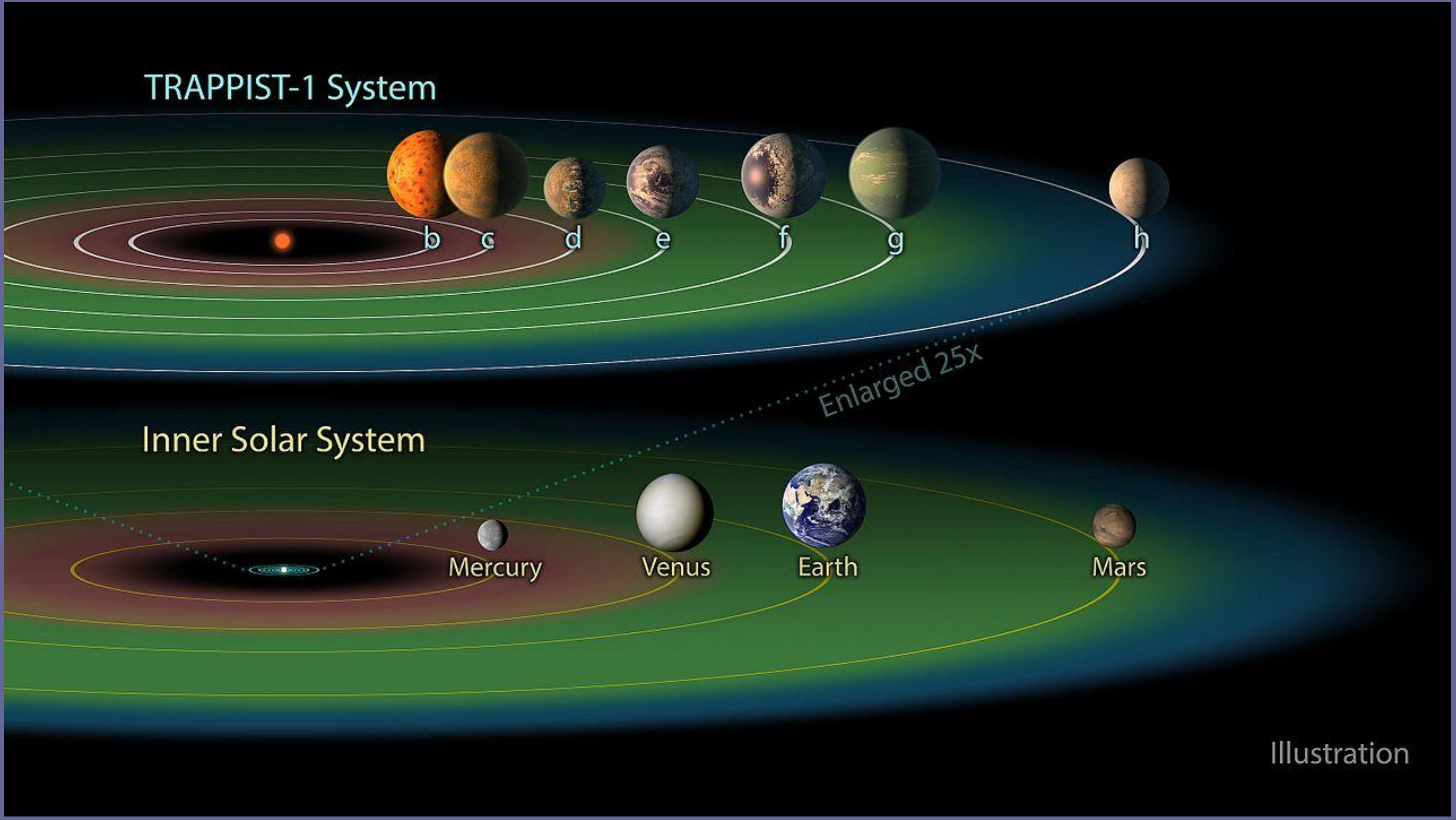


7.5 The Laplace resonance

- The Laplace resonance prevents Io's orbit to become circularized and maintains Io's volcanism.

7.5 The Laplace resonance

- Other example of (near) Laplace resonance: the exo-planets of Trappist-1:



7.5 The Laplace resonance

Exo-planet	P (days)	P_B/P	P_{n-1}/P_n
B	1.511	1	
C	2.421	5/8	5/8
D	4.050	3/8	5/3
E	6.100	1/4	3/2
F	9.206	1/6	3/2
G	12.353		4/3
H	18.766		3/2

- In such a tight planetary system, this resonance is essential for stability.

7.6 Perihelion precession in the Solar System

- To study long-term trends, we can approximate each planet of the Solar System as a ring of mass m_k and radius a_k
- The potential is:

$$U_{j,k} = -\frac{G\,m_k}{2\,\pi}\int_0^{2\,\pi}\frac{1}{|\vec{r_j}-\vec{r_k}|}d\phi$$

- For $a_j > a_k$

$$\frac{1}{|\vec{r_j}-\vec{r_k}|} = \frac{1}{a_j}\sum_{n=0}^{\infty}\left(\frac{a_k}{a_j}\right)^n P_n(\cos\phi)$$

- For $a_j < a_k$

$$\frac{1}{|\vec{r_j}-\vec{r_k}|} = \frac{1}{a_k}\sum_{n=0}^{\infty}\left(\frac{a_j}{a_k}\right)^n P_n(\cos\phi)$$

- Hence:

$$U_j = -\frac{G\,M_\odot}{r} - \sum_{n=0}^{\infty} s_n\left[\sum_{k<j}\frac{G\,m_k}{r}\left(\frac{a_k}{r}\right)^n + \sum_{j<k}\frac{G\,m_k}{a_k}\left(\frac{r}{a_k}\right)^n\right]$$

7.6 Perihelion precession in the Solar System

- In a nearly circular orbit, the deviations from a perfect circle are solution of the equation $$\ddot{\Delta r} + \left[-\frac{3\, f(a)}{a} - \left.\frac{d\, f}{dr}\right)_a \right] \Delta r = 0$$

  where $f(r)$ is a central force per unit mass.
- This leads to oscillations with a period $$P = \frac{2\,\pi}{\left[-\frac{3\, f(a)}{a} - \left.\frac{d\, f}{dr}\right)_a \right]^{1/2}}$$
- Applying this to the potential U_j, we obtain:

$$\left[3 + \left.\frac{d\, f}{dr}\right)_{a_j} \frac{a_j}{f(a_j)} \right]^{-1/2} = 1 + \frac{1}{2} \sum_{n=0}^{\infty} s_n\, n\,(n+1) \left[\sum_{k<j} \frac{m_k}{M_\odot} \left(\frac{a_k}{a_j} \right)^n + \sum_{j<k} \frac{m_k}{M_\odot} \left(\frac{a_j}{a_k} \right)^{n+1} \right]$$

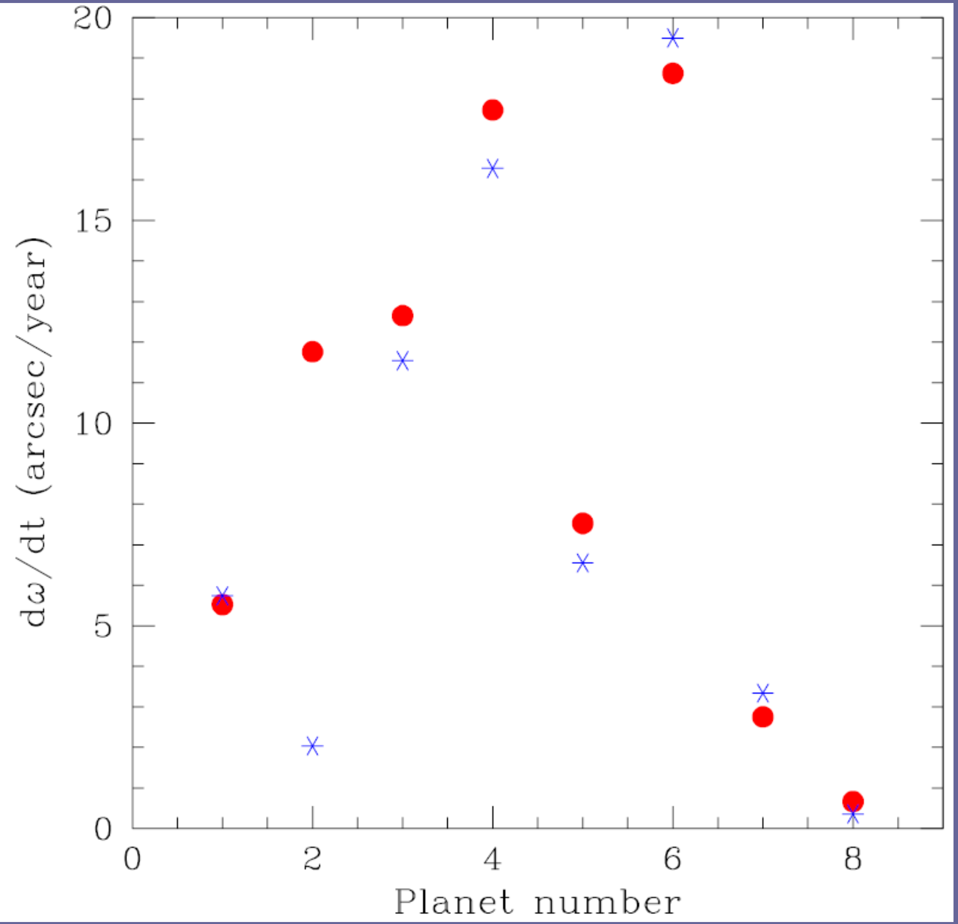
- This yields the rates of perihelion precession:

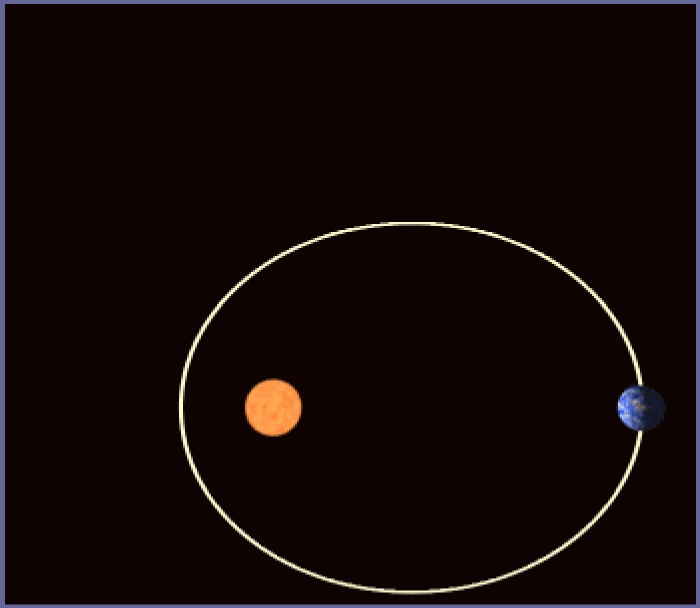
$$\dot{\omega} = \frac{n_j}{2} \sum_{n=0}^{\infty} s_n\, n\,(n+1) \left[\sum_{k<j} \frac{m_k}{M_\odot} \left(\frac{a_k}{a_j} \right)^n + \sum_{j<k} \frac{m_k}{M_\odot} \left(\frac{a_j}{a_k} \right)^{n+1} \right]$$

7.6 Perihelion precession in the Solar System

- Precession of perihelion :

$$\dot{\omega} = \frac{n_j}{2} \sum_{n=0}^{\infty} s_n \, n \, (n+1) \left[\sum_{k<j} \frac{m_k}{M_\odot} \left(\frac{a_k}{a_j} \right)^n + \sum_{j<k} \frac{m_k}{M_\odot} \left(\frac{a_j}{a_k} \right)^{n+1} \right]$$





Chapter VIII: The rotation of rigid celestial bodies

- Most objects in the Universe rotate.
- Here, we consider rigid body rotation (telluric planets, moons, asteroids,…): $\vec{\omega}$ is a constant vector.
- Let CM be the centre of mass, the angular momentum can be written:

$$
\begin{aligned}
\vec{L}_{CM} &= \int (\vec{r} \wedge \dot{\vec{r}})\, \rho(\vec{r})\, dV \\
&= \int [\vec{r} \wedge (\dot{\vec{r}}_{CM} + \vec{\omega} \wedge \vec{r})]\, \rho(\vec{r})\, dV \\
&= \int [\vec{r} \wedge (\vec{\omega} \wedge \vec{r})]\, \rho(\vec{r})\, dV \\
&= \mathcal{I}\, \vec{\omega}
\end{aligned}
$$

with the inertia tensor:

$\mathcal{I}_{ij} = -\int (x_i\, x_j)\, \rho\, dV$ for $i \neq j$, and $\mathcal{I}_{ii} = \sum_{j \neq i, j=1}^{3} \int x_j^2\, \rho\, dV$

8.1 Fundamental concepts

- In the principal axes of inertia, the inertia tensor is a diagonal matrix:

$$\mathcal{I} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

- Variation of the angular momentum:

$$\begin{aligned} \frac{d\,\vec{L}_{CM}}{dt} &= \frac{\delta\,\vec{L}_{CM}}{\delta t} + \vec{\omega} \wedge \vec{L}_{CM} \\ &= \int (\dot{\vec{r}} \wedge \dot{\vec{r}})\,\rho(\vec{r})\,dV + \int (\vec{r} \wedge \ddot{\vec{r}})\,\rho(\vec{r})\,dV \\ &= \int (\vec{r} \wedge \vec{F})\,dV = \vec{\mathcal{M}} \end{aligned}$$

$$\begin{aligned} A\,\dot{\omega}_1 + (C - B)\,\omega_2\,\omega_3 &= \mathcal{M}_1 \\ B\,\dot{\omega}_2 + (A - C)\,\omega_3\,\omega_1 &= \mathcal{M}_2 \\ C\,\dot{\omega}_3 + (B - A)\,\omega_1\,\omega_2 &= \mathcal{M}_3 \end{aligned}$$

8.1 Fundamental concepts

- Kinetic energy:

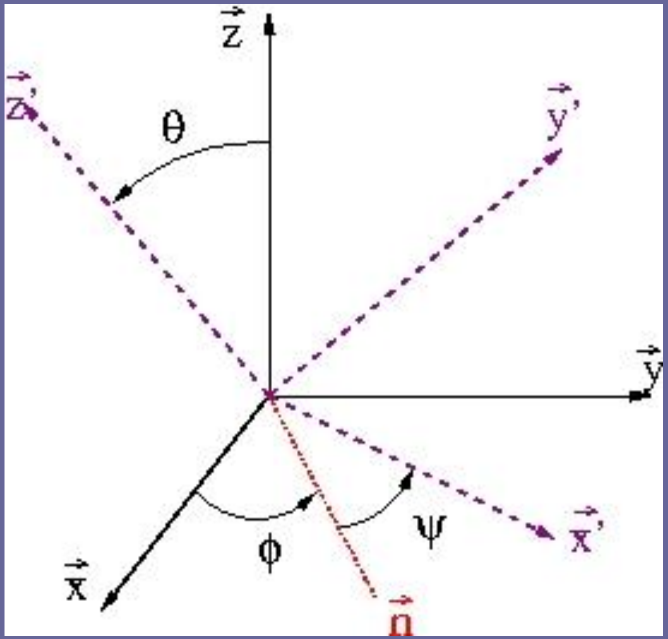
$$
\begin{aligned}
T &= \frac{1}{2}\int \dot{\vec{r}}\cdot\dot{\vec{r}}\,\rho(\vec{r})\,dV \\
&= \frac{1}{2}M\,\dot{r}_{CM}^2 + \frac{1}{2}\,\vec{\omega}\cdot\mathcal{I}\vec{\omega} = T_{CM} + T_{rot}
\end{aligned}
$$

- … in the principal axes of inertia:

$$
T_{\mathrm{rot}} = \frac{1}{2}\left[A\,\omega_1^2 + B\,\omega_2^2 + C\,\omega_3^2\right]
$$

8.1 Fundamental concepts

- Euler angles: *(x,y,z)* absolute frame of reference, *(x',y',z')* principal axes of inertia (rotating frame)



$$\vec{\omega} = \dot{\phi}\,\vec{e_z} + \dot{\theta}\,\vec{n} + \dot{\psi}\,\vec{e_z}'$$

$$\vec{e_z} = \cos\theta\,\vec{e_z}' + \sin\theta\,\sin\psi\,\vec{e_x}' + \sin\theta\,\cos\psi\,\vec{e_y}'$$

$$\vec{n} = \cos\psi\,\vec{e_x}' - \sin\psi\,\vec{e_y}'$$

$$\vec{\omega} = (\dot{\phi}\,\sin\theta\,\sin\psi + \dot{\theta}\,\cos\psi)\,\vec{e_x}' + (\dot{\phi}\,\sin\theta\,\cos\psi - \dot{\theta}\,\sin\psi)\,\vec{e_y}' + (\dot{\phi}\,\cos\theta + \dot{\psi})\,\vec{e_z}'$$

$$\begin{aligned} \omega_1 &= \dot{\phi}\,\sin\theta\,\sin\psi + \dot{\theta}\,\cos\psi \\ \omega_2 &= \dot{\phi}\,\sin\theta\,\cos\psi - \dot{\theta}\,\sin\psi \\ \omega_3 &= \dot{\phi}\,\cos\theta + \dot{\psi} \end{aligned}$$

8.1 Fundamental concepts

- For a freely rotating body (without external forces acting on it),

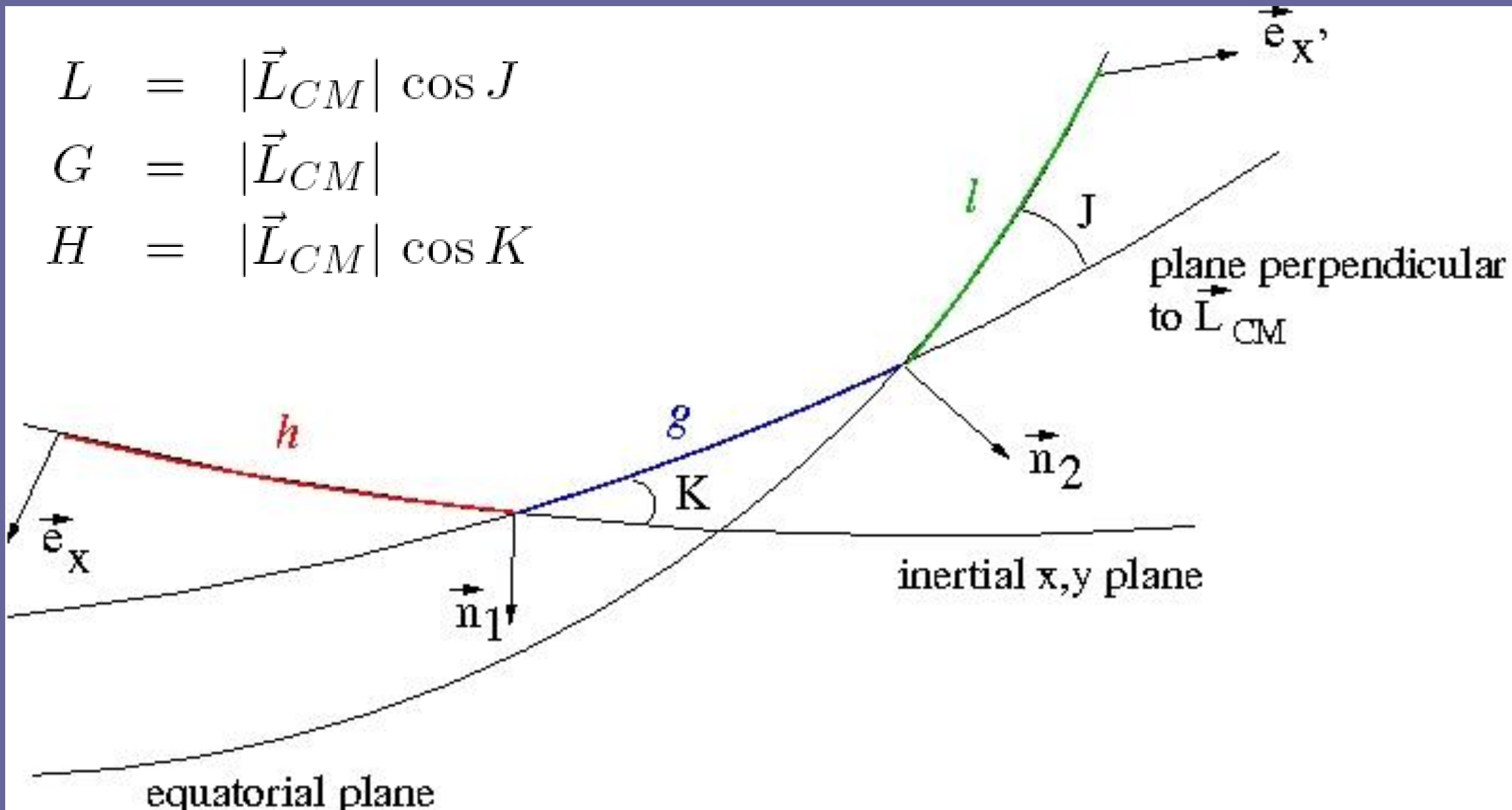
$$
\begin{aligned}
p_\phi &= \frac{\partial T}{\partial \dot{\phi}} = (A\,\omega_1\,\sin\psi + B\,\omega_2\,\cos\psi)\,\sin\theta + C\,\omega_3\,\cos\theta \\
p_\theta &= \frac{\partial T}{\partial \dot{\theta}} = A\,\omega_1\,\cos\psi - B\,\omega_2\,\sin\psi \\
p_\psi &= \frac{\partial T}{\partial \dot{\psi}} = C\,\omega_3
\end{aligned}
$$

- These expressions are very complex and we will rather use the Andoyer elements than the Euler angles.

8.2 The Andoyer elements

- Set of canonical elements: angles *l*, *g*, *h* and conjugated momenta: *L*, *G*, *H*

$$
\begin{aligned}
L &= |\vec{L}_{CM}| \cos J \\
G &= |\vec{L}_{CM}| \\
H &= |\vec{L}_{CM}| \cos K
\end{aligned}
$$



$$\vec{L}_{CM} = (G \sin J \sin l, G \sin J \cos l, G \cos J)$$

8.2 The Andoyer elements

- Expression of the angular velocity vector:

$$\vec{\omega} = \mathcal{I}^{-1}\,\vec{L}_{CM} = (A^{-1}G\,\sin J\,\sin l, B^{-1}G\,\sin J\,\cos l, C^{-1}G\,\cos J)$$

- Expression of the rotational kinetic energy:

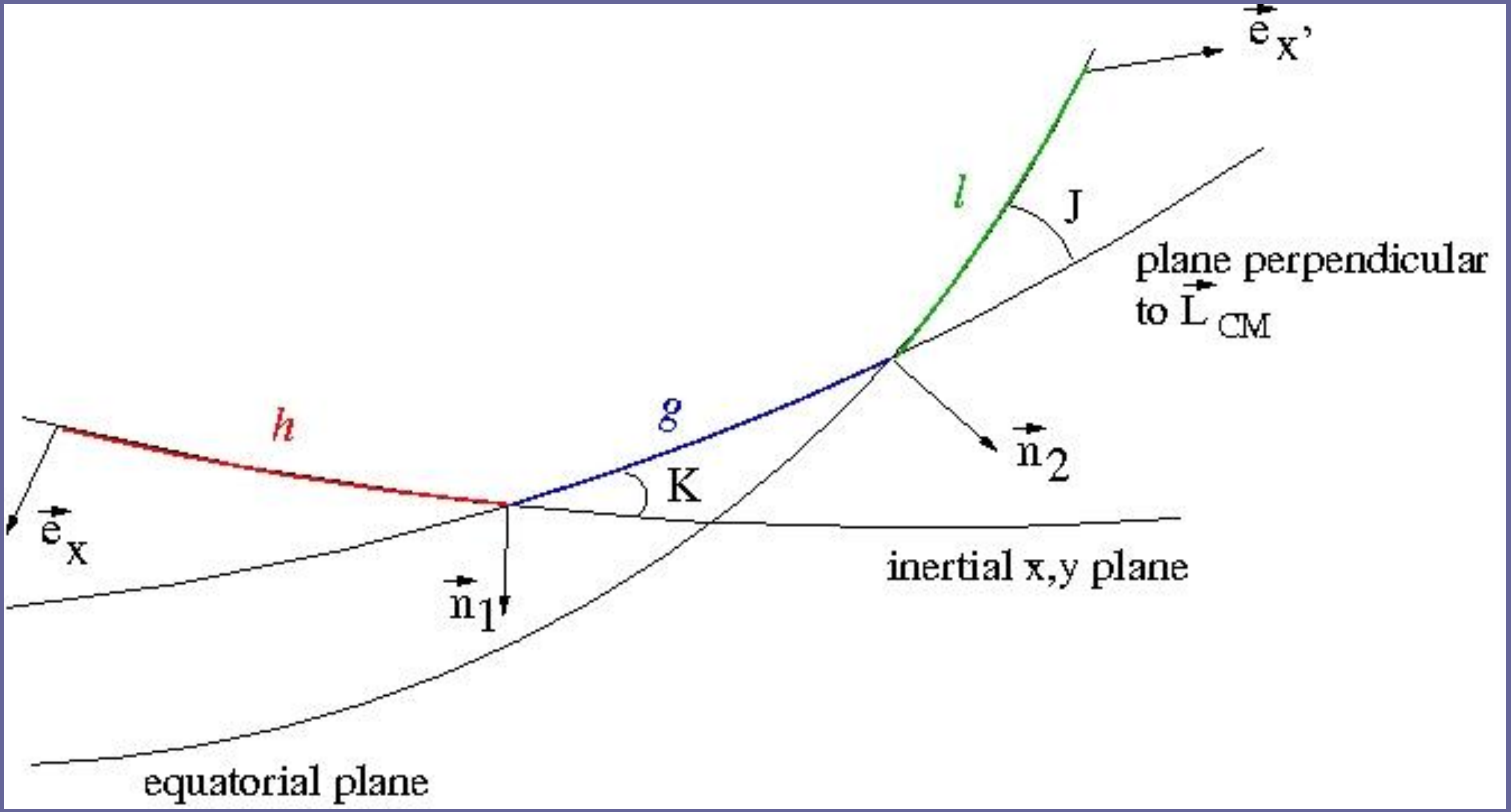
$$T = \frac{1}{2}\vec{\omega}\cdot\vec{L}_{CM} = \frac{1}{2}(G^2 - L^2)\,(A^{-1}\,\sin^2 l + B^{-1}\,\cos^2 l) + \frac{L^2}{2C}$$

- Hamilton’s canonical equations:

$$\begin{aligned}
\dot{L} &= -\frac{\partial\mathcal{H}}{\partial l} = (G^2 - L^2)\left[\frac{1}{B} - \frac{1}{A}\right]\sin l\,\cos l \\
\dot{G} &= -\frac{\partial\mathcal{H}}{\partial g} = 0 \\
\dot{H} &= -\frac{\partial\mathcal{H}}{\partial h} = 0 \\
\dot{l} &= \frac{\partial\mathcal{H}}{\partial L} = L\left[\frac{1}{C} - \frac{\sin^2 l}{A} - \frac{\cos^2 l}{B}\right] \\
\dot{g} &= \frac{\partial\mathcal{H}}{\partial G} = G\left[\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right] \\
\dot{h} &= \frac{\partial\mathcal{H}}{\partial H} = 0
\end{aligned}$$

8.2 The Andoyer elements

- Modified Andoyer elements: if K or J are close to zero:
  $p = l + g + h$, $q = g + h$, $s = -h$, $P = L$, $Q = G - L$, $S = G - H$



8.3 Perturbations due to an orbiting mass

- Consider a point-like mass *m*, on circular orbit about the body of mass *M* whose rotation we are studying.
- The potential writes: $$U(\vec{r'}) = -G\,m \int\int\int_V \frac{\rho(\vec{r})}{|\vec{r}-\vec{r'}|}\,dV$$
- Using the properties of the Legendre polynomials, we find that:

$$\begin{aligned}
\frac{1}{|\vec{r}-\vec{r'}|} &= (r^2 + r'^2 - 2\,r\,r'\,\cos\gamma)^{-1/2} \\
&= \frac{1}{r'}\left(1 - 2\,\frac{r}{r'}\,\cos\gamma + \frac{r^2}{r'^2}\right)^{-1/2} \\
&= \frac{1}{r'}\sum_{n=0}^{\infty}\left(\frac{r}{r'}\right)^n P_n(\cos\gamma)
\end{aligned}$$

$$\Rightarrow \quad U = \sum_{n=0}^{\infty} U_n$$

8.3 Perturbations due to an orbiting mass

- $U_0 = -GMm/r'$, $U_1 = 0$ and

$$U_2 = \frac{4}{3}\,\alpha\,n^2\,C - 2\,\alpha\,n^2\,C\,\frac{x'^2 + y'^2}{r'^2} - 4\,\beta\,n^2\,C\,\frac{x'^2 - y'^2}{r'^2}$$

with $\alpha = \frac{3}{4}\,\frac{m}{M+m}\,\frac{2\,C-A-B}{2\,C},\ \beta = \frac{3}{4}\,\frac{m}{M+m}\,\frac{B-A}{4\,C}$

- Suppose J = 0 and let λ be the longitude of the orbiting mass:

$$\begin{aligned}\frac{x'}{r'} &= \cos\left(h-\lambda\right)\,\cos\left(l+g\right) - \sin\left(h-\lambda\right)\,\sin\left(l+g\right)\,\cos K\\ \frac{y'}{r'} &= -\cos\left(h-\lambda\right)\,\sin\left(l+g\right) - \sin\left(h-\lambda\right)\,\cos\left(l+g\right)\,\cos K\end{aligned}$$

8.3 Perturbations due to an orbiting mass

- The Hamiltonian becomes:

$$\begin{aligned}\mathcal{H} \;=\;& \frac{L^2}{2\,C} + n\,\Lambda + \alpha\,n^2\,C\,\sin^2 K\,[1-\cos(2\,\lambda-2\,h)]\\ &-\beta\,n^2\,C\,[\cos(2\,\lambda-2\,h+2\,g+2\,l)\,(1-\cos K)^2\\ &+\cos(2\,\lambda-2\,h-2\,g-2\,l)\,(1+\cos K)^2+2\,\cos(2\,g+2\,l)\,(1-\cos^2 K)]\end{aligned}$$

- Using the modified Andoyer elements:

$$\begin{aligned}\mathcal{H} \;=\;& \frac{P^2}{2\,C} + n\,\Lambda + \alpha\,n^2\,C\,\sin^2 K\,[1-\cos(2\,\lambda+2\,s)]\\ &-\beta\,n^2\,C\,[\cos(2\,\lambda+2\,p+4\,s)\,(1-\cos K)^2\\ &+\cos(2\,\lambda-2\,p)\,(1+\cos K)^2+2\,\cos(2\,p+2\,s)\,(1-\cos^2 K)]\end{aligned}$$

8.3 Perturbations due to an orbiting mass

$$\mathcal{H} = \frac{P^2}{2\,C} + n\,\Lambda + \alpha\,n^2\,C\,\sin^2 K\,[1-\cos(2\,\lambda+2\,s)]$$
$$-\beta\,n^2\,C\,[\cos(2\,\lambda+2\,p+4\,s)\,(1-\cos K)^2$$
$$+\cos(2\,\lambda-2\,p)\,(1+\cos K)^2 + 2\,\cos(2\,p+2\,s)\,(1-\cos^2 K)]$$

- Precession:

$$\dot{s} = \frac{\partial\,\mathcal{H}}{\partial S} = \frac{-1}{P}\frac{\partial\,\mathcal{H}}{\partial\cos K}$$
$$= 2\,\alpha\,n^2\,\frac{C}{P}\cos K\,[1-\cos(2\,\lambda+2\,s)] - 2\,\beta\,n^2\,\frac{C}{P}\,[\cos(2\,\lambda+2\,p+4\,s)\,(1-\cos K)$$
$$-\cos(2\,\lambda-2\,p)\,(1+\cos K) + 2\,\cos(2\,p+2\,s)\,\cos K]$$

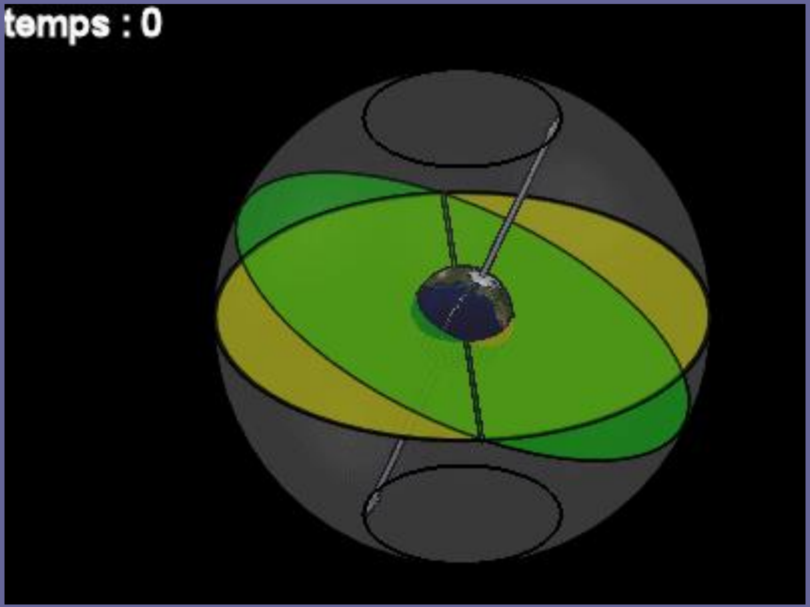
- Averaging over long timescales, we find:

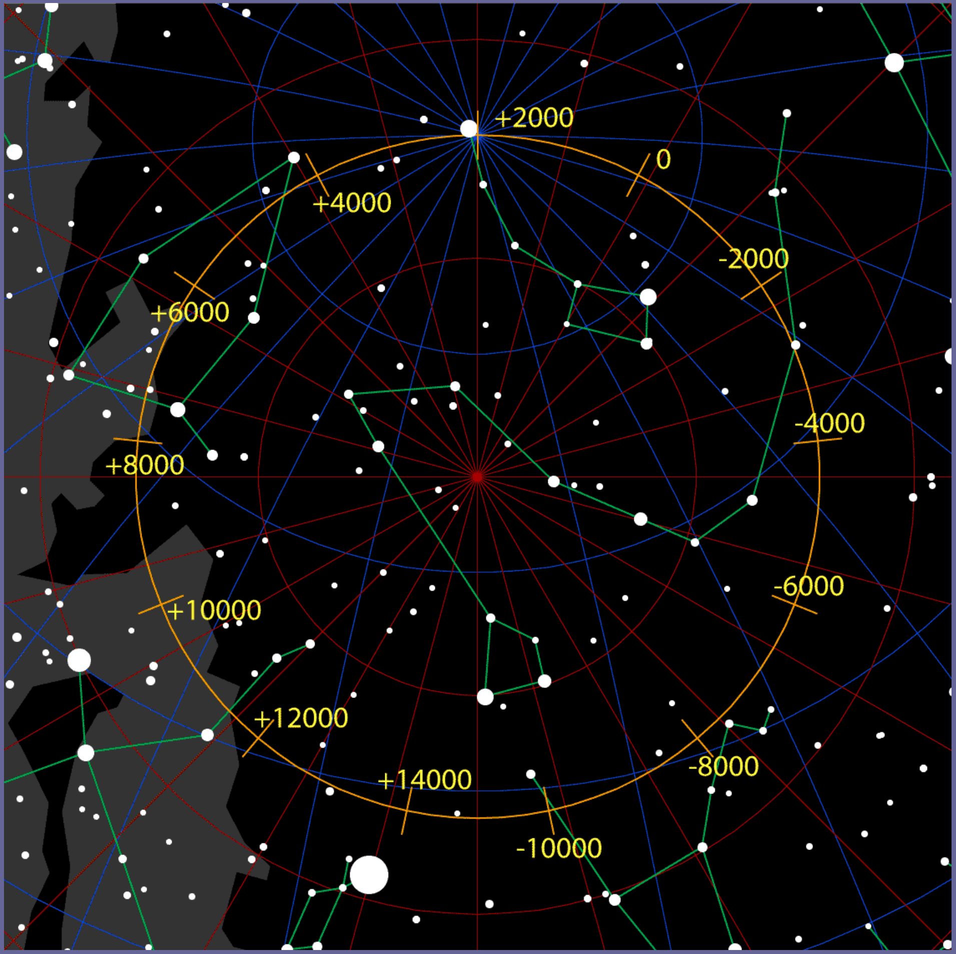
$$<\dot{s}>= 2\,\alpha\,n^2\,\frac{C}{P}\,\cos K$$

- The combined effect of the Sun and the Moon upon the Earth yields a precession period of 25645 years.

8.3 Perturbations due to an orbiting mass

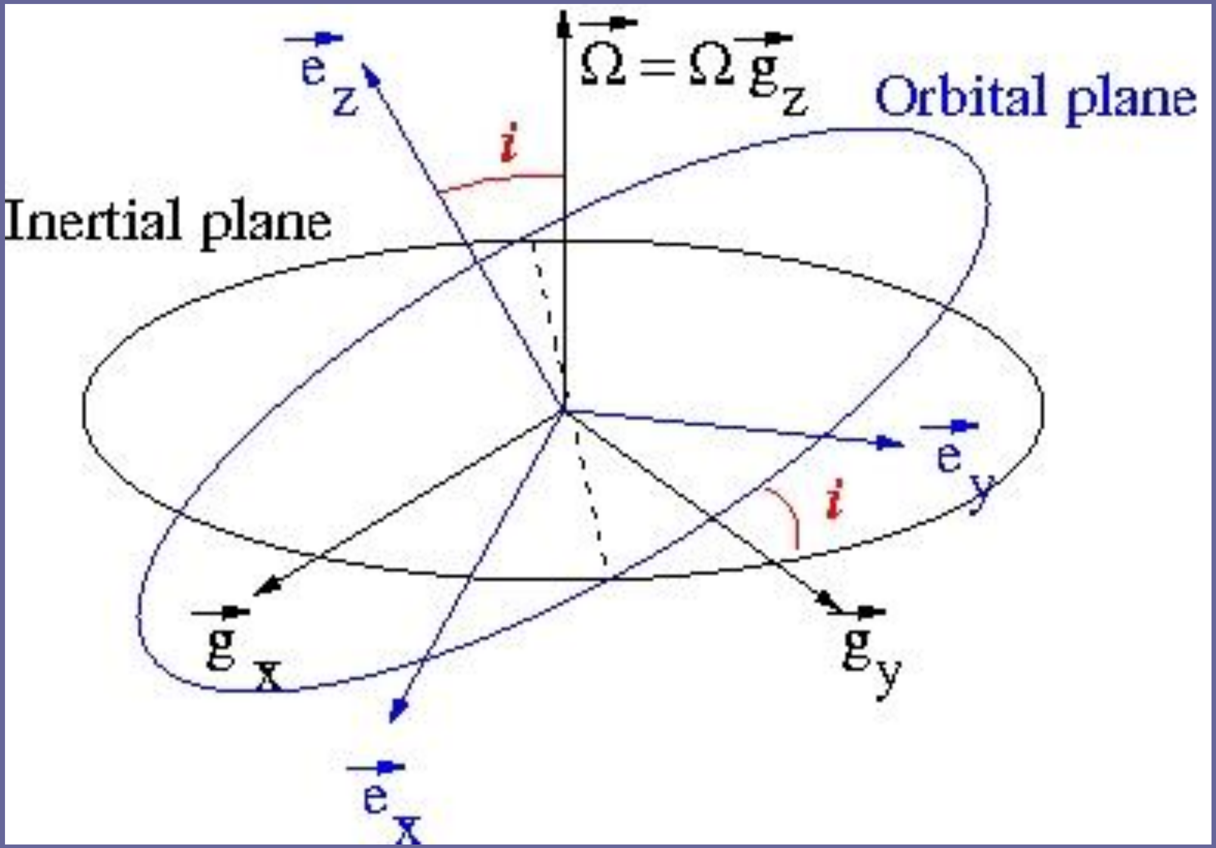
- Precession of the Earth's rotation axis:





8.4 The Cassini states

- Suppose that the orbital plane of the perturbing mass undergoes a slow precession at a rate $\vec{\Omega}$ with respect to the inertial frame of reference: $\vec{\omega}' = \vec{\omega} + \vec{\Omega}$



8.4 The Cassini states

- The kinetic energy now writes:

$$\vec{\omega}' = \vec{\omega} + \vec{\Omega} \quad \Rightarrow \quad T = \frac{1}{2}\,\vec{\omega}' \cdot \vec{L}_{CM} = \frac{1}{2}\,\vec{\omega} \cdot \mathcal{I}\,\vec{\omega} + \vec{\Omega} \cdot \mathcal{I}\,\vec{\omega} + \frac{1}{2}\,\vec{\Omega} \cdot \mathcal{I}\,\vec{\Omega}$$

$$\Rightarrow \quad \begin{aligned} \mathcal{H} &= \vec{\omega} \cdot \mathcal{I}\,\vec{\omega} + \vec{\Omega} \cdot \mathcal{I}\,\vec{\omega} - T + U \\ &= T - \vec{\Omega} \cdot \vec{L}_{CM} + U \end{aligned}$$

$$-\vec{\Omega} \cdot \vec{L}_{CM} = P\,\Omega\,(\sin i\,\sin K\,\cos s - \cos i\,\cos K)$$

$$\Rightarrow \quad \begin{aligned} \mathcal{H} = \; & \frac{P^2}{2\,C} + n\,\Lambda + P\,\Omega\,(\sin i\,\sin K\,\cos s - \cos i\,\cos K) \\ & + \alpha\,n^2\,C\,\sin^2 K\,[1 - \cos(2\,\lambda + 2\,s)] - \beta\,n^2\,C\,[\cos(2\,\lambda + 2\,p + 4\,s)\,(1 - \cos K)^2 \\ & + \cos(2\,\lambda - 2\,p)\,(1 + \cos K)^2 + 2\,\cos(2\,p + 2\,s)\,(1 - \cos^2 K)] \end{aligned}$$

8.4 The Cassini states

$$\begin{aligned}\mathcal{H} &= \frac{P^2}{2\,C} + n\,\Lambda + P\,\Omega\,(\sin i\,\sin K\,\cos s - \cos i\,\cos K)\\ &+\alpha\,n^2\,C\,\sin^2 K\,[1-\cos(2\,\lambda+2\,s)] - \beta\,n^2\,C\,[\cos(2\,\lambda+2\,p+4\,s)\,(1-\cos K)^2\\ &+\cos(2\,\lambda-2\,p)\,(1+\cos K)^2 + 2\,\cos(2\,p+2\,s)\,(1-\cos^2 K)]\end{aligned}$$

- Averaged over longer timescales and accounting for the rotation/revolution resonance, the Hamiltonian becomes:

$$\begin{aligned}\mathcal{H} &= \frac{P^2}{2\,C} + n\,\Lambda' - n\,P + P\,\Omega\,(\sin i\,\sin K\,\cos s - \cos i\,\cos K)\\ &+\alpha\,n^2\,C\,\sin^2 K - \beta\,n^2\,C\,\cos(2\,\sigma)\,(1+\cos K)^2\end{aligned}$$

$$\Rightarrow\quad \begin{aligned}\dot\sigma &= \frac{P}{C} - n + \Omega\,[\sin i\,\sin K\,\cos s - \cos i\,\cos K] + \frac{1-\cos K}{P}\,\frac{\partial\mathcal{H}}{\partial\cos K}\\ \dot s &= \frac{-1}{P}\,\frac{\partial\mathcal{H}}{\partial\cos K}\\ \dot P &= -2\,\beta\,n^2\,C\,(1+\cos K)^2\,\sin(2\,\sigma)\\ \dot S &= \Omega\,P\,\sin i\,\sin K\,\sin s\end{aligned}$$

8.4 The Cassini states

$$\begin{aligned}
\dot{\sigma} &= \frac{P}{C} - n + \Omega\left[\sin i \, \sin K \, \cos s - \cos i \, \cos K\right] + \frac{1-\cos K}{P}\frac{\partial \mathcal{H}}{\partial \cos K} \\
\dot{s} &= \frac{-1}{P}\frac{\partial \mathcal{H}}{\partial \cos K} \\
\dot{P} &= -2\,\beta\, n^2\, C\,(1+\cos K)^2 \, \sin(2\,\sigma) \\
\dot{S} &= \Omega\, P \, \sin i \, \sin K \, \sin s
\end{aligned}$$

- We obtain an equilibrium situation: $(\dot{\sigma} = \dot{s} = \dot{P} = \dot{S} = 0)$

if

$$\begin{aligned}
\sigma &= k\,\frac{\pi}{2} \\
s &= k'\,\pi \\
\frac{\partial \mathcal{H}}{\partial \cos K} &= 0 \\
P &= n\,C + \Omega\,C\,\cos(K - i)
\end{aligned}$$

$$\frac{\partial \mathcal{H}}{\partial \cos K} = \frac{-P\,\Omega}{\sin K}\left[\sin i \, \cos K \, \cos s + \cos i \, \sin K\right] - 2\,n^2\,C\left[\alpha\,\cos K + \beta\,(1+\cos K)\,\cos(2\,\sigma)\right]$$

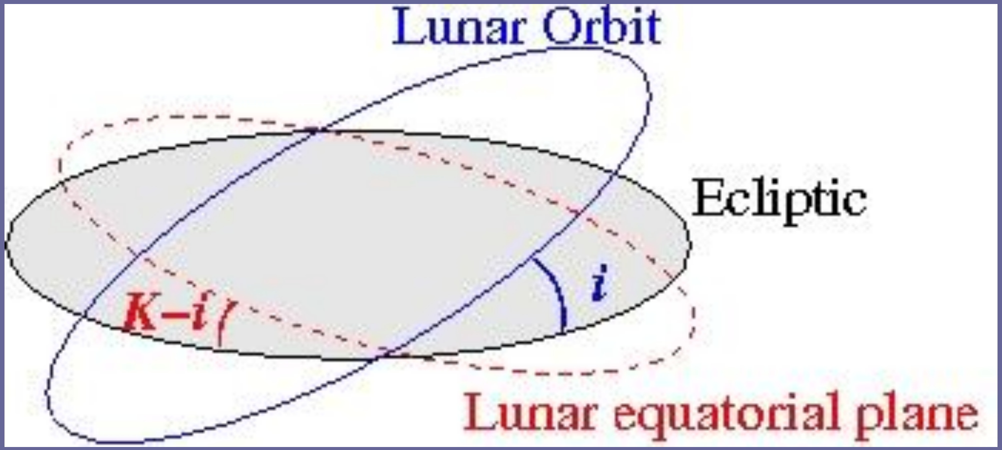
& $\dfrac{\partial \mathcal{H}}{\partial \cos K} = 0 \Rightarrow \dfrac{P\,\Omega}{\sin K}\left[\sin(i-K)\right] = 2\,n^2\,C\left[\alpha\,\cos K + \beta\,(1+\cos K)\right]$

8.4 The Cassini states

$$\frac{P\,\Omega}{\sin K}\,[\sin\,(i-K)] = 2\,n^2\,C\,[\alpha\,\cos K + \beta\,(1+\cos K)]$$

$$\Rightarrow \quad i = [1 + \frac{2\,n^2\,C}{P\,\Omega}(\alpha + 2\,\beta)]\,K$$

- For the Moon, *k = 0* et *k' = 1*: synchronous rotation and line of nodes aligned with the intersection between the lunar equator and the ecliptic:



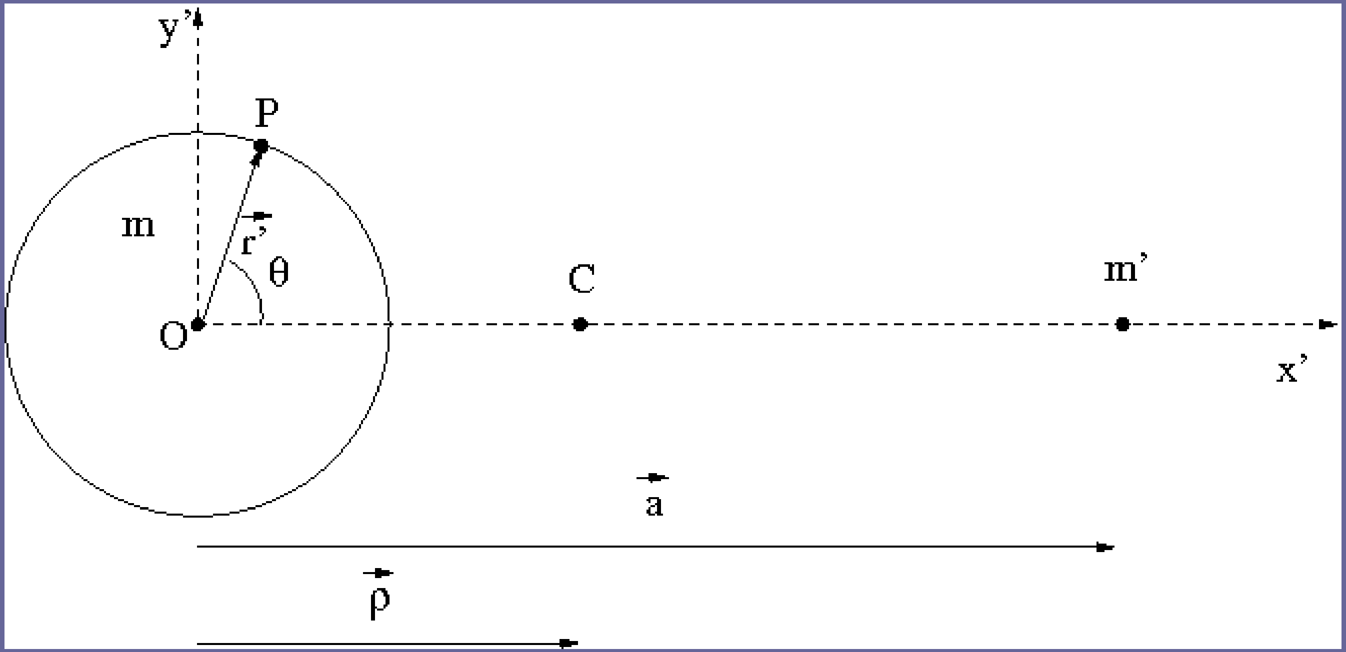
8.5 Tides

- Tides arise from spatial gradients in the gravitational potential.
- The tides produced by the Moon (and to a lesser extent by the Sun) have a spectacular effect on the level of the ocean's in some places such as Mont Saint-Michel:

- Tides are not restricted to the Earth-Moon system, but exist in many places in the Universe (stars, exoplanets, black-holes,…)

8.5 Tides

- Here we use a very simplified approach. Consider first that *m* does not rotate:



$$\vec{CP} = \vec{r} = \vec{CO} + \vec{OP} = \vec{r'} - \vec{\rho}$$

$$\Rightarrow \quad \ddot{\vec{r}} = \omega^2 \, \vec{\rho} = \frac{G\,m'}{|\vec{a} - \vec{r'}|^3} (\vec{a} - \vec{r'}) = -\vec{\nabla} \left(\frac{-G\,m'}{|\vec{a} - \vec{r'}|} \right)$$

8.5 Tides

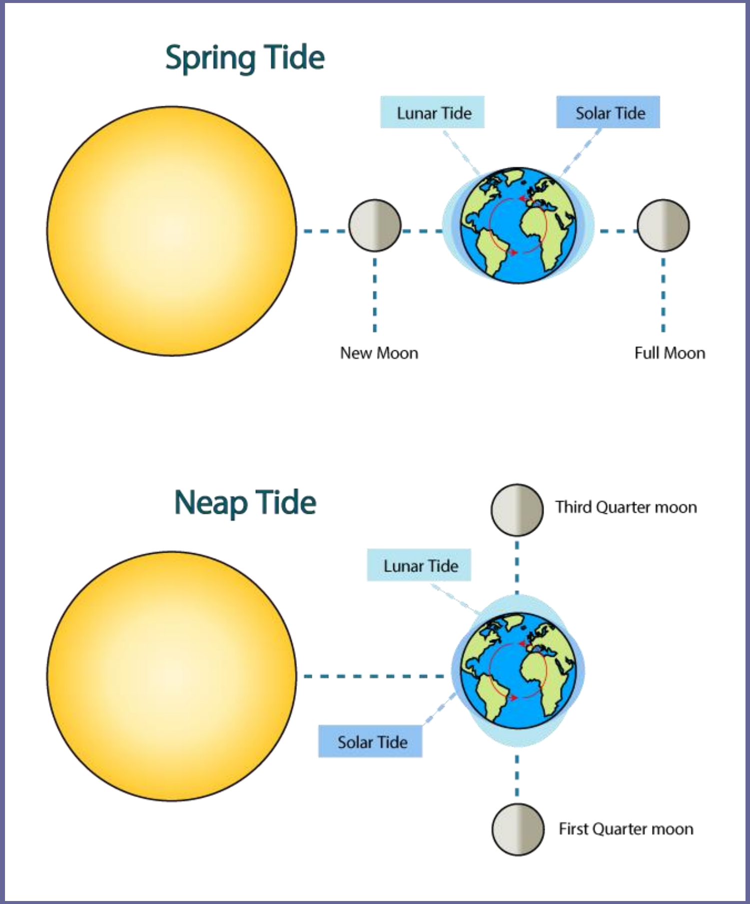
- Since
$$\vec{\rho} = \rho\, e_{\vec{x}'} = \rho\, \vec{\nabla}(x') = \rho\, \vec{\nabla}(r' \cos\theta)$$
$$\frac{G\,m'}{|\vec{a} - \vec{r}'|} \simeq \frac{G\,m'}{a} \left(1 + \frac{r'}{a} P_1(\cos\theta) + \frac{r'^2}{a^2} P_2(\cos\theta)\right)$$
we find that
$$\ddot{\vec{r}} = \omega^2\, \vec{\rho} = \frac{G\,m'}{|\vec{a} - \vec{r}'|^3} (\vec{a} - \vec{r}') = -\vec{\nabla} \left(\frac{-G\,m'}{|\vec{a} - \vec{r}'|}\right)$$
$\Rightarrow$ $\vec{\nabla} U = 0$ with
$$U = -\frac{G\,m'}{a} \left(1 + \frac{r'^2}{a^2} P_2(\cos\theta)\right)$$

- This tidal potential leads to a tidal deformation described by a spheroid:
$$R(\theta) = R \left(1 - \frac{2}{3}\, \epsilon\, P_2(\cos\theta)\right)$$
$$\epsilon = \frac{R(\frac{\pi}{2}) - R(0)}{R}$$

- For the Earth, the Moon produces $\varepsilon = -4.8\ 10^{-8}$.

8.5 Tides

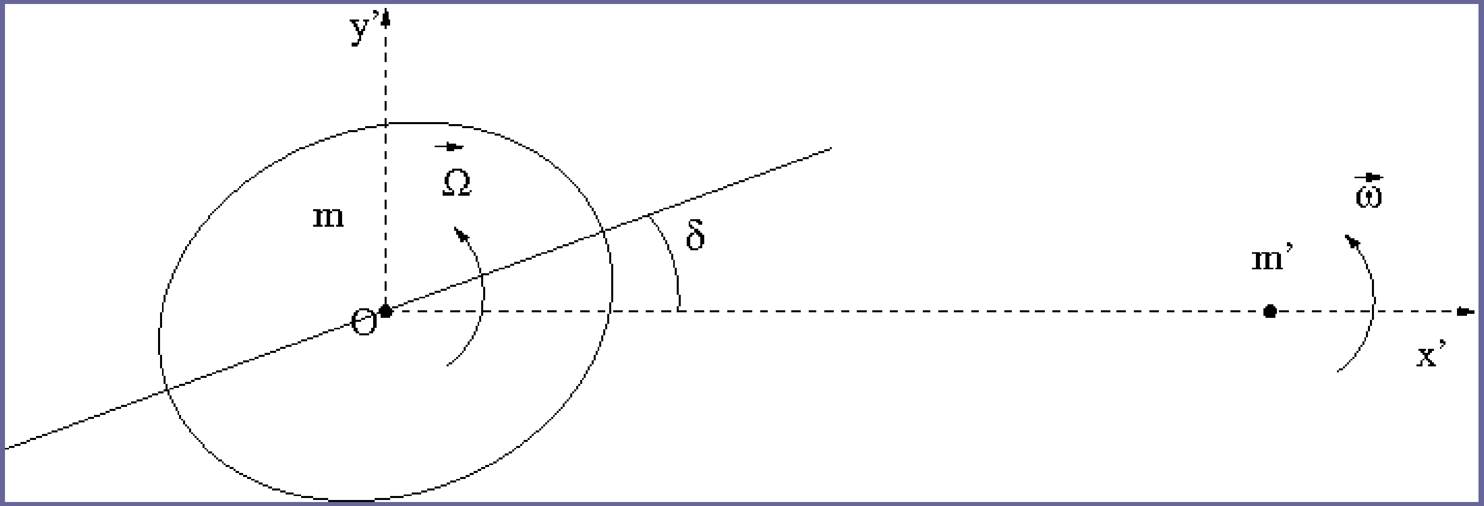
- The amplitude of the tides depends on the lunar phase (neap tides and spring tides):



- The most accurate method of tidal prediction (near coasts, at least) is to carefully measure tides over an extended period, and use harmonic analysis.

8.5 Tides

- The tidal bulge leads to an exchange of angular momentum between the Moon’s orbital motion and the Earth’s rotation.
- This is because, if we account for the Earth’s rotation, the tidal bulge lacks behind the Moon’s position by some small angle δ:



- The tidal bulge generates a potential outside the Earth, given by

$$U' = -\frac{G\,m}{a}\left(1 - \frac{2\,\epsilon\,R^2}{5\,a^2}\,P_2(\cos\left(\theta - \delta\right))\right)$$

8.5 Tides

- The torque produced on the Moon is given by

$$\vec{\mathcal{M}} = -m'\,\vec{a}\wedge\vec{\nabla}(U') = -m'\,a\,\vec{e_{x'}}\wedge\frac{1}{a}\left.\frac{\partial U'}{\partial\theta}\right)_{\theta=0}\vec{e_\theta}$$

$$= -G\,m\,m'\,\frac{6\,\epsilon}{5}\,\frac{R^2}{a^3}\,\cos\delta\,\sin\delta\,\vec{e_{z'}}$$

$$\simeq -G\,m\,m'\,\frac{6\,\epsilon}{5}\,\frac{R^2}{a^3}\,\delta\,\vec{e_{z'}}$$

- This leads to a slow increase of the Moon's distance.
- Because the total angular momentum of the Earth-Moon system remains constant, an identical, but opposite torque acts upon the Earth, slowing down its rotation:

$$\mathcal{I}_{z'z'}\,\dot{\Omega} = -\mathcal{M}$$

8.6 Spin-orbit coupling

- Consider an aspherical body of mass m_1 orbiting a spherical body of mass m_0 with $m_1 << m_0$.
- In the principal axes of inertia, the potential outside the aspherical mass becomes

$$U = \frac{G\, m_1}{r} - \frac{G\,(A+B+C)}{2\, r^3} + \frac{3\, G\,(A\, x^2 + B\, y^2 + C\, z^2)}{2\, r^5}$$

- The force and torque exerted by m_1 upon m_0 are:

$$\vec{F} = -m_0\, \vec{\nabla} U \qquad \vec{\mathcal{M}} = \vec{r} \wedge \vec{F}$$

- Since we consider an isolated system, the torque exerted by m_0 upon m_1 is $-\vec{\mathcal{M}}$
- If $\vec{\omega} = (0, 0, \dot{\phi}) = \dot{\phi}\, \vec{e_z}$, we obtain from Euler's equations:

$$C\, \ddot{\phi} = -\mathcal{M}_z = \frac{3\, G\, m_0\, (B-A)\, x\ y}{r^5}$$

8.6 Spin-orbit coupling

- Let us call ξ the true anomaly of m_1 on its orbit about m_0:

$$\ddot{\phi} = \frac{3\,G\,m_0}{2\,a^3}\,\frac{B-A}{C}\left[\frac{a^3}{r^3}\sin(2\,\xi)\cos(2\,\phi) - \frac{a^3}{r^3}\cos(2\,\xi)\sin(2\,\phi)\right]$$

$$= 8\,\beta\,n^2\left[\frac{a^3}{r^3}\sin(2\,\xi)\cos(2\,\phi) - \frac{a^3}{r^3}\cos(2\,\xi)\sin(2\,\phi)\right]$$

- With $\beta = \frac{3}{4}\frac{m}{M+m}\frac{B-A}{4\,C}$. From Chapter 6, we know:

$$\left(\frac{a}{r}\right)^3\cos(2\,\xi) = -\frac{e}{2}\cos M + \cos(2\,M) + \frac{7\,e}{2}\cos(3\,M) + \mathcal{O}(e^2)$$

$$\left(\frac{a}{r}\right)^3\sin(2\,\xi) = -\frac{e}{2}\sin M + \sin(2\,M) + \frac{7\,e}{2}\sin(3\,M) + \mathcal{O}(e^2)$$

- This leads to:

$$\ddot{\phi} = 8\,\beta\,n^2\left[\frac{-e}{2}\sin(n\,t - 2\,\phi) + \sin(2\,n\,t - 2\,\phi) + \frac{7\,e}{2}\sin(3\,n\,t - 2\,\phi)\right]$$

8.6 Spin-orbit coupling

$$\ddot{\phi} = 8\,\beta\,n^2\left[\frac{-e}{2}\,\sin\left(n\,t - 2\,\phi\right) + \sin\left(2\,n\,t - 2\,\phi\right) + \frac{7\,e}{2}\,\sin\left(3\,n\,t - 2\,\phi\right)\right]$$

- This equation is highly non-linear. We can use a representation of the conditions at pericentre, i.e. when $M = n\,t = k\,2\pi$.
- Let us define $\eta = \phi - q\,n\,t$ where $q=k_o/k_s$ is either 0.5, 1.0 or 1.5.

$$\begin{aligned}\ddot{\eta} \;&=\; -8\,\beta\,n^2\left\{\left[\frac{-e}{2}\,\cos\left[(2\,q-1)\,n\,t\right] + \cos\left[(2\,q-2)\,n\,t\right] + \frac{7\,e}{2}\,\cos\left[(2\,q-3)\,n\,t\right]\right]\,\sin\left(2\,\eta\right)\right.\\ &+\left.\left[\frac{-e}{2}\,\sin\left[(2\,q-1)\,n\,t\right] + \sin\left[(2\,q-2)\,n\,t\right] + \frac{7\,e}{2}\,\sin\left[(2\,q-3)\,n\,t\right]\right]\,\cos\left(2\,\eta\right)\right\}\end{aligned}$$

- If we average over k_s orbital cycles, we obtain:

$$\begin{aligned} q = 0.5 \quad &\Rightarrow \quad \ddot{\eta} = 4\,\beta\,e\,n^2\,\sin\left(2\,\eta\right)\\ q = 1.0 \quad &\Rightarrow \quad \ddot{\eta} = -8\,\beta\,n^2\,\sin\left(2\,\eta\right)\\ q = 1.5 \quad &\Rightarrow \quad \ddot{\eta} = -28\,\beta\,e\,n^2\,\sin\left(2\,\eta\right)\end{aligned}$$

8.6 Spin-orbit coupling

- We thus obtain:

$$
\begin{aligned}
q = 0.5 &\Rightarrow (n^{-1}\dot{\phi} - 0.5)^2 + 4\,\beta\,e\,\cos(2\,\phi - n\,t) = E \\
q = 1.0 &\Rightarrow (n^{-1}\dot{\phi} - 1)^2 - 8\,\beta\,\cos(2\,\phi - 2\,n\,t) = E \\
q = 1.5 &\Rightarrow (n^{-1}\dot{\phi} - 1.5)^2 - 28\,\beta\,e\,\cos(2\,\phi - 3\,n\,t) = E
\end{aligned}
$$

- This leads to spin-orbit resonances (e.g. q=1.5 for Mercury).