

Stellar Structure

SPAT0044

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Goals of the course

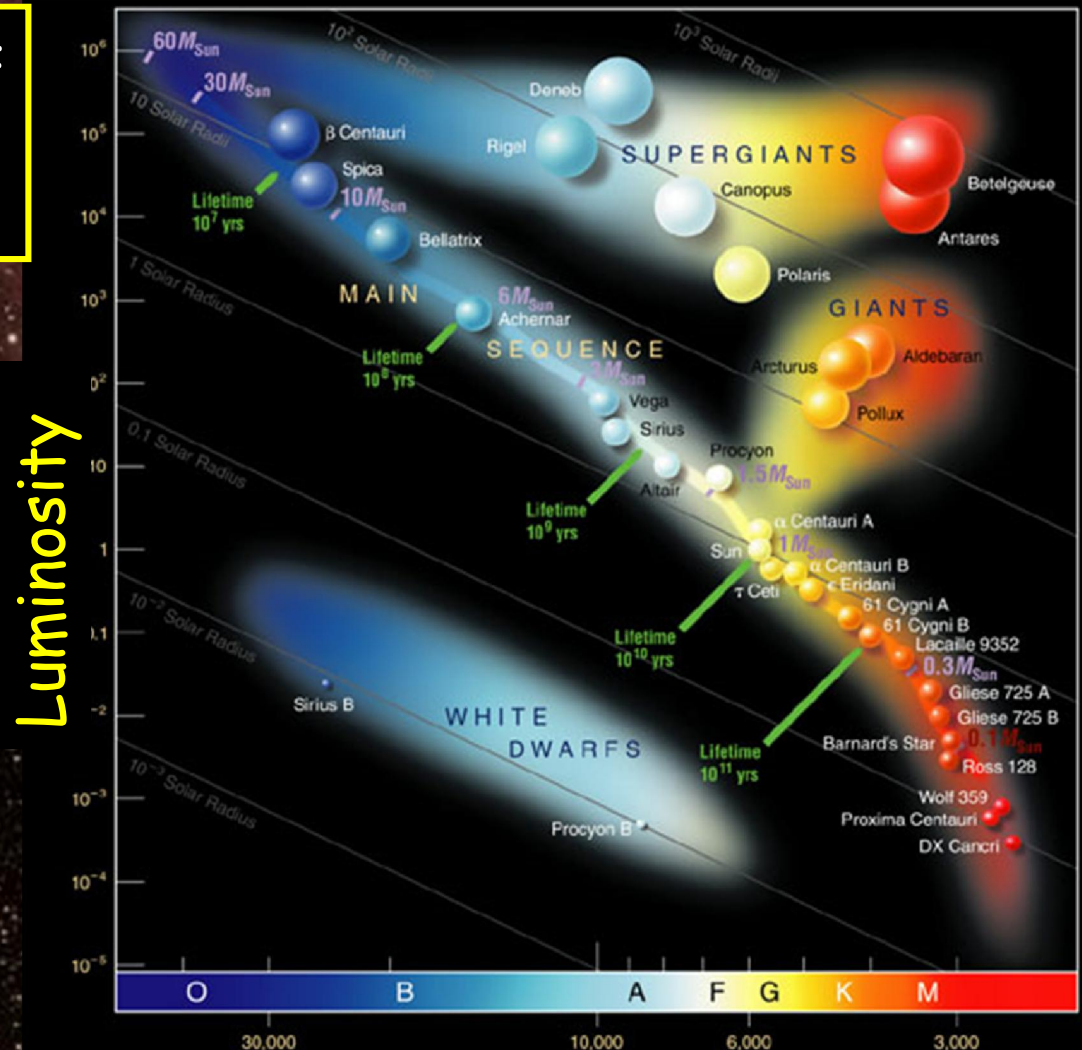
- **Physical** processes inside the stars
stars = laboratories of fundamental physics
 - Quantum physics (radiation - matter) et nuclear physics
 - Thermodynamics and quantum statistical physics
 - Hydrodynamics
- Presentation of simple approximations enabling a **mathematical** analysis of the stellar properties
- Link theory - observations

The Hertzsprung-Russell diagram

L = Luminosity =
Power radiated
by the star

Stars of
different masses,
different ages

Different positions
in the HR diagram

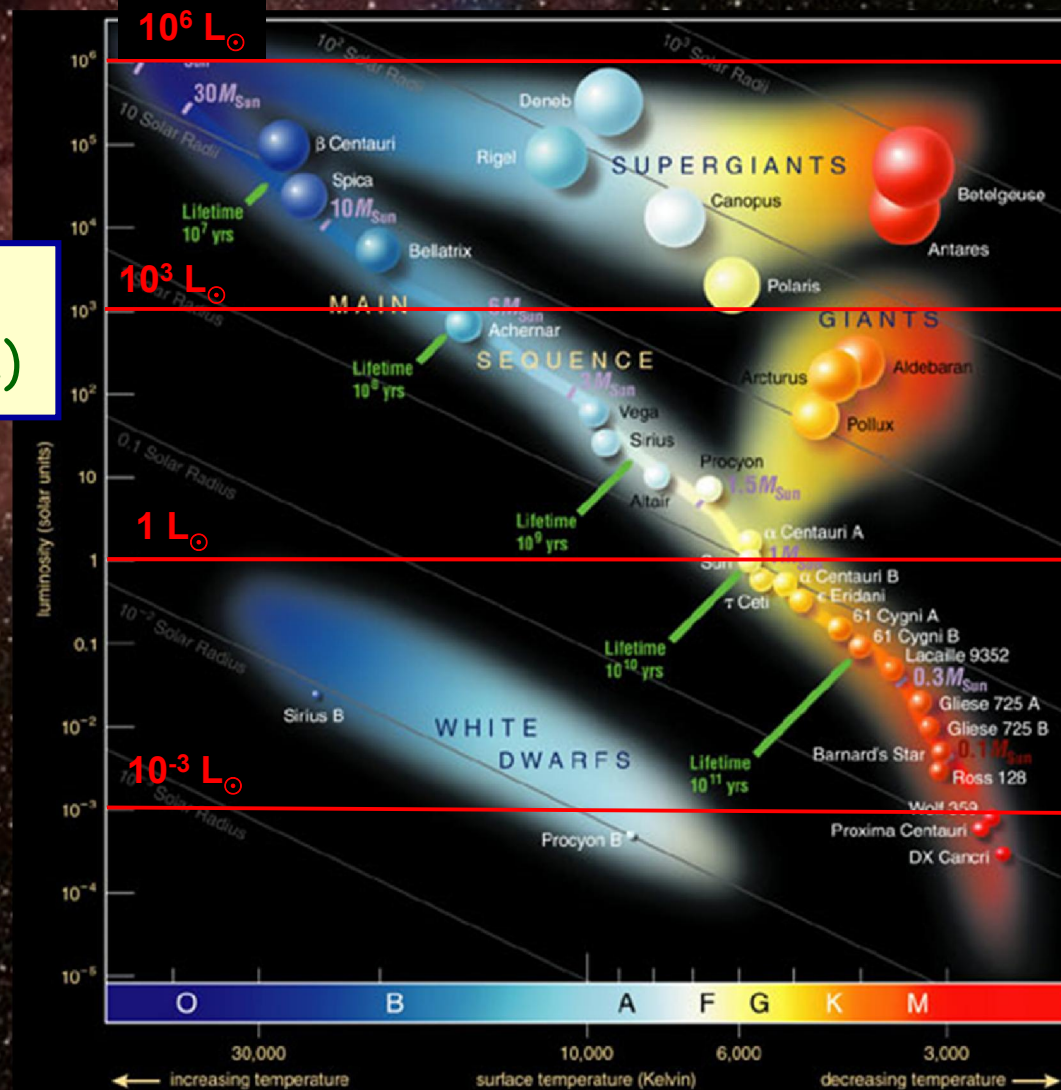


Color - spectral type

Effective temperature $\sim$ surface temperature

Luminosity of stars

Magnitude
log (Luminosity L)



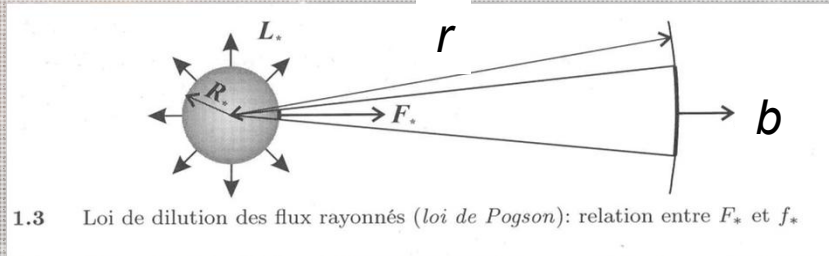
Log (Effective Temperature)

Luminosity and magnitude (ordonnée du diagramme HR)

- **Luminosity L** : Total **power** radiated by the star
- In a star of radius R :

$$L = 4\pi R^2 \cdot \int_0^{\infty} F_{\lambda} d\lambda$$

- **b** : energy **flux** crossing a surface perpendicular to the line of sight
[W/m²]



$$b = \frac{L}{4\pi r^2} = \frac{R^2}{r^2} F$$

For a given luminosity, b decreases with the distance.



In order to measure the luminosity, the distance must be known

Luminosity and magnitude

- Apparent magnitude m :

$$m = -2.5 \log b + \text{Cst} \quad m_1 - m_2 = -2.5 \log \frac{b_1}{b_2}$$

- Absolute magnitude M :

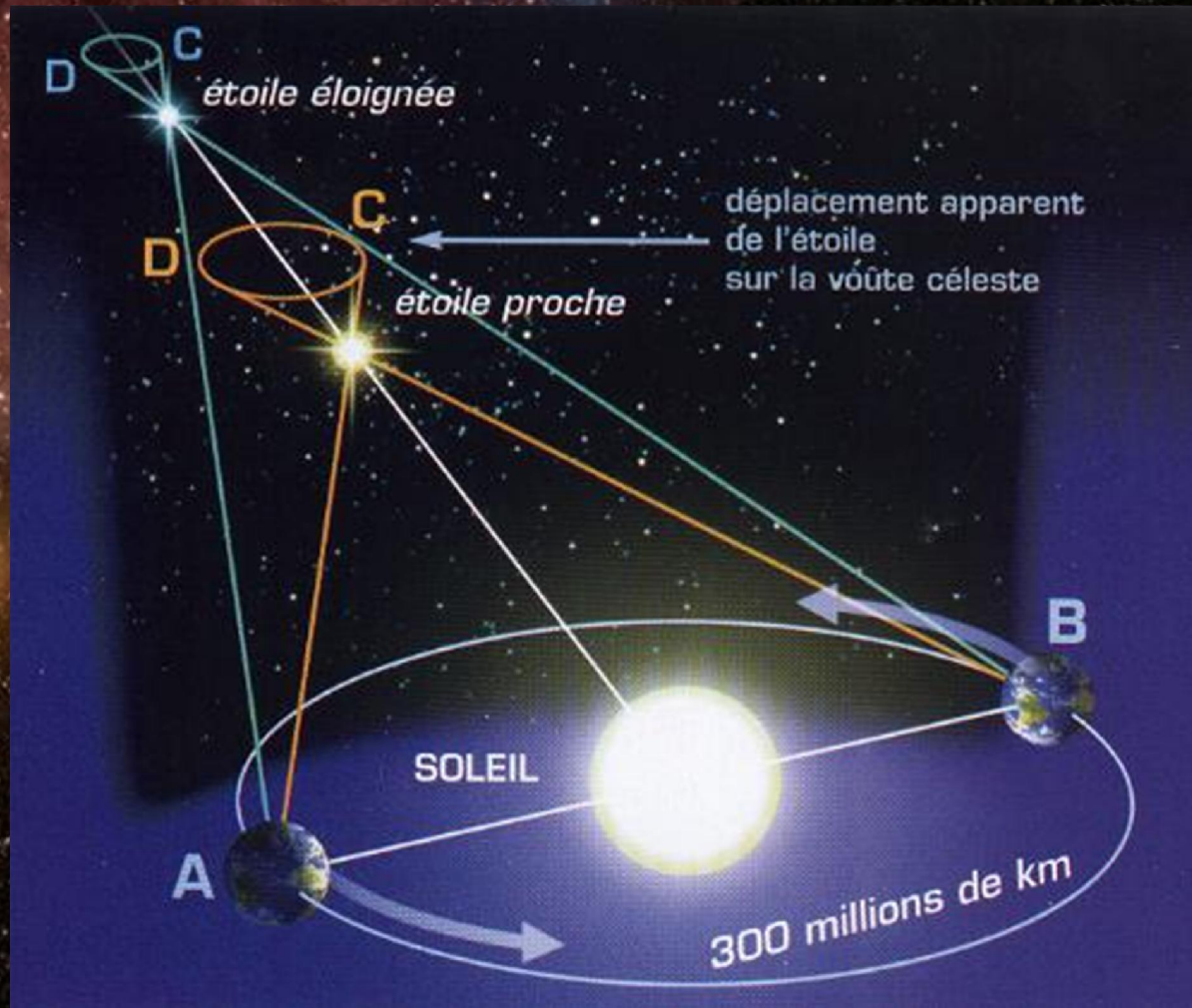
The magnitude at a reference distance of 10 parsecs (pc)

$$\frac{B}{b} = \left(\frac{r_{pc}}{10} \right)^2 \quad M = -2.5 \log B + \text{Cst}$$

$$M - m = 5 - 5 \log r_{pc}$$

In order to convert apparent magnitudes into absolute magnitudes, the distance must be known

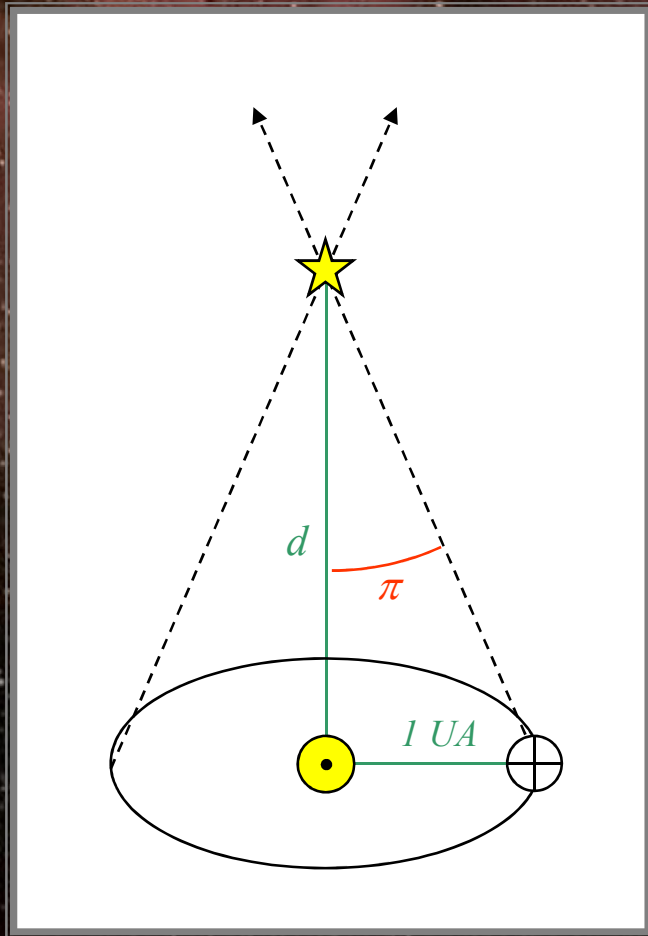
Distance measurement: the annual parallax method



Triangulation base = Earth-Sun distance

The parsec

1 pc = distance of a star having an annual parallax of 1 arcsecond (")



The **Astronomical Unit (UA)** is the semi-major axis of the earth-sun orbit.

$$d_{UA} = 1/\pi_{rad} = 206\,264.8/\pi''$$

$$1 \text{ Parsec} = 1 \text{ Pc}$$

$$= 206\,264.8 \text{ UA}$$

$$\cong 3 \times 10^{13} \text{ km} \cong 3.26 \text{ light years}$$

$$d_{pc} = 1/\pi''$$

The monochromatic flux F_λ

Qty of energy / u. time / u. surface / u. length (wavelength)

$F_\lambda d\lambda$: flux of energy between λ and $\lambda+d\lambda$

$\int_{\lambda_1}^{\lambda_2} F_\lambda d\lambda$: Flux in the wavelength interval $[\lambda_1, \lambda_2]$

The bolometric flux F

Flux integrated over the whole spectrum :

$$F = \int_0^\infty F_\lambda d\lambda$$

Photométrie

Les mesures photométriques se font à travers des **filtres** qui isolent une partie du spectre électromagnétique

→ mesure du flux reçu dans une certaine bande spectrale

→ une magnitude est toujours donnée en référence à un filtre

$$Ex : m_B = -2.5 \log(F_B) + \text{cst} \quad F_B = \int w_B(\lambda) F_\lambda d\lambda$$



Couleurs

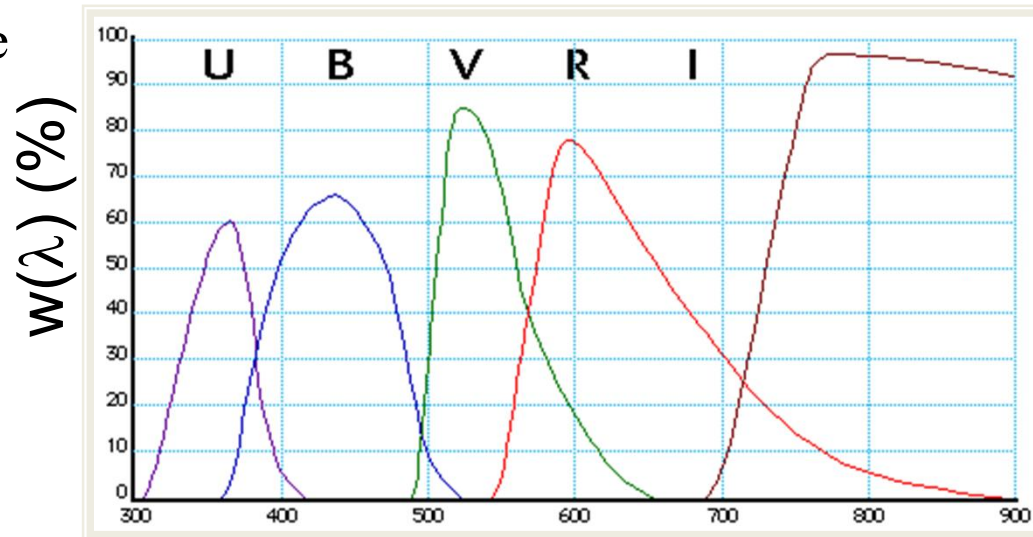
Pour quantifier la couleur d'une étoile on définit des **indices de couleur**

$$Ex : m_B - m_V = M_B - M_V$$

indépendant de la distance car la dilution géométrique ne dépend pas de la longueur d'onde.

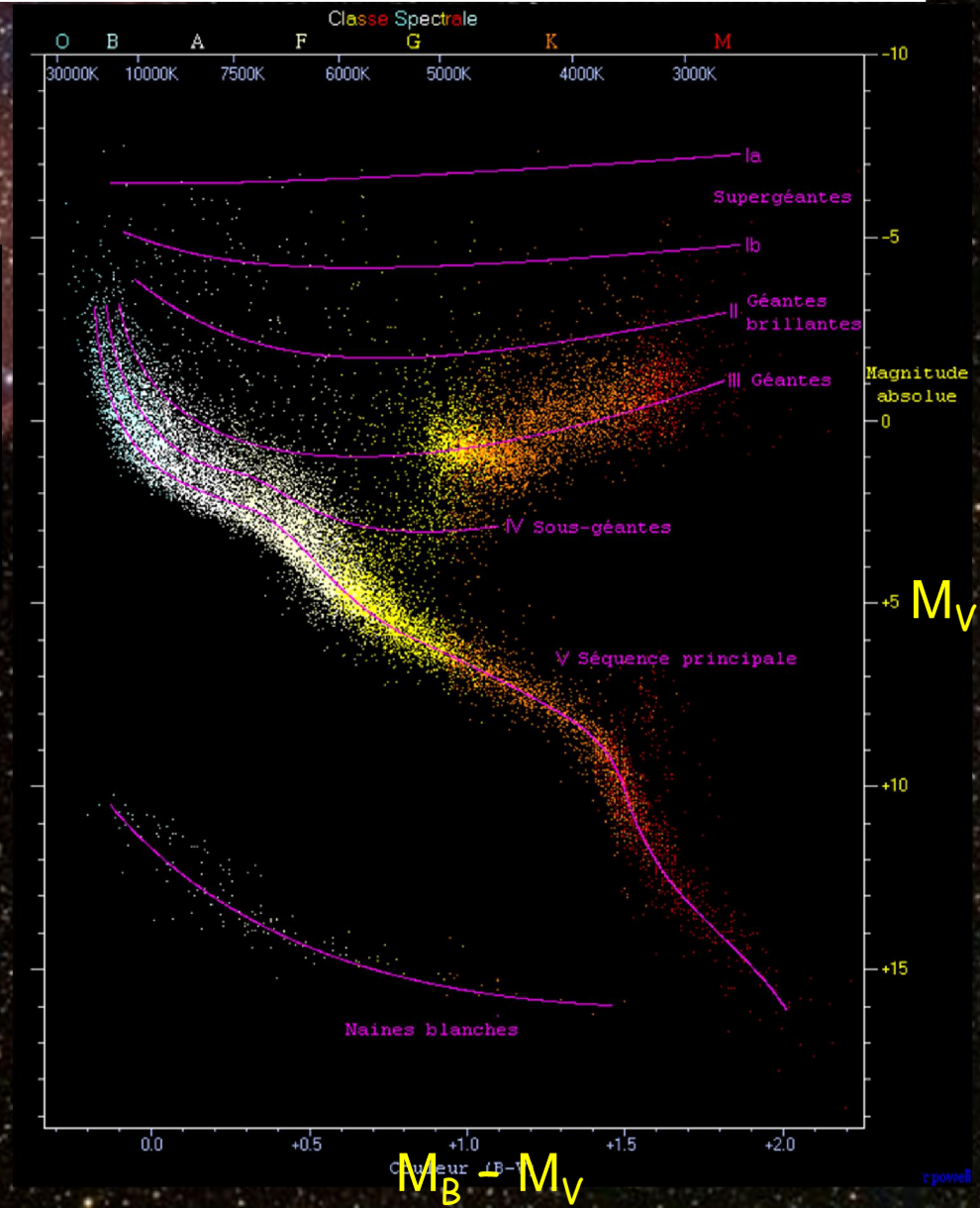
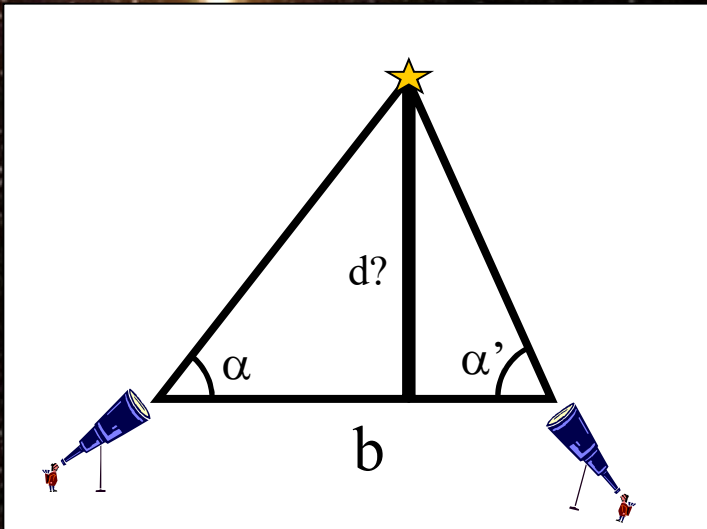
Mais rougissement interstellaire ...

Courbes de transmission des filtres UBVRI



Hertzsprung-Russell diagram

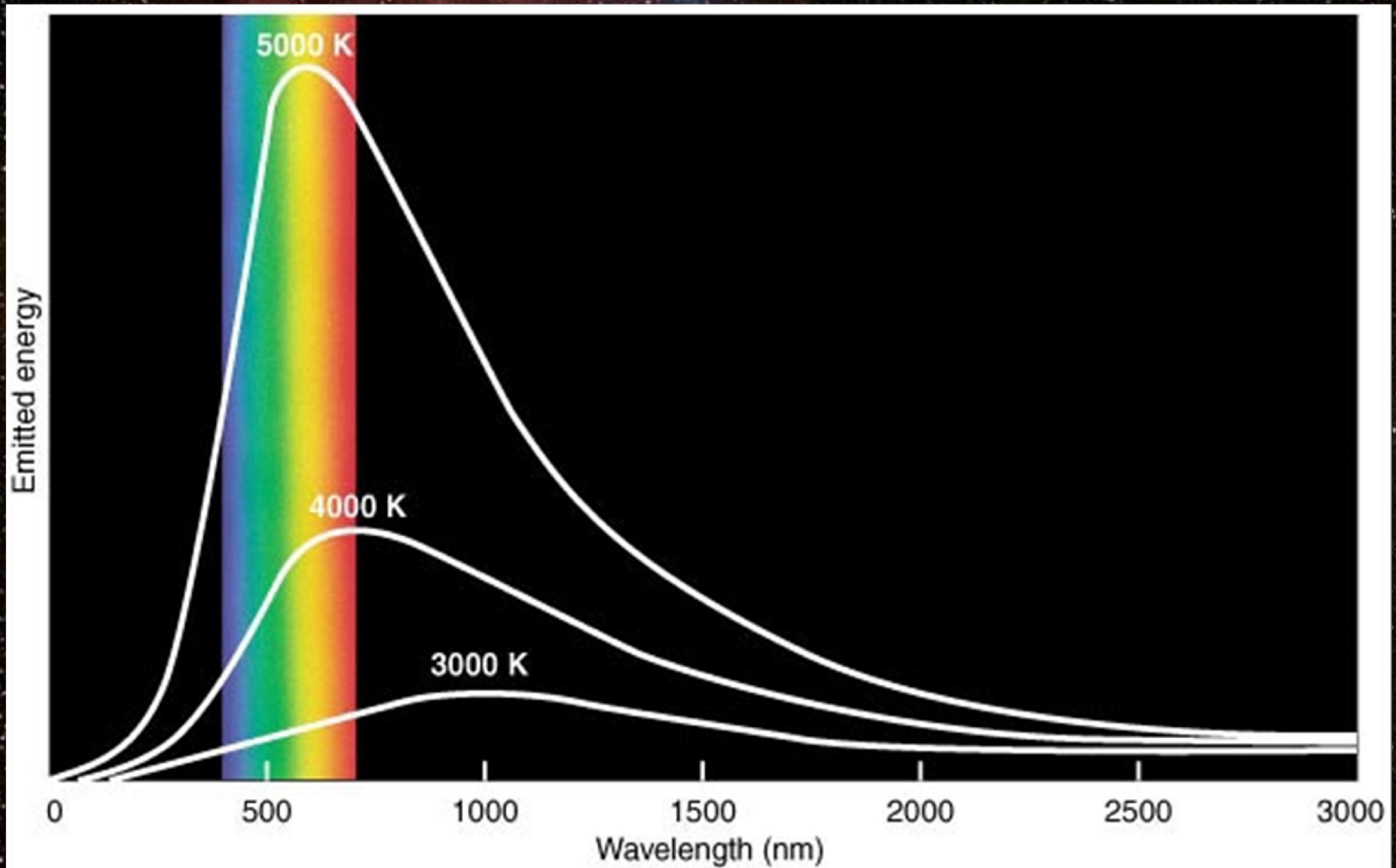
Stars whose distance was measured by the Hipparcos satellite (parallax method)



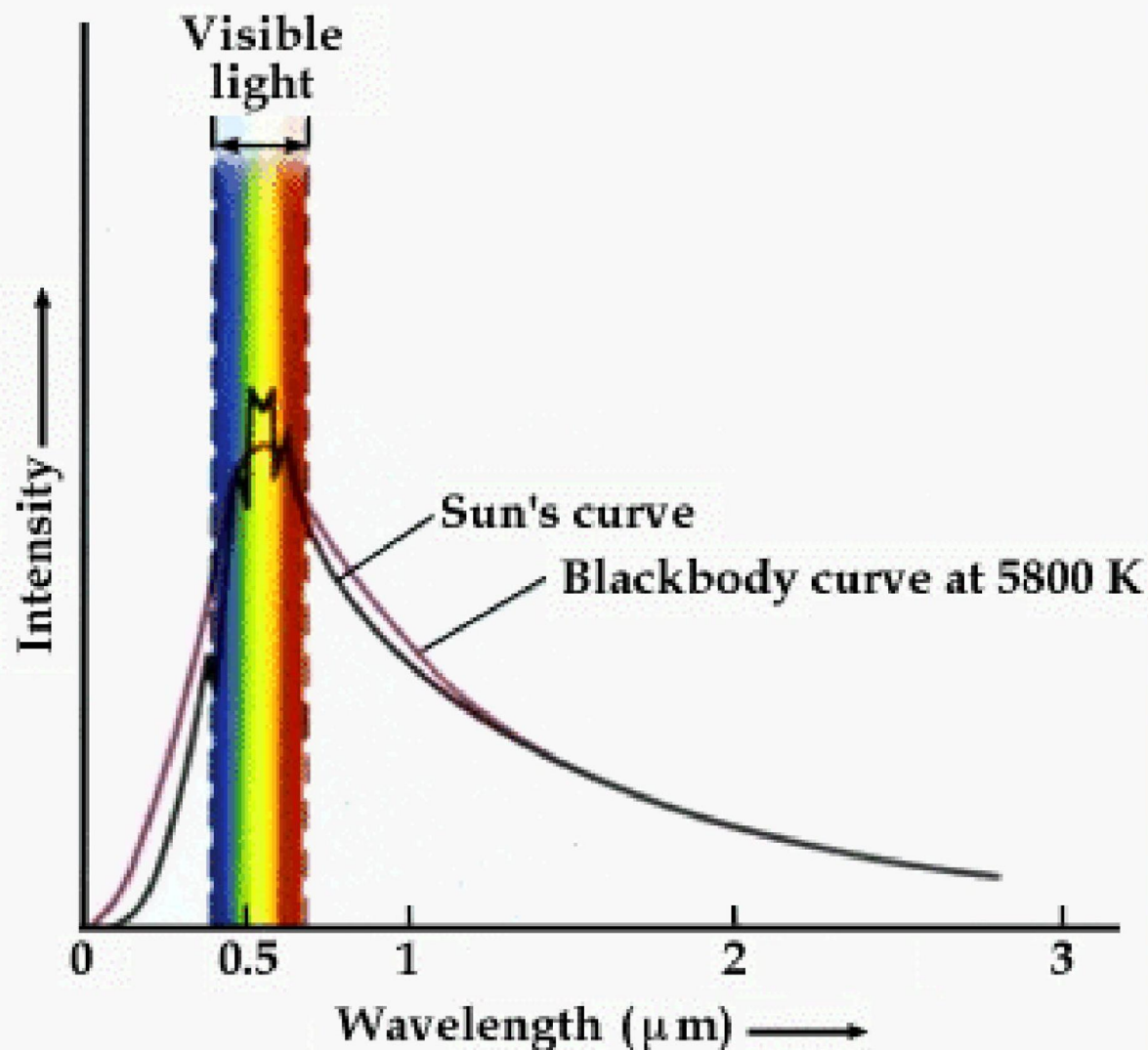
Luminosity and magnitude

- Luminosity $L = 4\pi R^2 F$ (W)
- $b = L / 4\pi r^2$ (W/m²)
- Apparent magnitude $m_1 - m_2 = -2,5 \log (b_1/b_2)$
- $B = b (r_{pc}/10)^2$
- Absolute magnitude $M - m = 5 - 5 \log r_{pc}$

The black body spectrum



Stellar spectrum ~ Black body



Black body laws

- Planck function

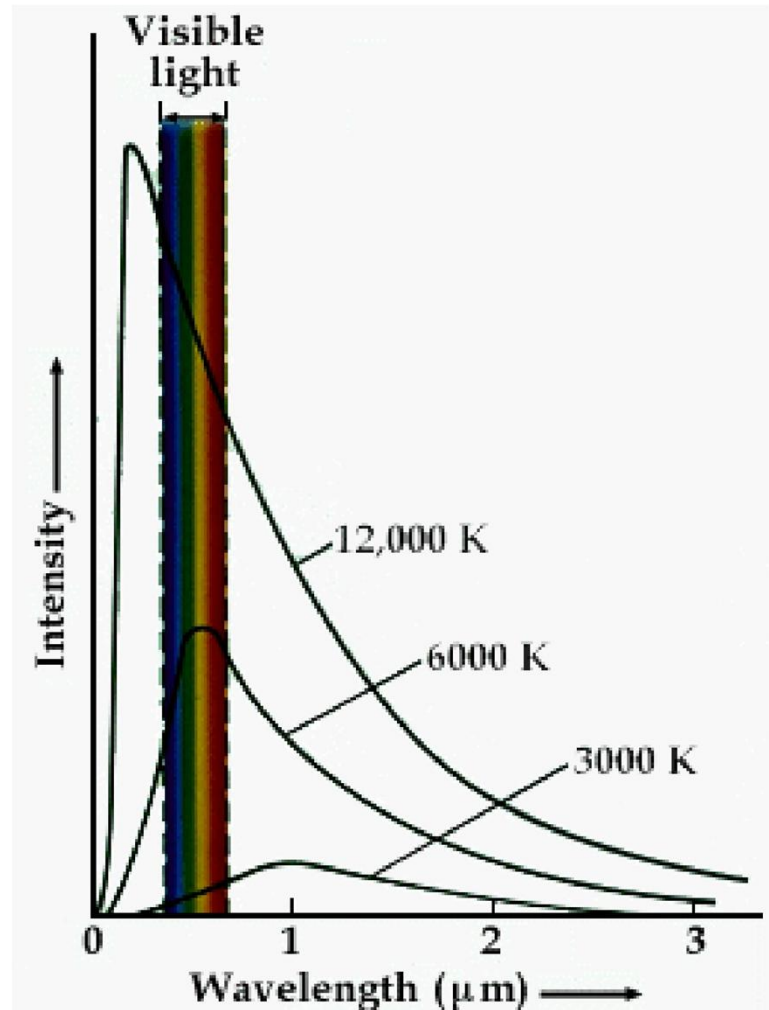
$$F_{\lambda} = \frac{2\pi hc^2}{\lambda^5} \frac{1}{\exp(hc/\lambda kT) - 1}$$

- Stefan-Boltzmann law

$$F = \int_0^{\infty} F_{\lambda} d\lambda = \sigma T^4$$

- Wien law

$$\lambda_{\max}(cm) = \frac{0,29}{T(K)}$$



Effective temperature: Definition

Definition 1 :

A: body emitting from its surface a bolometric flux **F**

The effective temperature of A is by definition :

$$T_{eff} \equiv \left(\frac{F}{\sigma} \right)^{1/4}$$

Effective temperature: Definition

Any body : $F = \sigma T_{\text{eff}}^4$

Black body : $F = \sigma T^4$

Definition 2 :

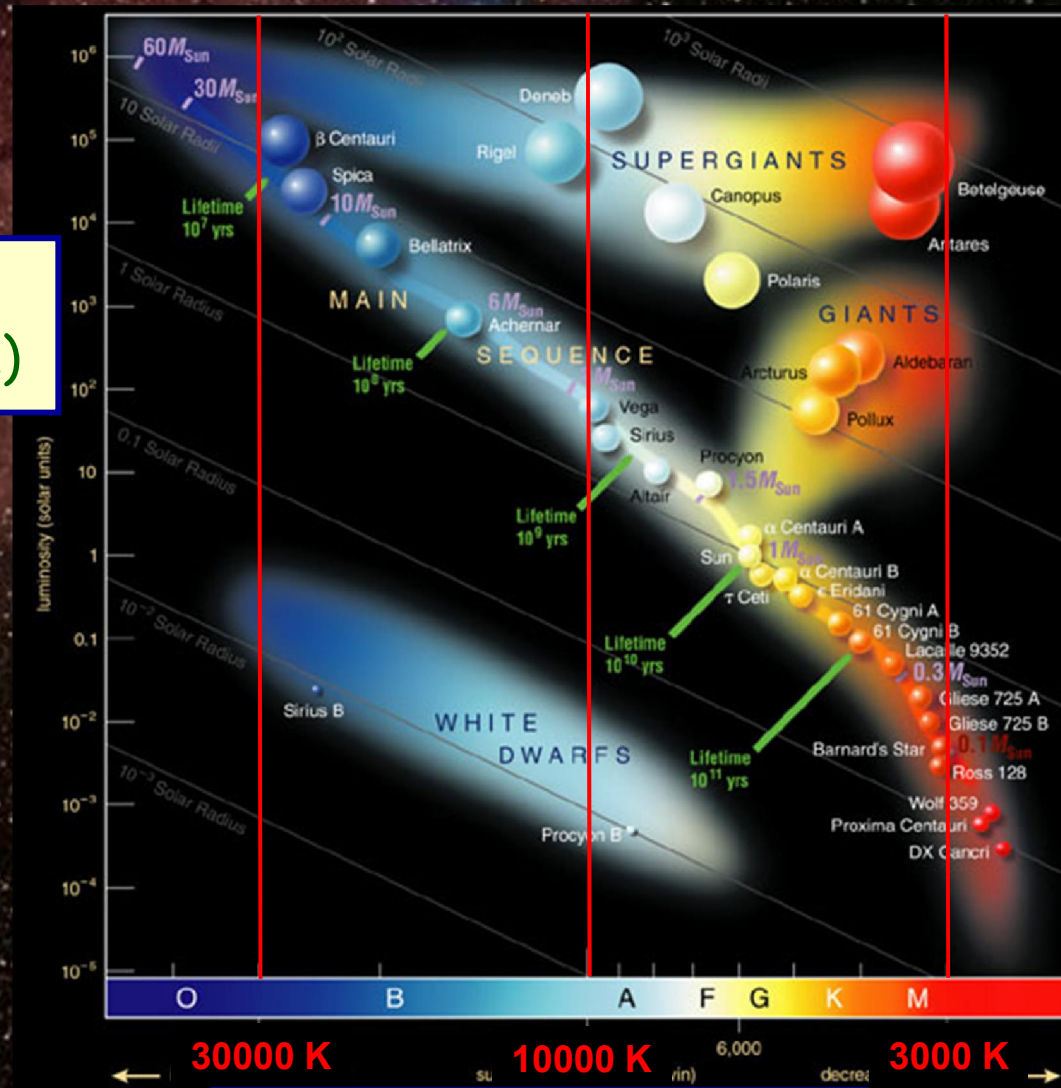
The effective temperature of an emitting body is the temperature of a hypothetical black body emitting the same bolometric flux.

For a spherical symmetric body of radius R and luminosity L , we have by definition :

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4$$

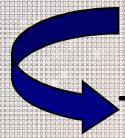
Effective temperatures of stars

Magnitude
log (Luminosity L)



Size of stars

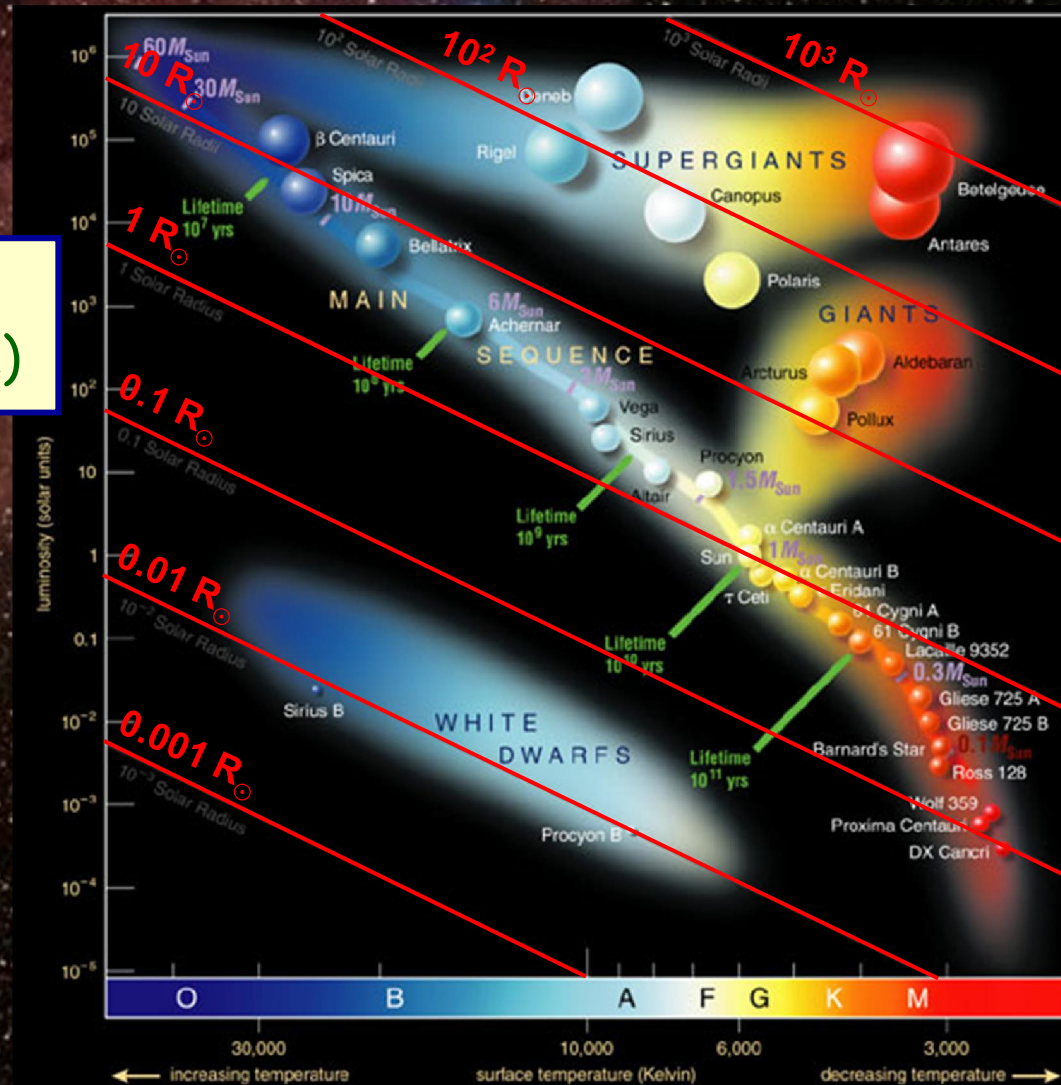
• **Size** of stars :
$$F = \frac{L}{4\pi R^2} = \sigma T_{\text{eff}}^4 \rightarrow \log L = 2 \log R + 4 \log T_{\text{eff}} + \text{Cst}$$



- **Main Sequence stars** have a restricted range of radii
- **Supergiants** are thousands times larger than the Sun
- **Giants** are ten-hundred times larger than the Sun
- **White dwarfs** are one hundred times smaller than the Sun

Stellar sizes

Magnitude
log (Luminosity L)



Log (Effective Temperature)

Hertzsprung-Russell diagram

The main sequence

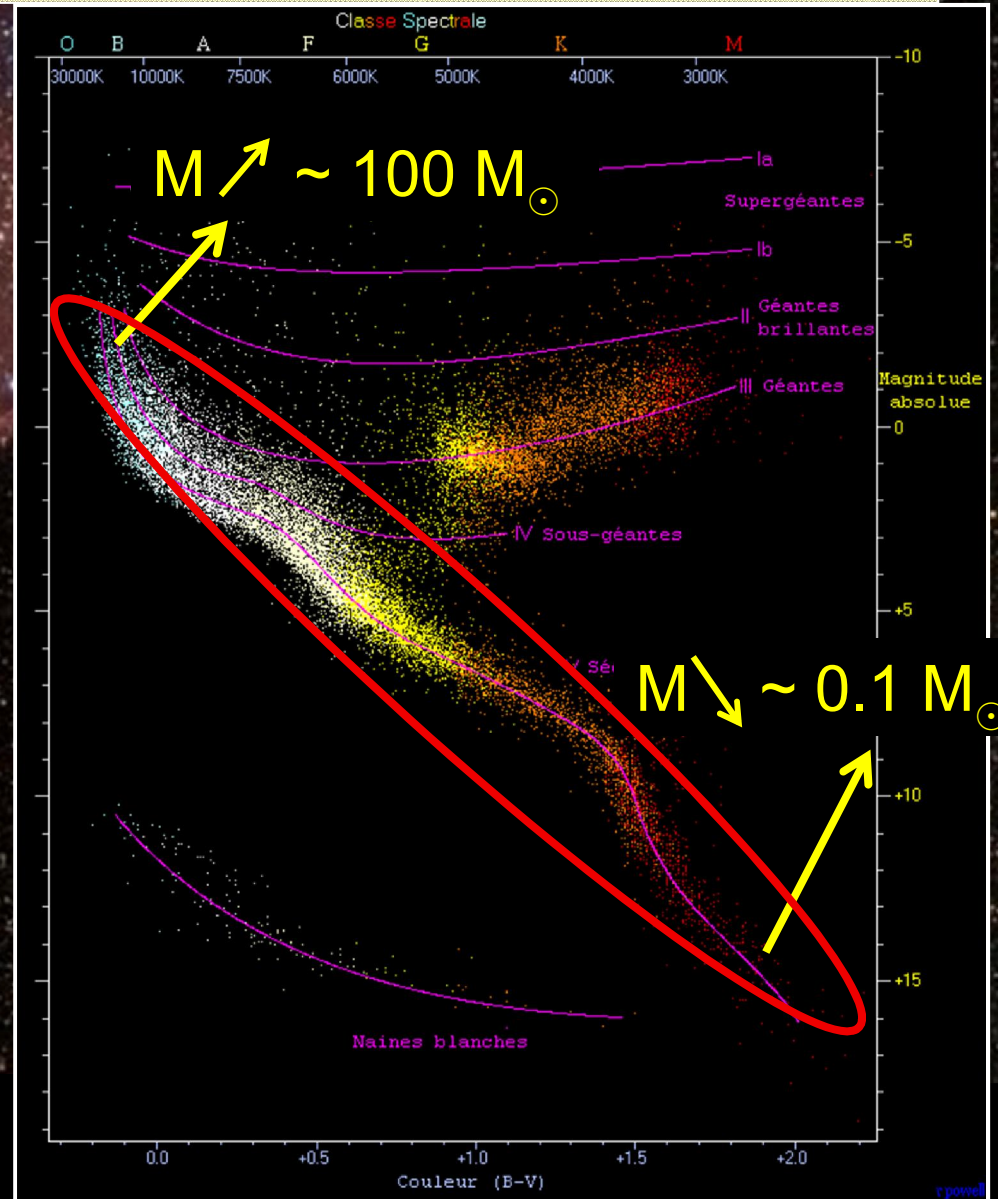
Stellar source of energy:
Thermonuclear reactions

Longest Phase : Fusion of
hydrogen into helium

Most stars are in this phase

They occupy a specific region
of the HR diagram:
The main sequence

Our Sun is a main sequence star

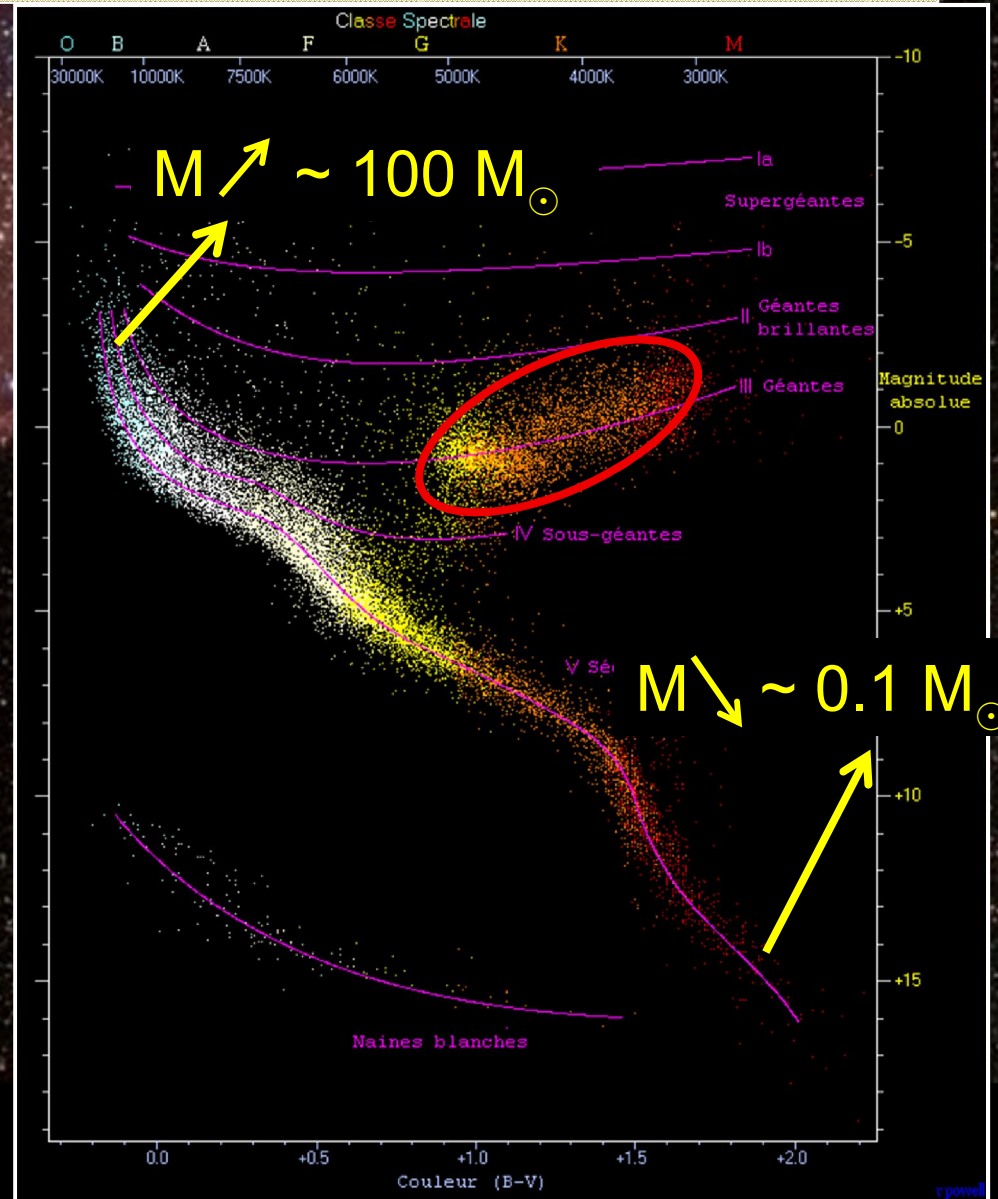


Hertzsprung-Russell diagram

Red giants

The core of a red giant is very dense,
its envelope is huge
(can encompass the earth orbit)

Phase of core helium fusion
into carbon and oxygen,
and/or shell hydrogen fusion

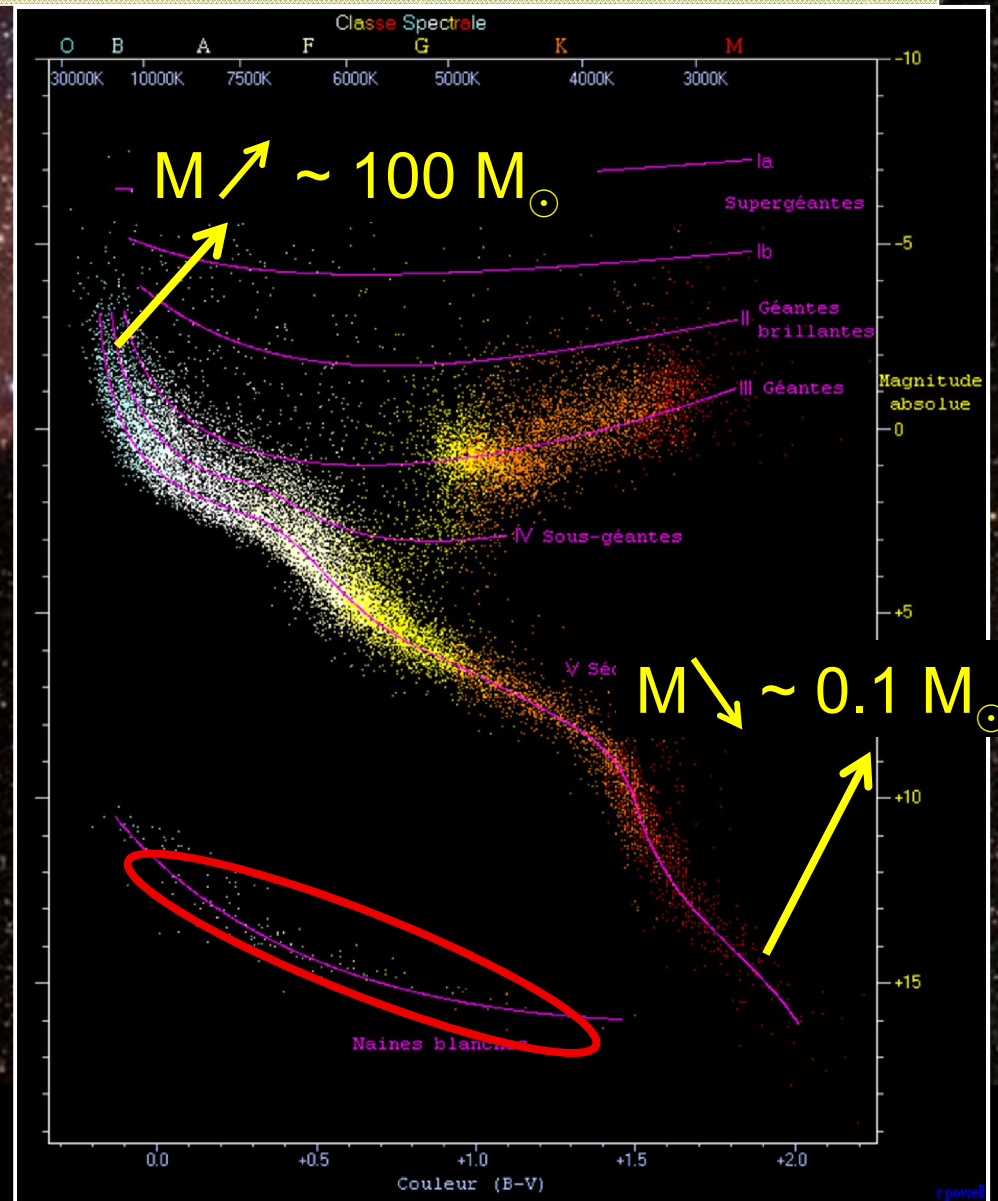


Hertzsprung-Russell diagram

White dwarfs

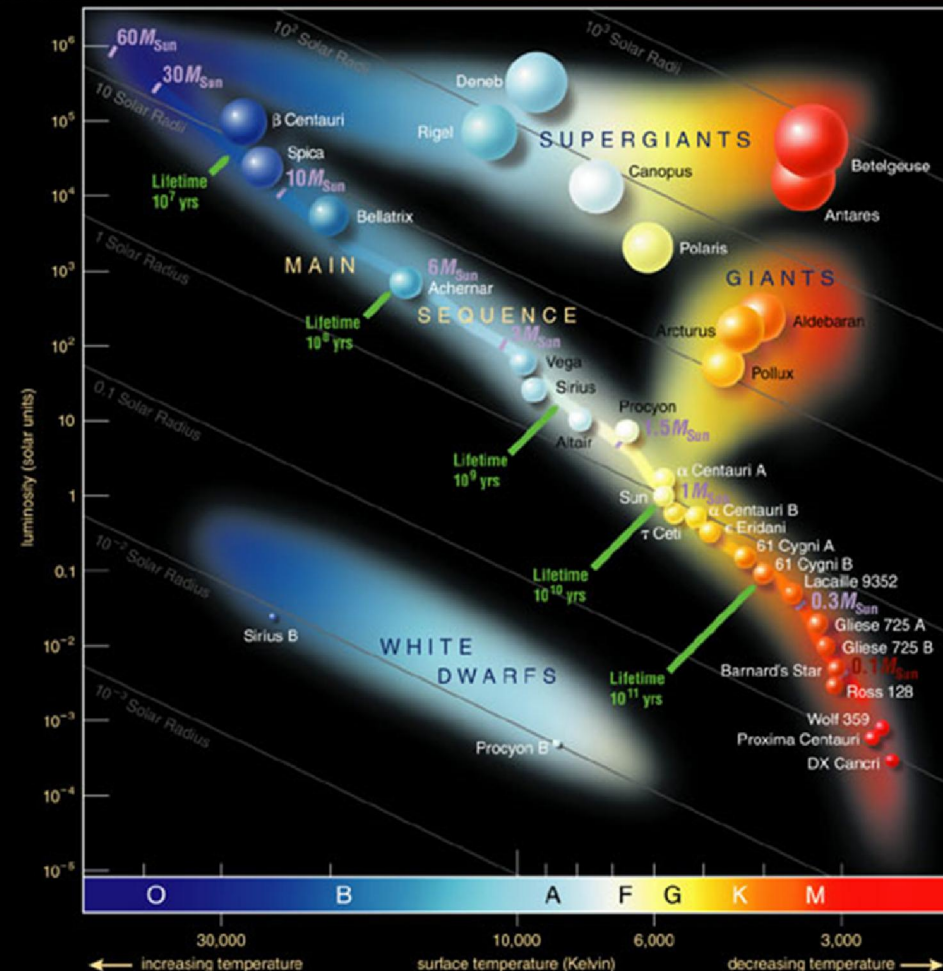
End of the life of our Sun:
Small and dense
($R \sim 0.01 R_{\odot}$, $\rho \sim 1 \text{ ton} / \text{cm}^3$)

Slow cooling ...



Stellar masses

Determination of the masses in binary systems: **Third Kepler law**



- Main sequence : $0.1 - 100 M_{\odot}$
- Mass of giants : $1 - 10 M_{\odot}$
- Mass of supergiants : $\geq 20 M_{\odot}$
- Mass of white dwarfs : $\approx 0.5 M_{\odot}$

• Mass-luminosity relation in main sequence stars

$$L \sim M^{v(>1)}$$

Characteristics of stars

$$0.08 M_{\odot} < M < 100 M_{\odot}$$

$$\dots 0 \dots < L < 10^6 L_{\odot}$$

$$3000 \text{ K} < T_{\text{eff}} < 100000 \text{ K}$$

$$\dots 0 \dots < R < 1000 R_{\odot}$$

$$\text{Pop I} - \text{Pop II} \Rightarrow X, Y, Z$$

Our Sun

$$\text{Radius} = 0.697 \times 10^6 \text{ km}$$

Can be measured, knowing:

- Its distance d
- Its angular diameter θ

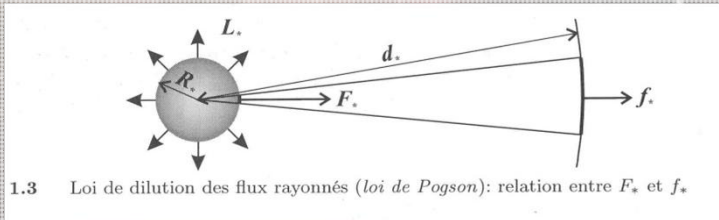
$$\Rightarrow R = d \tan \theta/2$$

$$\text{Luminosity} = 3.826 \times 10^{26} \text{ W}$$

Can be measured, knowing :

- Its distance d
- The flux f received on earth

$$\Rightarrow L = 4\pi d^2 f$$



$$\text{Effective temperature} = 5777 \text{ K}$$

~ Surface temperature

Obtained from its definition :

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4$$

$$\text{Mass} = 1.989 \times 10^{30} \text{ kg}$$

From the third Kepler law:

$$GM = 4\pi^2 a^3/P^2$$

Internal characteristics of stars

Spherical Symmetry

→ Only one space dimension: r

Masse

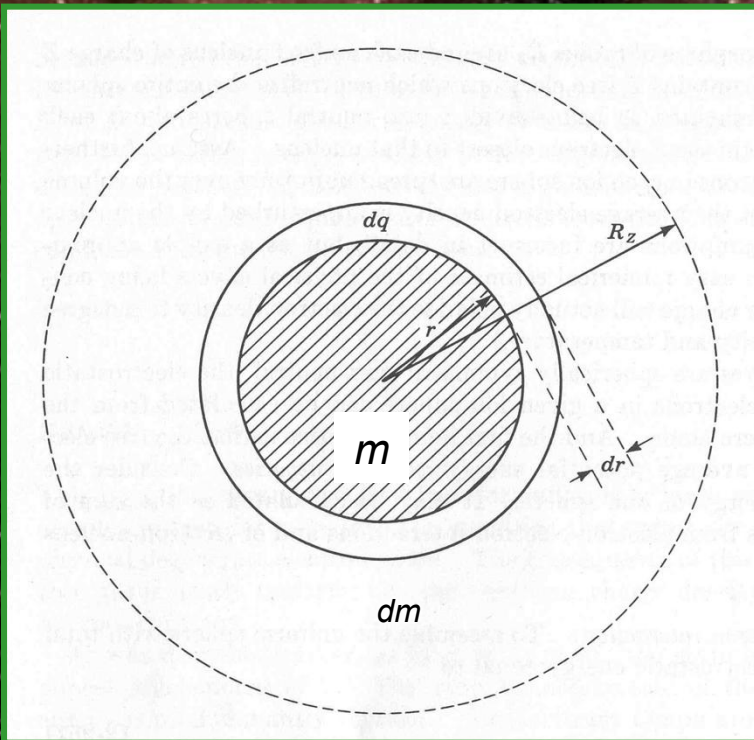
Mass of each spherical shell of thickness dr

$$dm = \rho dV = \rho 4\pi r^2 dr$$

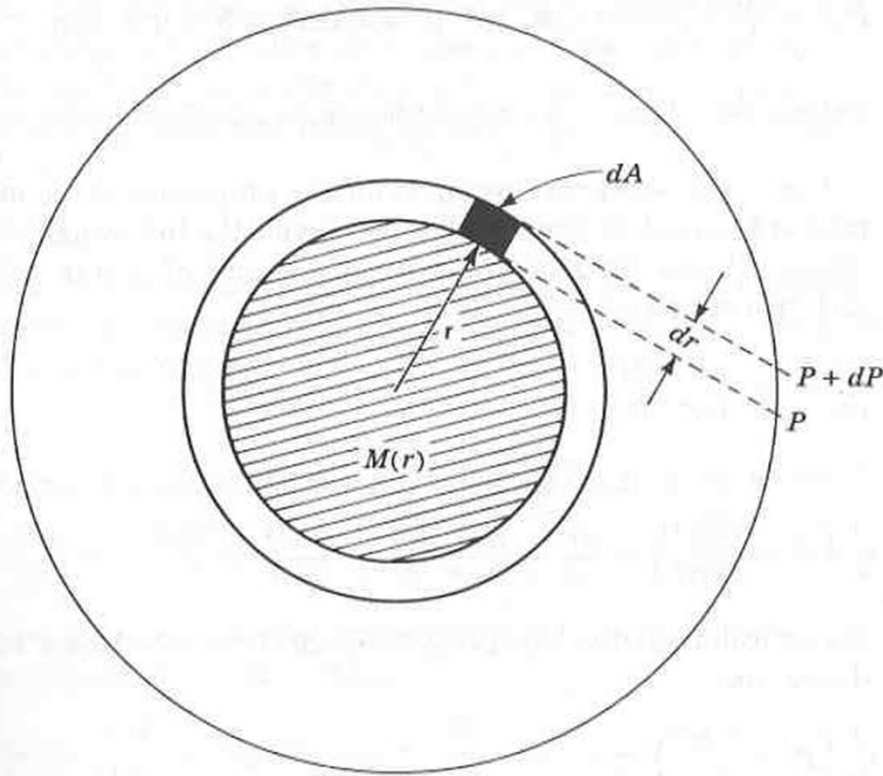
$$\frac{dm}{dr} = 4\pi \rho r^2$$

Sun :

$$\begin{aligned} \langle \rho \rangle &= \frac{3}{4\pi} \frac{M}{R^3} \\ &\sim 1.4 \times 10^3 \text{ kg/m}^3 \end{aligned}$$

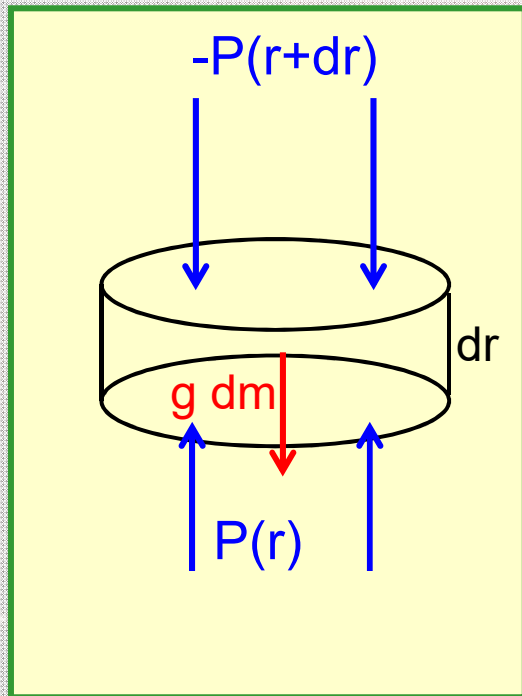


A gas sphere in hydrostatic equilibrium



The resultant force on
each material volume
 $= 0$

Sphere of gas in hydrostatic equilibrium



Resultant force on an infinitesimal volume
(1 m² section) :

Pressure : $-P(r+dr) + P(r)$ (surface force)

Weight : $-\rho \, g \, dr$ (volume force)

Equilibrium of the forces :

$$\sum \vec{F} = 0 \quad \boxed{\frac{dP}{dr} = -\rho \, g = -\rho \frac{Gm}{r^2}}$$

Integrated form :
($P(R) = 0$)

$$\boxed{P(r) = \int_r^R g \, \rho \, dr = \int_{m(r)}^M \frac{Gm}{4 \pi r^4} dm}$$

Fast rotation and hydrostatic equilibrium

Stars can rotate fast $\Rightarrow$ centrifugal force:

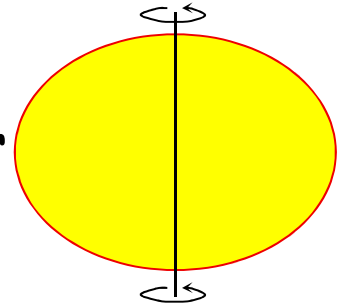
$$\nabla P / \rho = -\nabla \phi + \Omega^2 r \sin \theta \vec{e}_s \Rightarrow \text{Deformation of the star}$$

$\Rightarrow$ All quantities ($T, P, \rho, \vec{F}, \dots$) depend
on 2 space dimensions (r, θ) $\Rightarrow$ much harder to solve

Critical speed : if $V_{eq} = \Omega R > V_{crit} \approx (GM/R)^{1/2}$,

Centrifugal force $>$ gravity $\Rightarrow$ the envelope is expelled

Some stars (Be stars, ...) are not far away from that !



Influence of rotation

Affects the stellar
structure and evolution :
- 2D structure
- Transport processes

Affects spectroscopic and photometric
observables (T_{eff} , ...),
Because at the photosphere: $F(\theta), g_e(\theta)$

Fast rotation and hydrostatic equilibrium

Baroclinity:

$$\begin{aligned}s &= r \sin\theta \\ \vec{e}_s &= \nabla s \\ s \vec{e}_s &= (1/2) \nabla(s^2)\end{aligned}$$

Rotational of the equilibrium equation $\nabla P/\rho = -\nabla\phi + \Omega^2 s \vec{e}_s$

$$\nabla \times (\nabla P/\rho) = -(1/\rho^2) \nabla \rho \times \nabla P = (1/2) \nabla(\Omega^2) \times \nabla(s^2)$$

↪ For a rigid or cylindrical rotation ($\Omega(s)$): $\nabla \rho \times \nabla P = 0$

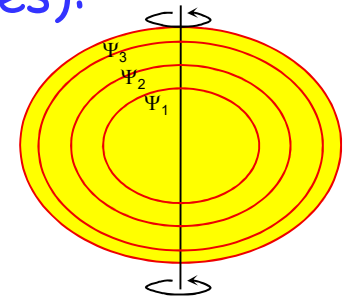
→ Isobars = Iso-density = Isotherms : $P(\Psi)$, $\rho(\Psi)$, $T(\Psi)$, ...

$$\Psi = \text{Total Potential} = \phi - \int_0^s \Omega^2(s) s ds$$

Else (more complex differential rotation of real cases):

Isobars $\neq$ Iso-density $\neq$ Isotherms

↪ Baroclinity



Sun, simplified model of constant density :

$$P_c = \frac{3}{8\pi} \frac{G M^2}{R^4}$$
$$\sim 5 \times 10^{14} \text{ Pa}$$

Out of equilibrium state:

$$dm \vec{\gamma} = \sum \vec{F} = 0 \quad \text{Radial acceleration} = -\frac{1}{\rho} \frac{dP}{dr} - \frac{G m}{r^2}$$

Dynamical time
~ free fall time
(pressure suppressed)

Order of magnitude:

$$\frac{R}{t_{\text{dyn}}^2} = \frac{GM}{R^2}$$



$$t_{\text{dyn}} = \sqrt{\frac{R^3}{GM}}$$

Sun : $t_{\text{dyn}} = 26 \text{ min} \ll \text{stellar evolution time}$

During the main part of the stellar life, hydrostatic equilibrium
Exception : Explosion of a supernova (dynamical instability)

State of the stellar matter

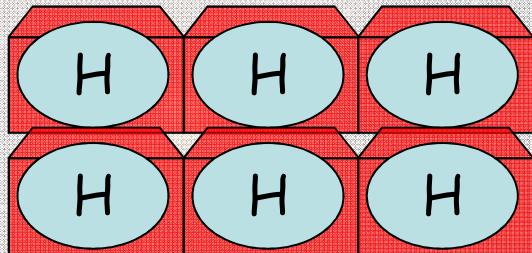
$$P_c = \frac{3}{8\pi} \frac{G M^2}{R^4}$$

$$\sim 5 \times 10^{14} \text{ Pa}$$

$$\langle \rho \rangle = \frac{3}{4\pi} \frac{M}{R^3}$$

$$\sim 2 \times 10^3 \text{ kg/m}^3$$

Hydrogen atoms (neutral)
in contact



$$d_0 \sim 10^{-10} \text{ m}$$



$$\rho = \frac{1.67 \times 10^{-27}}{(10^{-10})^3} \text{ kg/m}^3$$

But is the atoms repulsion
sufficient to support the
weight of the gas column ?

$$P_{\text{el}} \simeq \frac{e^2 / (4\pi\epsilon_0 d_0^2)}{d_0^2}$$

$$\approx 2 \times 10^{12} \text{ Pa}$$

The matter
is ionized

State of the stellar matter

The stellar matter is ionized: plasma

Mean distance
between particles $\sim 10^{-10} \text{ m} \gg 10^{-15} \text{ m}$ Size of nuclei

Equation of state of an ideal gaz: $P_g \simeq \frac{k \rho T}{\mu m_u}$

Estimation of the core temperature

$$T_c \simeq \frac{P_g \mu m_u}{k \rho} \simeq 10^7 \text{ K}$$

Sufficient temperature for the
thermonuclear fusion of hydrogen nuclei

$$T_c \simeq 10^7 \text{ K} \gg T_{\text{surf}} = 5800 \text{ K}$$

Transport of energy from the center to the surface

1) Radiation

Sources of opacity

In the stars: $kT \sim h\nu < \text{keV} \Rightarrow$ electronic transitions

Bound-bound transitions

Bound-free

Free-free

Diffusion

Mean free path of photons $\sim \text{cm}$
In stellar interiors!

Stellar interiors are completely opaque
and in thermodynamic equilibrium

Transport of energy (from the center to the surface)

1) Radiation

In stellar interiors:

Mean free path of photons :

$$\ell = (\kappa \rho)^{-1} \approx 2\text{cm}$$

$\ll$

$dr / d\ln T, dr/d\ln P, \dots$

Opaque interior

Matter-radiation
thermodynamic equilibrium

Planck law :

$$I_{\lambda} = \frac{2hc^2}{\lambda^5} \frac{1}{\exp(hc/\lambda kT) - 1}$$

Stefan law :

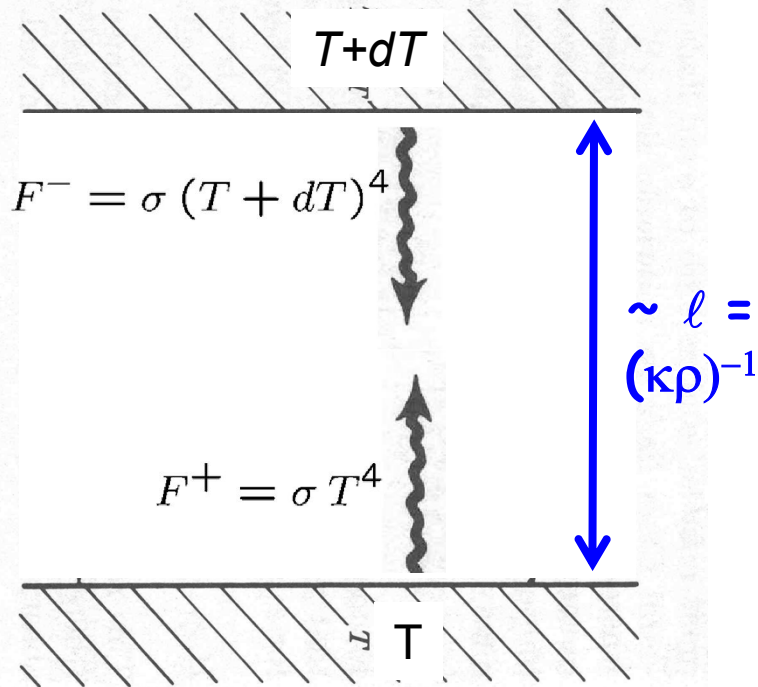
$$I = \sigma T^4 / \pi$$

$$T_c \simeq 10^7 \text{ K} \gg T_{\text{surf}} = 5800 \text{ K}$$

Transport of energy from the center to the surface

1) Radiation

Approximative proof



$$\begin{aligned} F &= F^+ - F^- \\ &= -\sigma((T + dT)^4 - T^4) \\ &= -\sigma \frac{dT^4}{dT} dT = -\frac{\sigma}{\kappa\rho} \frac{dT^4}{dT} \end{aligned}$$

But the mean free paths in different directions were not appropriately taken into account

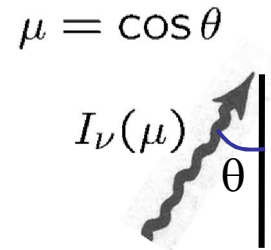
$$F = -\frac{4acT^3}{3\kappa\rho} \frac{\partial T}{\partial r}$$

1) Radiation

Rigorous proof

Solution of the radiative transfer equation:

$$I_\nu(\tau) = \int_\tau^\infty B_\nu(T(\tau')) e^{-\tau+\tau'} d\tau' \simeq B_\nu(T(\tau))$$



Radiative transfer equation

$$\frac{\mu}{\kappa_\nu \rho} \frac{dI_\nu}{dr} = B_\nu - I_\nu$$

* $2\pi\mu$ and integration

$$\begin{aligned} 2\pi \int_{-1}^1 \frac{\mu^2}{\kappa_\nu \rho} \frac{dI_\nu}{dr} d\mu &\simeq 2\pi \frac{1}{\kappa_\nu \rho} \frac{dB_\nu}{dr} \int_{-1}^1 \mu^2 d\mu \\ &= 2\pi \int_{-1}^1 \cancel{B_\nu} \mu d\mu - F_\nu \end{aligned}$$

2/3

0

$$F_\nu = -\frac{4\pi}{3\kappa_\nu \rho} \frac{dB_\nu}{dT} \frac{dT}{dr}$$

$$F = \int_0^\infty F_\nu d\nu = -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dr}$$

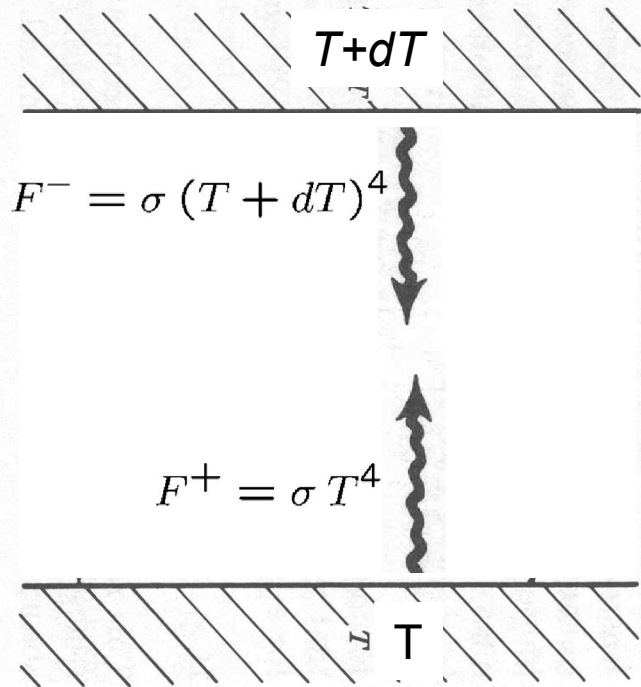
Rosseland mean

$$\rightarrow \frac{1}{\kappa} = \frac{\pi}{acT^3} \int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} d\nu$$

Transport of energy (from the center to the surface)

1) Radiation

$$\begin{aligned} L &= 4\pi r^2 F \\ &= -\frac{16\pi r^2 a c T^3}{3\kappa\rho} \frac{dT}{dr} \end{aligned}$$



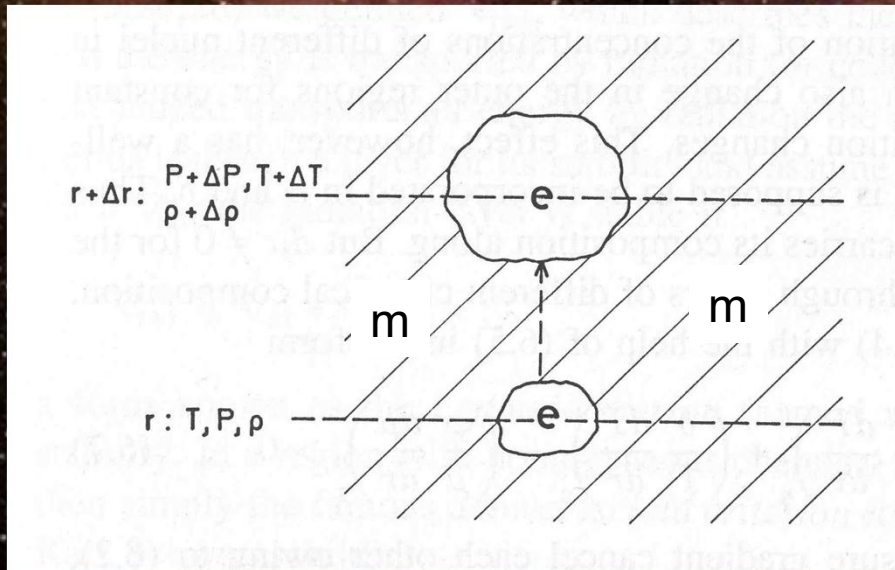
$T_{\text{center}} \gg T_{\text{surface}}$
Gradient of $T \rightarrow L$
Whatever the source of energy

For a fixed L
 $\kappa \nearrow \Rightarrow |dT/dr| \nearrow$
 $T_{\text{surf}} = 5800 \text{ K}$

Example: partial ionization zones

Transport of energy (from the center to the surface)

2) Convection



If $(\Delta \rho)_e < (\Delta \rho)_m$



Archimedes force
upwards



Convectively unstable
medium

Transport of energy (from the center to the surface)

2) Convection

Convectively unstable medium
if and only if :

$$\rho_e < \rho_m$$

Temperature criterion ?

Pressure equilibrium:

$$P_e = P_m$$

Ideal gas
 $P = R \rho T / \mu$



$$P_e = R \rho_e T_e / \mu_e = P_m = R \rho_m T_m / \mu_m$$



$$\rho_e < \rho_m$$

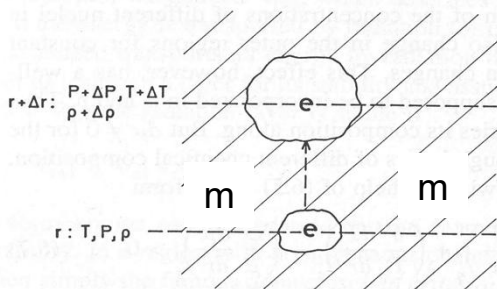


$$T_e / \mu_e > T_m / \mu_m$$

If $\mu_e = \mu_m$



$$T_e > T_m$$



Transport of energy (from the center to the surface)

2) Convection

Convectively unstable medium if and only if :

$$T_e > T_m$$



Transport of energy !

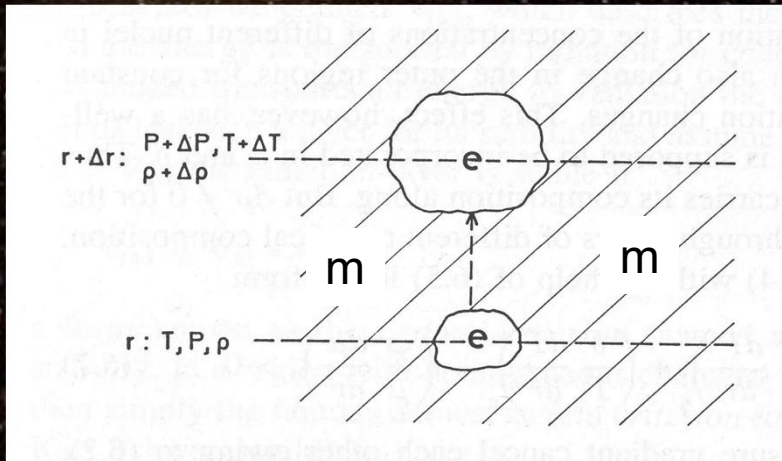
$$(\Delta T/T)_e > (\Delta T/T)_m$$

$$\Delta T < 0$$

$$|d\ln T/dr|_e < |d\ln T/dr|_m$$

$$\Delta P < 0$$

$$(d\ln T/d\ln P)_e < (d\ln T/d\ln P)_m$$



Transport of energy (from the center to the surface)

2) Convection

Convectively unstable medium if and only if :

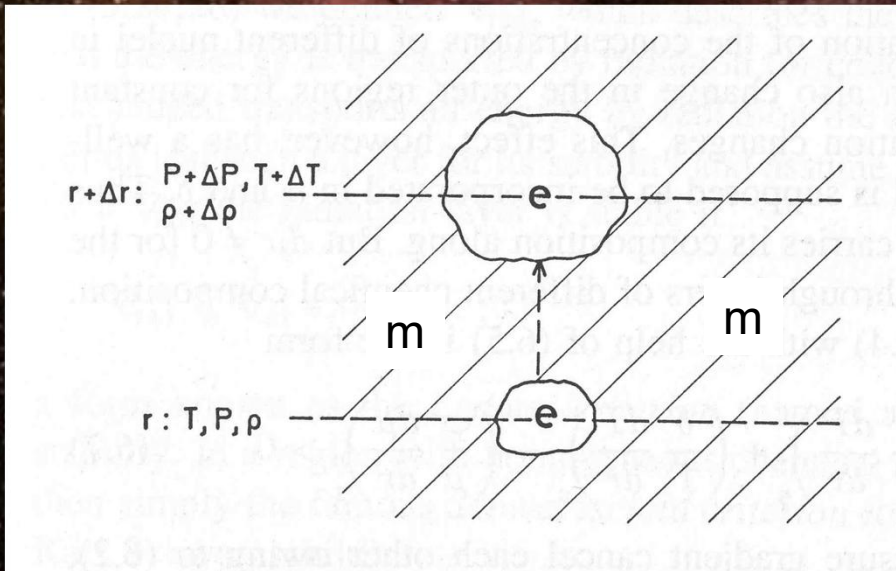
$$(d\ln T/d\ln P)_e < (d\ln T/d\ln P)_m$$

Efficient convection

→ Negligible radiative loss
 $(d\ln T/d\ln P)_e = \partial \ln T / \partial \ln P|_s \equiv \nabla_{ad}$

$$(d\ln T/d\ln P)_m \equiv \nabla > \nabla_{ad}$$

Convectively unstable medium if and only if :



Transport of energy (from the center to the surface)

Convection + radiation

$$\frac{dT}{dr} = - \frac{3 \kappa \rho L_R}{16 \pi a c r^2 T^3} \quad \div \quad T \, d \ln P / dr = - T \rho g / P$$

$$\frac{d \ln T}{d \ln P} = \frac{3 \kappa P L_R}{16 \pi a c G m T^4}$$

$$\nabla_{rad} \equiv \frac{3 \kappa P L}{16 \pi a c G m T^4}$$

∇_{rad} = radiative gradient = fictive gradient required to transport the whole energy by radiation

Schwarzschild criterion:
Convectively unstable medium



$$\nabla_{rad} > \nabla_{ad}$$

Transport of energy (from the center to the surface)

Schwarzschild criterion:
Convectively unstable medium



$$\nabla_{\text{rad}} > \nabla_{\text{ad}}$$

Convection typically
appears when :

$$\nabla_{\text{rad}} \equiv \frac{3 \kappa P L}{16 \pi a c G m T^4}$$



L/m
Very high

Nuclear reactions
only near the center

Convective core

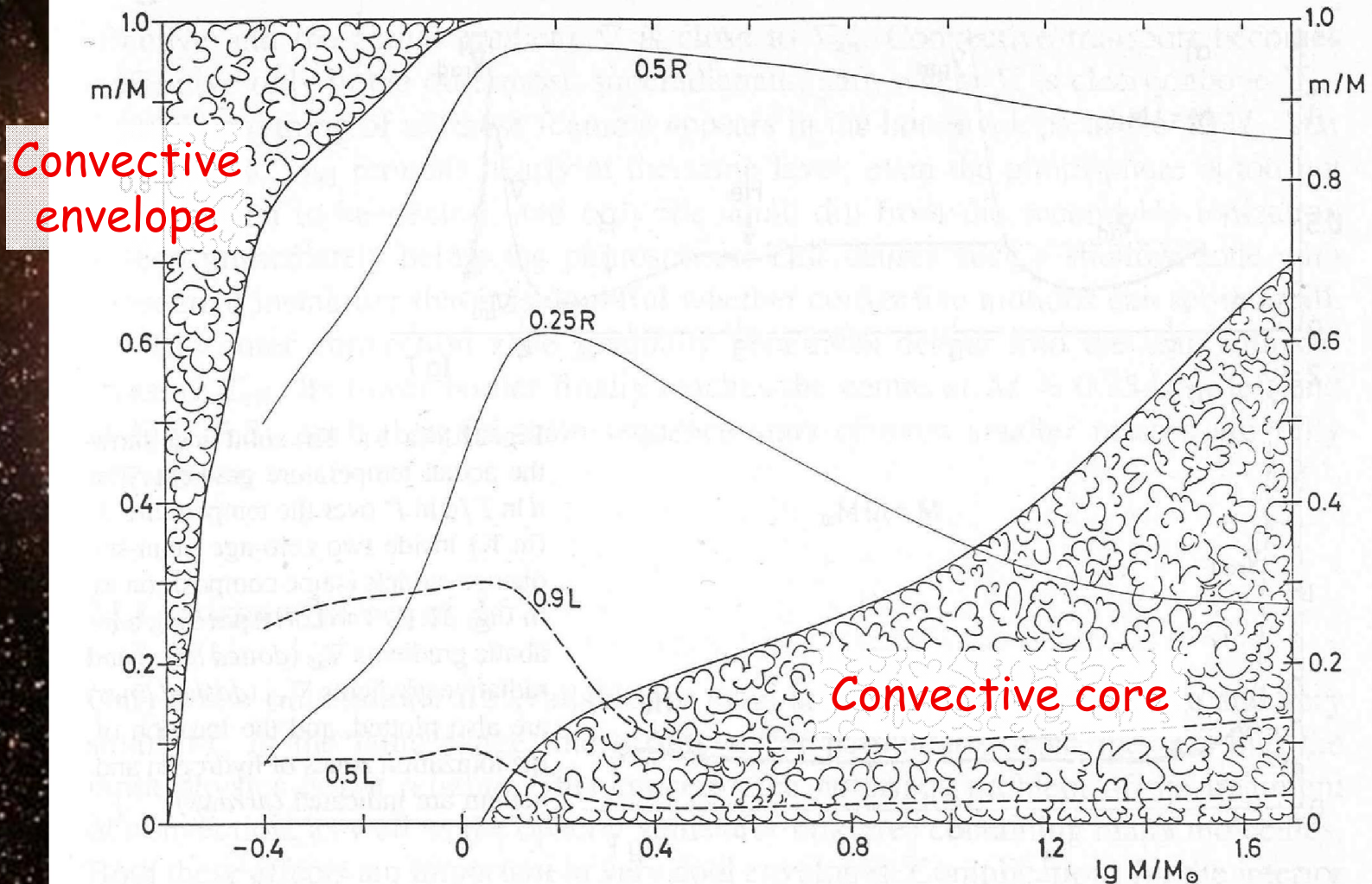


κ (opacity)
very high

Partial ionization zones
near the surface

Convective envelope

Transport of energy (from the center to the surface)



Transport of energy (from the center to the surface)

How efficient is this energy transport ?

$$F_c = \rho c_p \overline{V_{c,r}} \Delta T \quad (\text{Enthalpy flux})$$
$$\approx \rho c_p T (P/\rho)^{1/2} \alpha^2 (\nabla - \nabla_{\text{ad}})^{3/2}$$

Mixing-length theory
(very approximative !)

Very high heat capacity

A small $(\nabla - \nabla_{\text{ad}})$ is sufficient

$$\nabla \simeq \nabla_{\text{ad}} \simeq 2/5$$

In the deep layers where
convective transport
is very efficient

Fully ionized gas

Equation of energy conservation

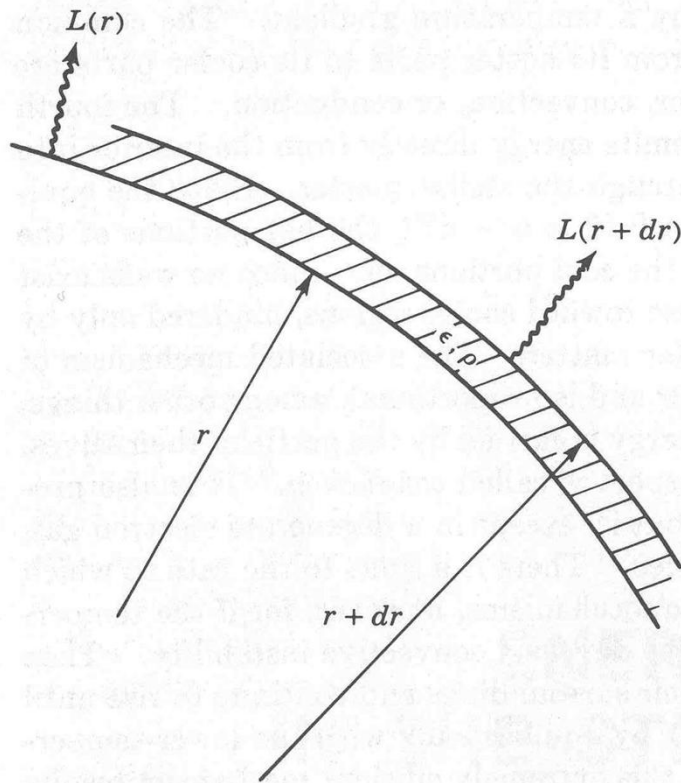
Energy balance :

Volume source of energy inside stars : **nuclear reactions**

On définit ϵ_n comme le taux de production d'énergie nucléaire par unité de masse et de temps

$\epsilon_n dm$ est la quantité d'énergie produite ds la coquille sphérique de masse dm par unité de temps

$$\epsilon_n = \epsilon_n(\rho, T, X_i)$$



Equation de conservation de l'énergie

Bilan d'énergie :

Quantité d'énergie échangée par la coquille sphérique avec l'extérieur / unité de temps :

$$L(r+dr) - L(r) = dL = \partial L / \partial r dr$$

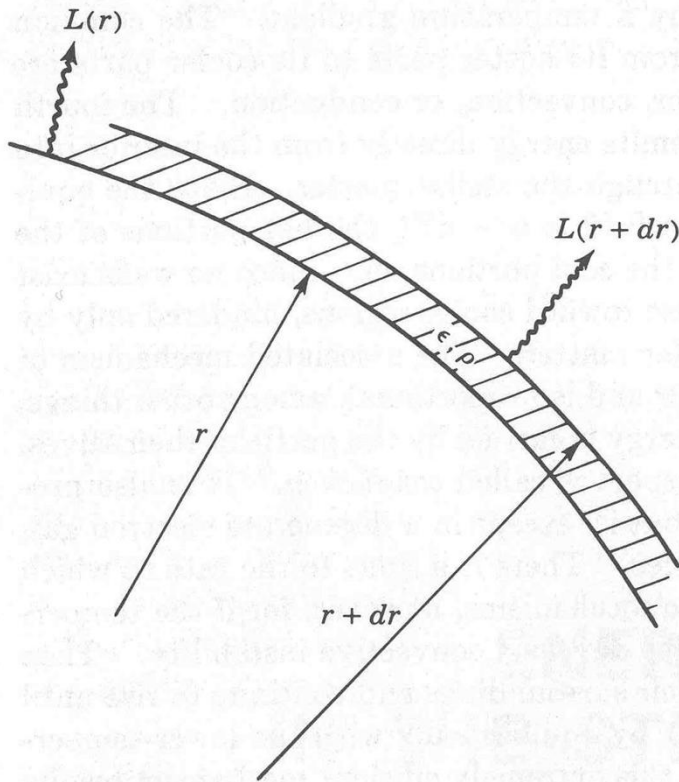
A l'équilibre thermique :

Σ énergie fournie =
 Σ énergie perdue



$$dL = \varepsilon_n dm \Rightarrow dL/dm = \varepsilon_n$$

$$\Rightarrow dL/dr = 4\pi r^2 \rho \varepsilon_n$$



Equation de conservation de l'énergie

→ Energie interne

Hors équilibre thermique :

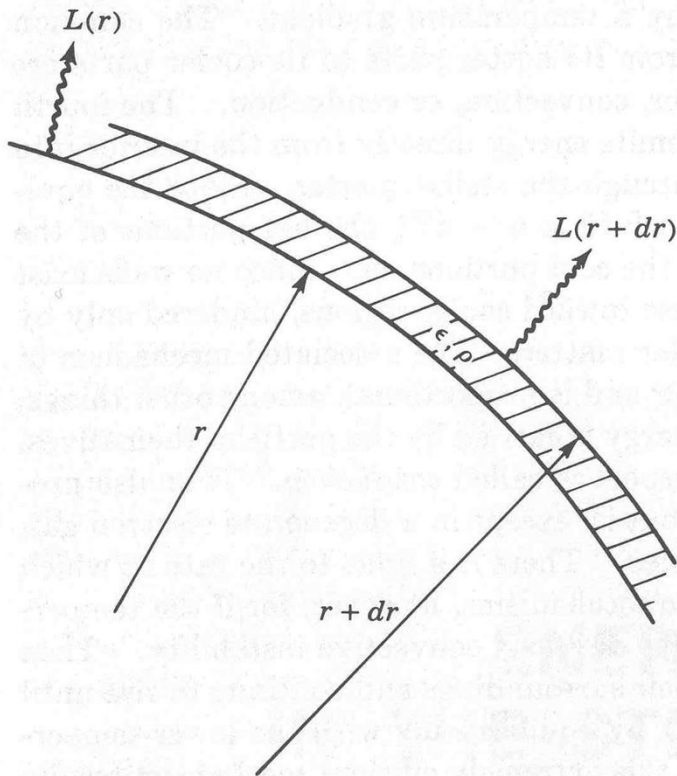
Puissance de dilatation

$$\begin{aligned}\varepsilon_n dm - dL &= \frac{dQ}{dt} dm \\ &= \left(\frac{du}{dt} + P \frac{dv}{dt} \right) dm = T \frac{ds}{dt} dm\end{aligned}$$

$$\frac{dL}{dm} = \varepsilon_n - T \frac{ds}{dt}$$

Apport d'énergie venant de la contraction gravifique (voir théorème du Viriel)

$\varepsilon_{\text{grav}}$



La Rotation

La baroclinité:

$$r \sin\theta \vec{e}_s = (1/2) \nabla(r^2 \sin^2\theta)$$

Rotationnel de l'équation d'équilibre $\nabla P/\rho = -\nabla\phi + \Omega^2 r \sin\theta \vec{e}_s$

$$\nabla \times (\nabla P/\rho) = -(1/\rho^2) \nabla \rho \times \nabla P = (1/2) \nabla(\Omega^2) \times \nabla(r^2 \sin^2\theta)$$

↪ Pour une rotation solide ou cylindrique ($\Omega(r \sin\theta)$): $\nabla \rho \times \nabla P = 0$

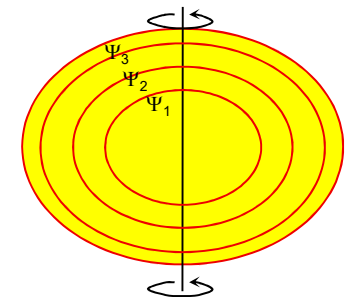
→ Isobares = Iso-densité = Isothermes : $P(\Psi)$, $\rho(\Psi)$, $T(\Psi)$, ...

$$\Psi = \text{Potentiel total} = \phi - \frac{1}{2} \Omega^2 r^2 \sin^2 \theta, \text{ (Rot. solide)}$$

Sinon (rotation différentielle plus complexe):

Isobares $\neq$ Iso-densité $\neq$ Isothermes

↪ Baroclinité



Processus non-standards: La Rotation

Le paradoxe de Von Zeipel:

Considérons la zone radiative d'une étoile en rotation solide

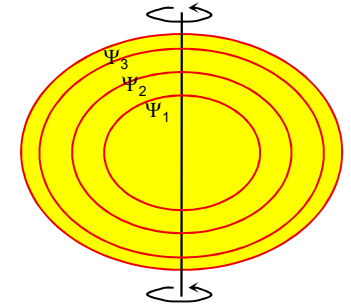
→ $P(\Psi)$, $\rho(\Psi)$, $T(\Psi)$, ...

$$\vec{F}_{\text{rad}} = -\chi \vec{\nabla} T = -\left\{ \chi(\Psi) \frac{dT}{d\Psi} \right\} \vec{\nabla} \Psi,$$

$$\vec{\nabla} \cdot \vec{F}_{\text{rad}} = -\chi \frac{dT}{d\Psi} \nabla^2 \Psi - \vec{\nabla} \left(\chi \frac{dT}{d\Psi} \right) \cdot \vec{\nabla} \Psi,$$

$$= -\chi \frac{dT}{d\Psi} [4\pi G \rho - 2\Omega^2] - \frac{d}{d\Psi} \left(\chi \frac{dT}{d\Psi} \right) |\vec{\nabla} \Psi|^2,$$

Non-constant sur
les équipotentielles



Quel que soit $T(\Psi)$, il est impossible d'avoir $\nabla \cdot (\vec{F}_{\text{rad}}) = 0$ partout !

→ Déséquilibre thermique :

$$\rho T \left[\partial_t S + \vec{V} \cdot \vec{\nabla} S \right] = -\vec{\nabla} \cdot \vec{F}_{\text{rad}} + \rho \epsilon \neq 0,$$

Cas stationnaire : $\vec{V} \cdot \nabla S \neq 0$

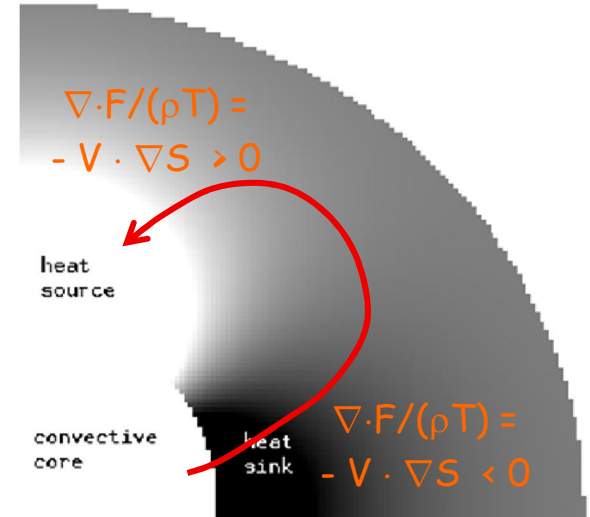
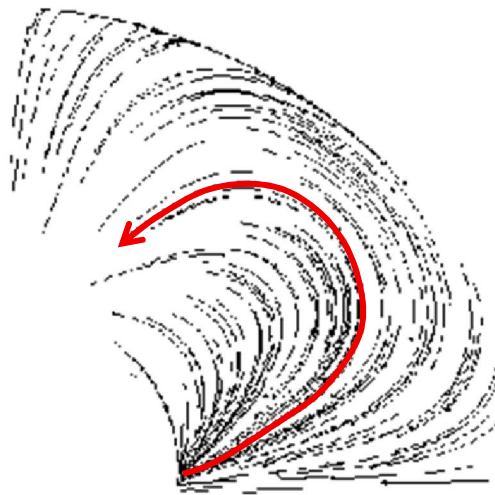
→ Circulation méridienne

Processus non-standards: La Rotation

Circulation méridienne

Mécanisme de transport à grande échelle:

- 1) Transport du moment cinétique
- 2) Transport des éléments chimiques



Temps caractéristique associé à son établissement:

temps d'Eddington-Sweet:
$$t_{ES} = \frac{R}{U} \approx \frac{GM^2}{RL} \frac{g}{\Omega^2 R}$$

Ensuite, la circulation s'adapte pour assurer le transport du moment angulaire requis par les conditions aux limites (vents, accrétion, effet de marée dans les binaires proches, ...)

Equations de structure

Conservation
de la masse

$$\frac{dm}{dr} = 4\pi \rho r^2$$

Equilibre
hydrostatique

$$\frac{dP}{dr} = -\frac{G\rho m}{r^2}$$

4 équations

7 inconnues ($m, \rho, P, T, L, \kappa, \varepsilon$)

$$\rho = \rho(T, P, X_i), \quad \kappa = \kappa(T, P, X_i), \\ \varepsilon = \varepsilon(T, P, X_i)$$

X_i : composition chimique

Transfert
d'énergie

Radiation

Convection

$$\frac{dT}{dr} = -\frac{3\kappa\rho L}{16\pi ac r^2 T^3}$$

$$\frac{dT}{dr} = -\nabla_{\text{ad}} \frac{Gm\rho}{r^2} \frac{T}{P}$$

Conservation
de l'énergie

$$\frac{dL}{dr} = 4\pi \rho r^2 (\varepsilon_n + \varepsilon_{\text{grav}})$$

Equations de structure

Conservation
de la masse

$$\frac{dr}{dm} = \frac{1}{4\pi \rho r^2}$$

Equilibre
hydrostatique

$$\frac{dP}{dm} = -\frac{G m}{4\pi r^4}$$

4 équations

7 inconnues ($r, \rho, P, T, L, \kappa, \varepsilon$)

$$\rho = \rho(T, P, X_i), \quad \kappa = \kappa(T, P, X_i), \\ \varepsilon = \varepsilon(T, P, X_i)$$

X_i : composition chimique

Transfert
d'énergie

Radiation

Convection
(efficace)

$$\frac{dT}{dm} = -\frac{3 \kappa L}{64 \pi^2 a c r^4 T^3}$$

$$\frac{dT}{dm} = -\nabla_{\text{ad}} \frac{Gm}{4\pi r^4} \frac{T}{P}$$

Conservation
de l'énergie

$$\frac{dL}{dm} = \varepsilon_n + \varepsilon_{\text{grav}}$$

Conditions aux limites

4 équations différentielles

$\Rightarrow$

4 conditions aux limites

Au centre

Masse : $m(0) = 0$ ou $r(0) = 0$
Luminosité : $L(0) = 0$

A la photosphère

Raccord avec un modèle d'atmosphère

Température : $T = T_{\text{eff}} = (L/4\pi R^2\sigma)^{1/4}$
Pression : $P = P(T_{\text{eff}}, \log g, X_i) = f(L, R, M, X_i)$
 ≈ 0

Equations d'état et Composition chimique

Les étoiles sont composées pour l'essentiel d'

- **Hydrogène** (fraction en masse initiale $X \simeq 0.7$)
- **Helium** (fraction en masse initiale $Y \simeq 0.28$)
- Éléments lourds (« métaux »: Fe, C, N, O, ...)
fraction en masse totale $Z \simeq 0.015$ pour le soleil)

Le gaz est presque complètement ionisé à l'intérieur mais les interactions coulombiennes entre ions et électrons affectent peu les trajectoires des particules:

$$(3/2) kT = E_{\text{cin}} > e^2/d \text{ (énergie potentielle coulombienne)}$$

→ **Gaz parfait**

Equation of State

Gaz parfait

$$P_{\text{gaz}} = n k T = \frac{k \rho T}{\mu m_u}$$

Poids moléculaire
moyen (sans dim.)

Mélange de plusieurs
éléments complètement
ionisés

$$P_{\text{gaz}} = P_e + \sum_i P_i$$

$$n = n_e + \sum_i n_i = \sum_i (1+Z_i) n_i$$

$$n_i = \rho_i / (A_i m_u) = \rho X_i / (A_i m_u)$$

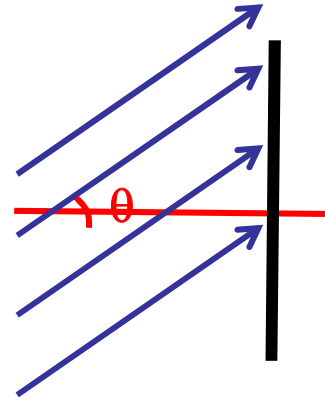
$$\mu = \left(\sum_i X_i (1+Z_i) / A_i \right)^{-1}$$

L'intégrale de pression

Pression = flux d'impulsion normale à une surface

Particules de mêmes direction :

$$P = n p v \cos^2 \theta$$



Gaz isotrope :

Fraction de particules faisant un angle entre θ et $\theta+d\theta$: $\sin \theta d\theta / 2$

Pression des particules d'angles $\in [\theta, \theta+d\theta]$ et impulsions $\in [p, p+dp]$:

$$dP(\theta, p) = n(p) p v \cos^2 \theta \sin \theta / 2 dp d\theta$$

Pression totale:

$$P = \int_0^\infty n(p) p v dp \int_0^\pi \cos^2 \theta \sin \theta d\theta / 2 = \frac{1}{3} \int_0^\infty v p n(p) dp$$

Equation of State

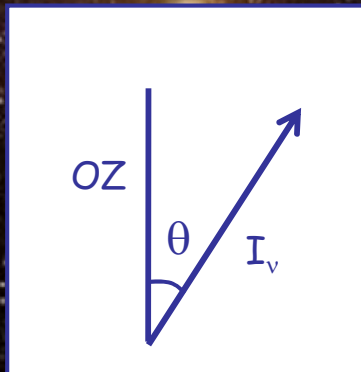
Pression de radiation

Pression = flux d'impulsion

Impulsion d'un photon
projetée sur OZ :

$$p_z(\nu) = \cos \theta \, h\nu/c$$

Nbre de phot.
/m³/4π rad



$$\begin{aligned}
 P_R &= 2\pi \int_0^\infty \int_0^\pi p_z(\nu) \frac{n(\nu)}{4\pi} c \cos \theta \sin \theta d\theta d\nu \\
 &= \frac{1}{2} \int_0^\infty h\nu n(\nu) d\nu \int_0^\pi \cos^2 \theta \sin \theta d\theta \\
 &= \frac{1}{3} \int_0^\infty h\nu n(\nu) d\nu = \frac{1}{3} \int_0^\infty E_\nu d\nu \\
 &= \frac{1}{3} a T^4
 \end{aligned}$$

Cas général non
isotrope

Tenseur

Corps noir

Densité
d'énergie

Equation of State

Pression totale

$$P = \frac{k\rho T}{\mu m_u} + \frac{1}{3}aT^4$$

Dans les étoiles « chaudes »,
 $P_R = (1/3) aT^4$ peut devenir significatif
comparativement à P_{gaz} .

Equation of State

Pression totale

$$P = \frac{k\rho T}{\mu m_u} + \frac{1}{3}aT^4$$

Mais, gaz de fermions

Principe d'exclusion de Pauli

On ne peut avoir plus de 2 fermions identiques dans le volume de phase : $dp_x dp_y dp_z dx dy dz = h^3$

Equation of State

Quantum degeneracy of the electron gas

Distribution de Boltzmann

Nbre d'électrons dans dV , impulsion entre p et $p+dp$

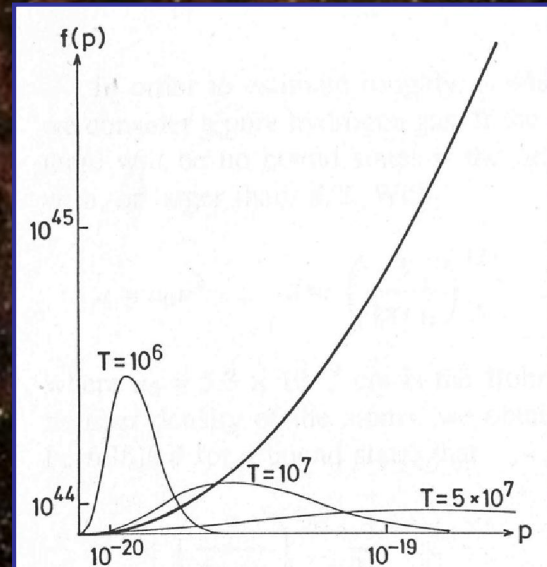
$$\frac{f(p)dpdV}{4\pi p^2 dpdV} \leq \frac{2}{h^3}$$

$$f(p)dpdV = n_e \frac{4\pi p^2}{(2\pi m_e kT)^{3/2}} \exp\left(-\frac{p^2}{2m_e kT}\right) dp dV$$

Volume occupé dans l'espace des phases:

$$4\pi p^2 dp dV$$

$$f(p) \leq 8\pi p^2 / h^3$$



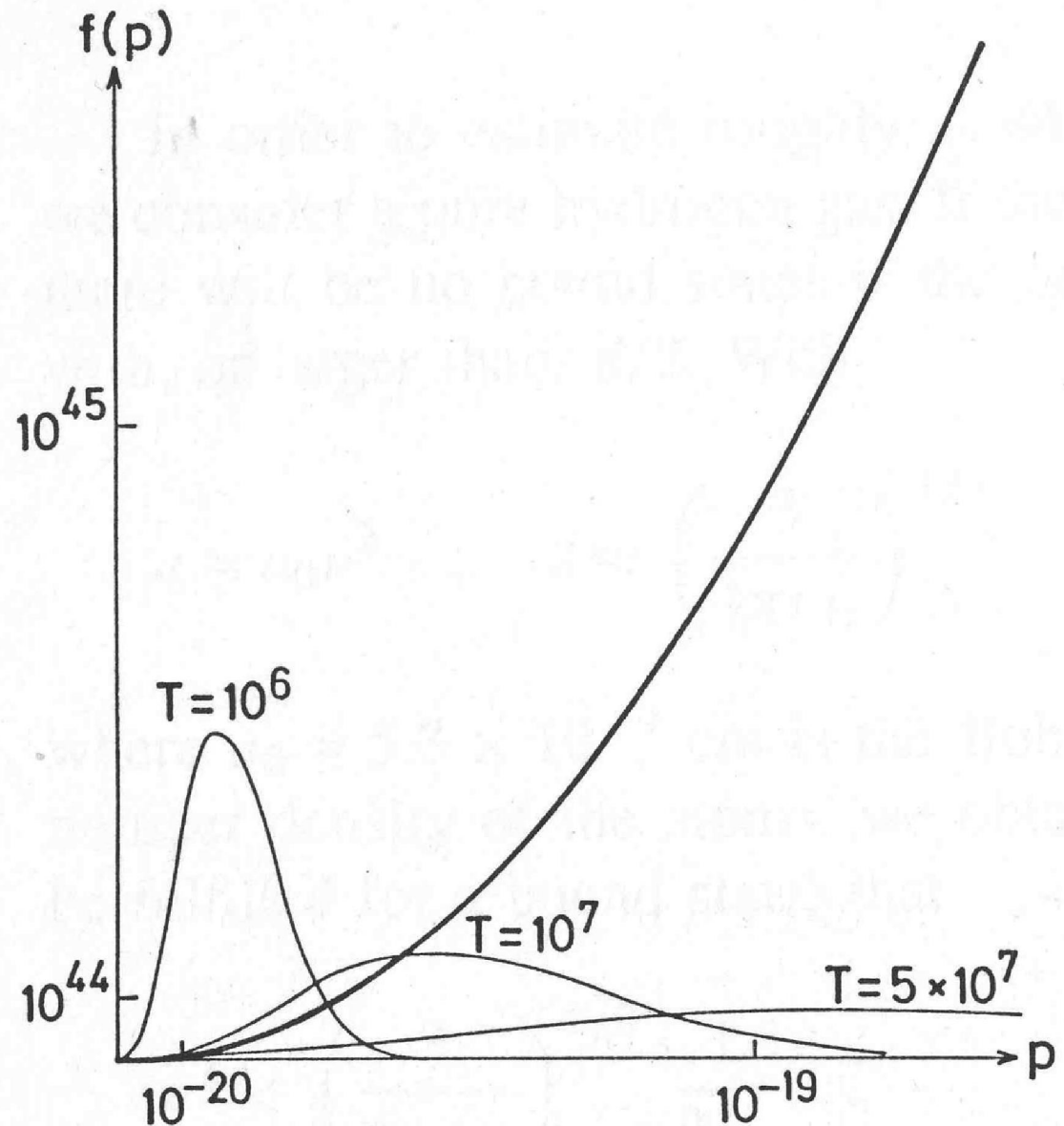
Equation

Quantum degene

Distribution de Boltzmann

Nbre d'électrons dans dV , impulsion entre p et $p+dp$

$$\frac{f(p)dpdV}{4\pi p^2 dpdV} \leq \frac{2}{h^3}$$



Equation of State

Quantum degeneracy of the electron gas

Distribution de Boltzmann

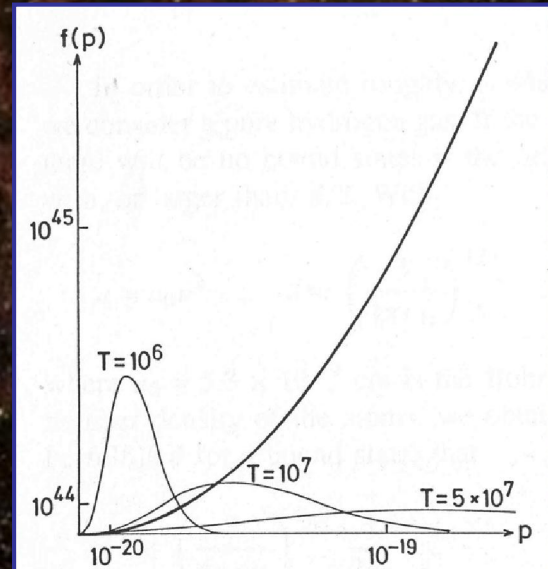
$$f(p)dpdV = n_e \frac{4\pi p^2}{(2\pi m_e kT)^{3/2}} \exp\left(-\frac{p^2}{2m_e kT}\right) dp dV \quad .$$

$$\frac{f(p)dpdV}{4\pi p^2 dpdV} \leq \frac{2}{h^3}$$

$$f(p) \leq 8\pi p^2 / h^3$$

La dégénérescence apparaît quand $\rho \nearrow \nearrow$ ou $T \searrow \searrow$

→ Statistique de Fermi-Dirac



Equation of State

Quantum degeneracy of the electron gas

Critère de dégénérescence

Volume spatial

$$V_S = \frac{1}{n_e} = \frac{\mu_e m_u}{\rho}$$

Volume quantique
(principe d'incertitude)

$$V_Q \simeq \lambda^3 = \frac{h^3}{(m_e v)^3}$$

Pas de dégénérescence
ssi :

$$V_Q \ll V_S$$

$$\frac{h^3}{(m_e v)^3} \ll \frac{\mu_e m_u}{\rho}$$

$$m v^2 = 3 k T$$

$$\frac{\rho T^{-3/2}}{\mu_e} \ll \frac{m_u (3 m_e k)^{3/2}}{h^3} = 2.385 \times 10^{-5} \quad (\text{MKSA})$$

La dégénérescence
apparaît d'abord pour
les électrons (m_e)

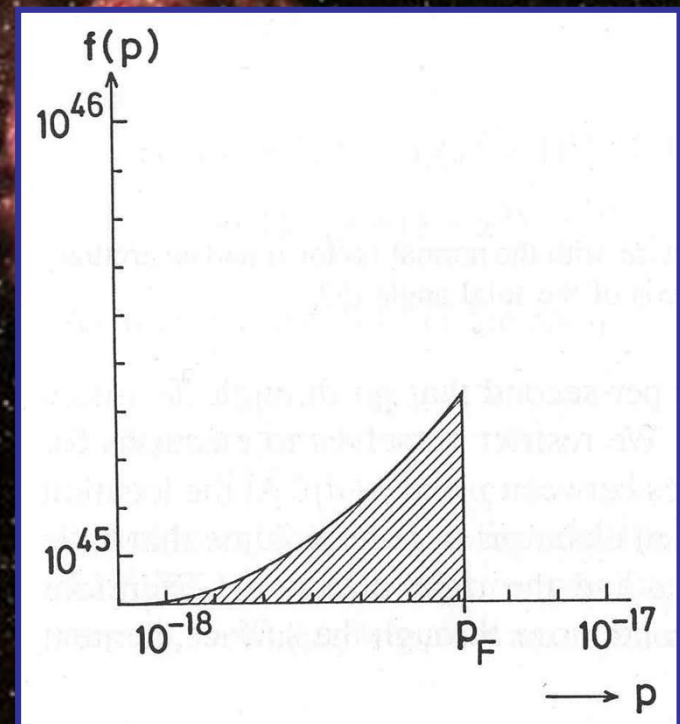
Equation of State

Quantum degeneracy of the electron gas

A simple example:
completely degenerated gas
($T \rightarrow 0$ limit)

$$\begin{aligned} f(p) &= \frac{8\pi p^2}{h^3} & \text{si } p \leq p_F \\ f(p) &= 0 & \text{sinon} \end{aligned}$$

Fermi momentum



Equation of State

Quantum degeneracy

Mean molecular weight

$$n_e = \int_0^{p_F} \frac{8\pi p^2}{h^3} dp = \frac{8\pi p_F^3}{3h^3}$$

$$\rho = m_u \mu_e n_e = \frac{8\pi m_u \mu_e p_F^3}{3h^3}$$

$$\begin{aligned} P_e &= 2\pi \int_0^{p_f} \int_0^\pi p_z \frac{f(p)}{4\pi} v_z \sin \theta d\theta dp \\ &= 2\pi \int_0^{p_f} \int_0^\pi (p \cos \theta) \left(\frac{8\pi p^2}{4\pi h^3} \right) \left(\frac{p \cos \theta}{m_e} \right) \sin \theta d\theta dp \\ &= \frac{8\pi}{15h^3} \frac{p_F^5}{m_e} \end{aligned}$$

$$P_e = \frac{8\pi}{15h^3 m_e} \left(\frac{3h^3}{8\pi m_u} \right)^{5/3} \left(\frac{\rho}{\mu_e} \right)^{5/3}$$

Non-relativistic case

Equation of State

Quantum degeneracy

$$n_e = \int_0^{p_F} \frac{8\pi p^2}{h^3} dp = \frac{8\pi p_F^3}{3h^3} \quad \rho = m_u \mu_e n_e = \frac{8\pi m_u \mu_e p_F^3}{3h^3}$$

$$\begin{aligned} P_e &= 2\pi \int_0^{p_F} \int_0^\pi p_z \frac{f(p)}{4\pi} v_z \sin \theta d\theta dp \\ &= 2\pi \int_0^{p_F} \int_0^\pi (p \cos \theta) \left(\frac{8\pi p^2}{4\pi h^3} \right) \left(\frac{p \cos \theta}{m_e} \right) \sin \theta d\theta dp \\ &= \frac{2\pi c p_F^4}{3h^3} \end{aligned}$$

~~$c \cos \theta$~~

$$P_e = \frac{2\pi c}{3h^3} \left(\frac{3h^3}{8\pi m_u} \right)^{4/3} \left(\frac{\rho}{\mu_e} \right)^{4/3}$$

Extremely
relativistic case

Equation of State

Quantum degeneracy

$$n_e = \int_0^{p_F} \frac{8\pi p^2}{h^3} dp = \frac{8\pi p_F^3}{3h^3} \quad \rho = m_u \mu_e n_e = \frac{8\pi m_u \mu_e p_F^3}{3h^3}$$

$$P_e = 2\pi \int_0^{p_f} \int_0^\pi p_z \frac{f(p)}{4\pi} v_z \sin \theta d\theta dp$$

Partly relativistic
case


$$v_z = v \cos \theta = \frac{p \cos \theta}{\sqrt{m_e^2 + p^2/c^2}}$$

Equation of State

Calculus of the electronic density
and the mean molecular weight per electron

$$\begin{aligned} n_e &= \sum_i Z_i n_i \\ &= \frac{\rho}{m_u} \sum_i Z_i \frac{X_i}{A_i} \end{aligned}$$

$$n_i = \rho_i / (A_i m_u) = \rho X_i / (A_i m_u)$$

Hydrogen (X), Helium (Y), metals (for the metals, $Z_i/A_i \simeq \frac{1}{2}$)

$$n_e = \frac{\rho}{m_u} (X + Y/2 + (1 - X - Y)/2) = \frac{\rho}{m_u} \frac{1 + X}{2}$$

$$\mu_e = \frac{\rho}{n_e m_u} \simeq \frac{2}{1 + X}$$

$$\mu_e = (\sum_i X_i Z_i / A_i)^{-1}$$

Equation of State

Quantum degeneracy of the electron gas

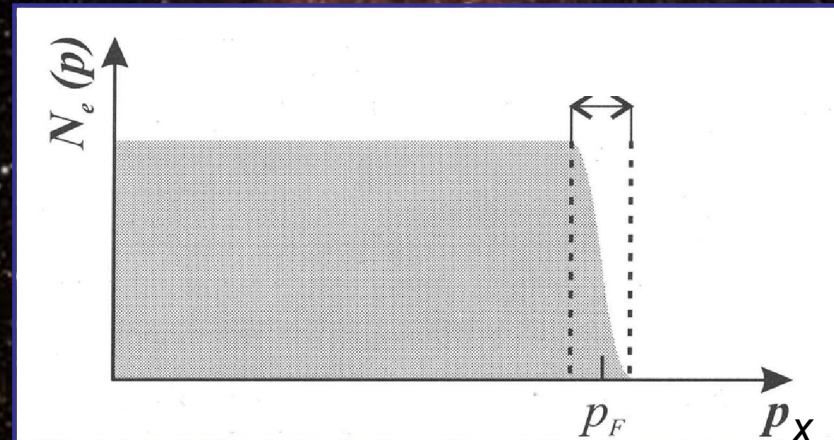
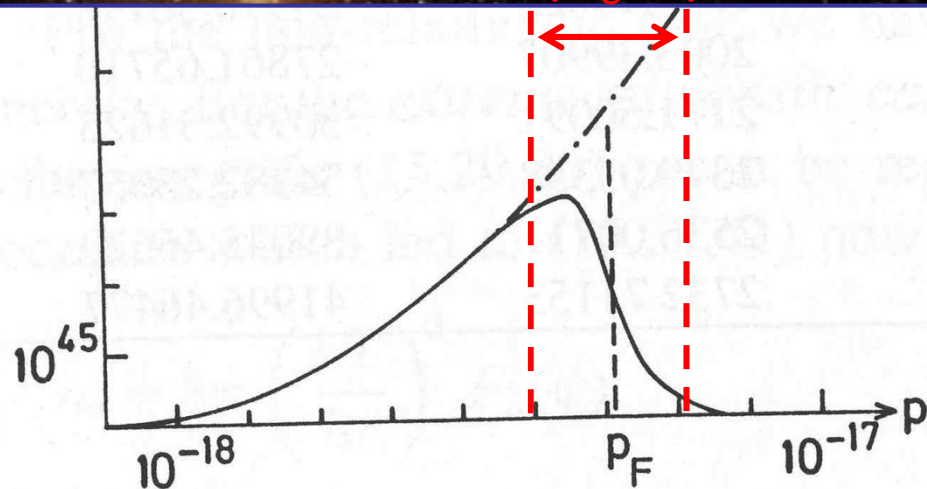
Fermi-Dirac distribution

$$f(p)dpdV = \frac{8\pi p^2 dpdV}{h^3} \frac{1}{1 + e^{E/kT - \psi}}$$

Pressure and density are quasi insensitive to the temperature

$$\Rightarrow \partial \ln P / \partial \ln T|_p \simeq 0$$

$(m_e kT)^{1/2}$



Equation of State

Radiation pressure:

$$P_R = \frac{1}{3} a T^4 \geq P_g = \frac{k \rho T}{\mu m_u}$$

$$\log T \geq \frac{1}{3} \log \rho + \text{cst.}$$

Degeneracy:

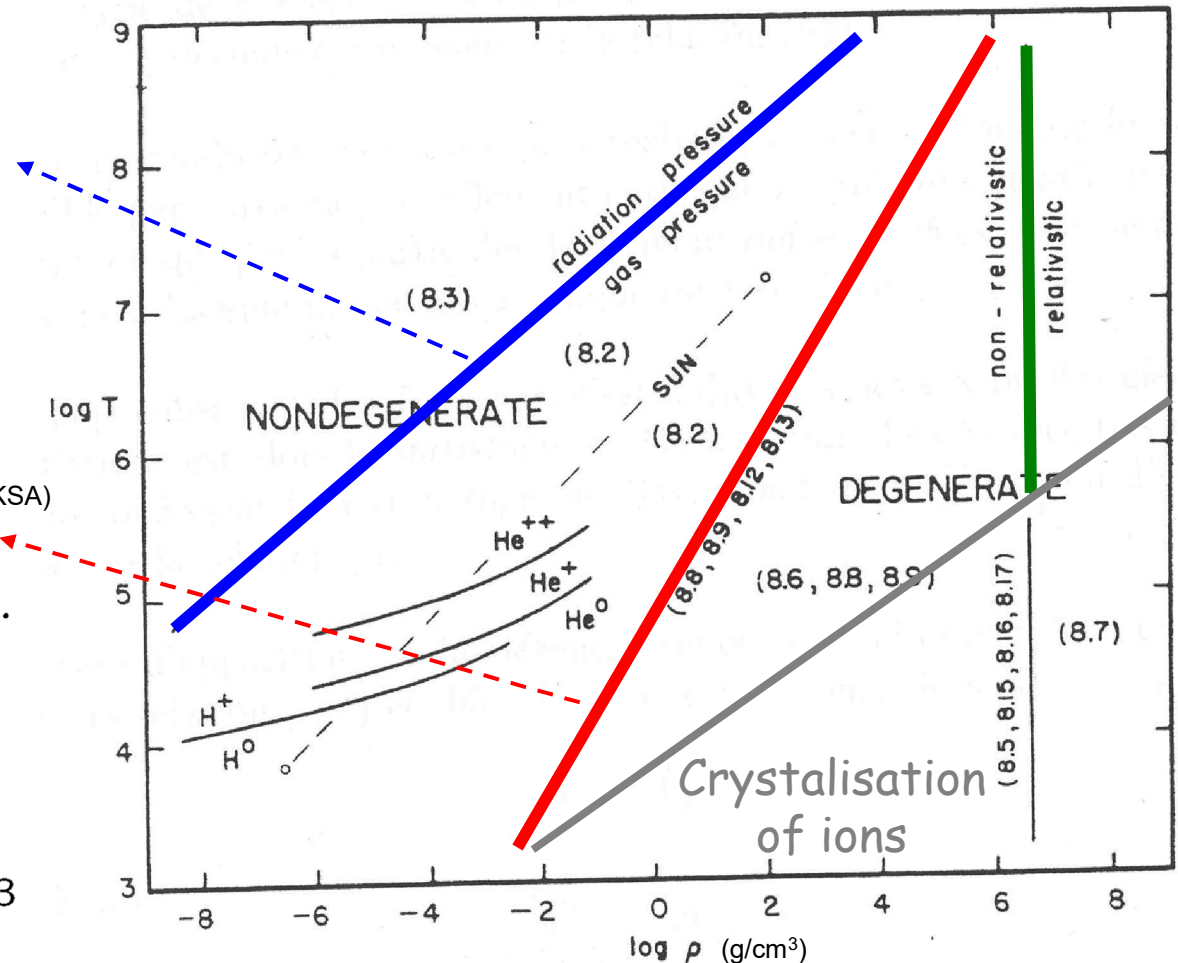
$$\frac{\rho T^{-3/2}}{\mu_e} \geq 2.385 \times 10^{-5} \text{ (MKSA)}$$

$$\log T \leq \frac{2}{3} \log \rho + \text{cst.}$$

Relativistic degeneracy:

$$\rho \geq \frac{8\pi m_u \mu_e (m_e c)^3}{3h^3}$$

$$= 9.74 \times 10^8 \mu_e \text{ kg/m}^3$$



Equation of State

Non-degenerated :

$$P = \frac{k\rho T}{\mu m_u} + \frac{1}{3}aT^4$$

Degenerated :

$$\frac{\rho T^{-3/2}}{\mu_e} \geq 2.385 \times 10^{-5}$$


$$P_e \approx K \left(\frac{\rho}{\mu_e} \right)^{5/3}$$

Non-relativistic degeneracy


$$\rho > 1000 \text{ kg/cm}^3$$

$$P_e \approx K^* \left(\frac{\rho}{\mu_e} \right)^{4/3}$$

Relativistic degeneracy

Opacity

$$\frac{dT}{dr} = - \frac{3 \kappa \rho L}{16 \pi a c r^2 T^3}$$

Mean Rosseland opacity

$$\frac{1}{\kappa} = \frac{\pi}{a c T^3} \int_0^\infty \frac{1}{\kappa_\nu} \frac{dB_\nu}{dT} d\nu$$

$$\kappa_\nu = f(\nu, T, \rho, X_i)$$

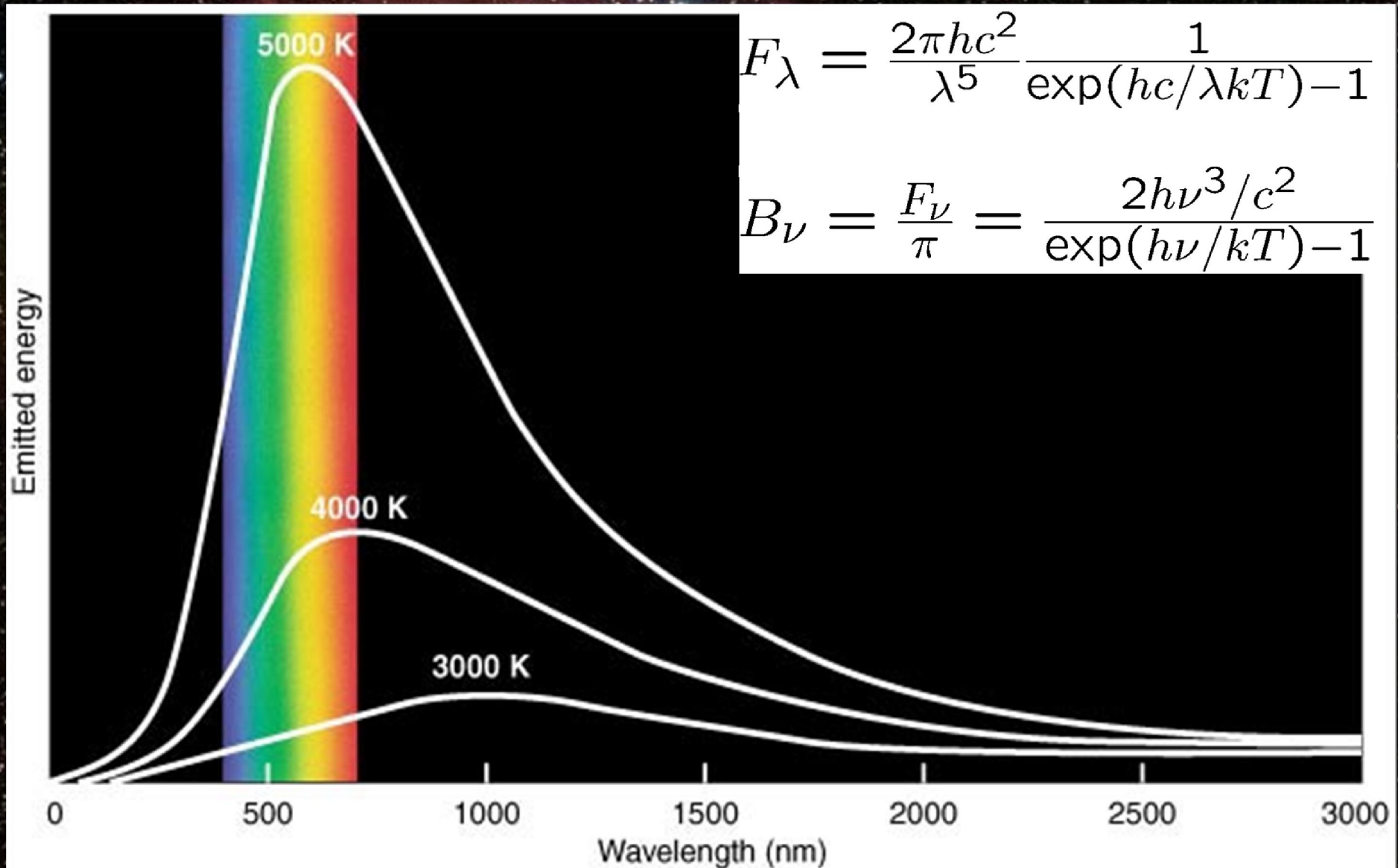
$$\kappa = \kappa(T, \rho, X_i)$$

Depends on the frequency

Parts of the spectrum which have the highest weight:

- Maximum of the Planck distribution : $\nu \approx kT/h$
- Less opaque regions (harmonic mean)

Planck function (Black body)

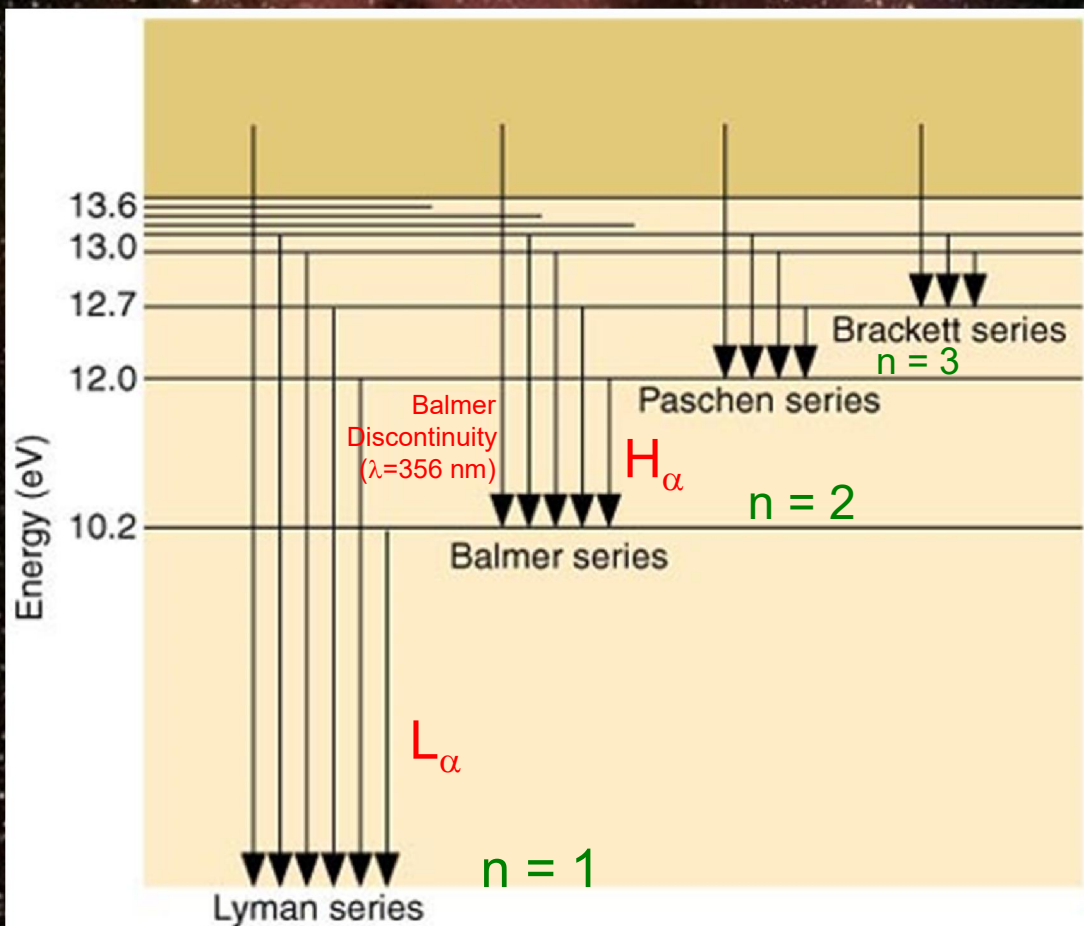


Electronic levels of Hydrogen

$$E_n \simeq -1/n^2 (m_e e^4 / 2\hbar^2) = -(1/n^2) h c R_\infty$$



Rydberg constant



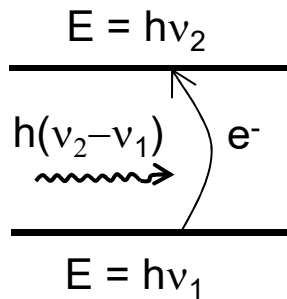
$$\lambda^{-1} = R_\infty (1/k^2 - 1/n^2)$$

k = 1 : Lyman series (L_α, L_β, ...)

k = 2 : Balmer series (H_α, H_β, ...)

Opacity

Bound-bound transitions



Absorption if
 $\nu \approx (E_2 - E_1)/h$

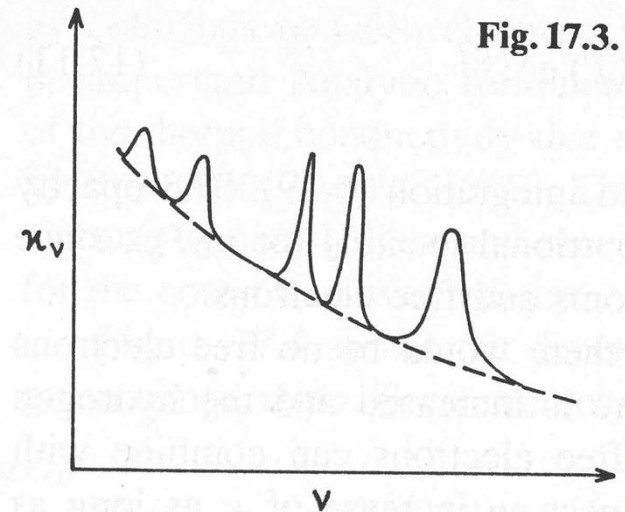


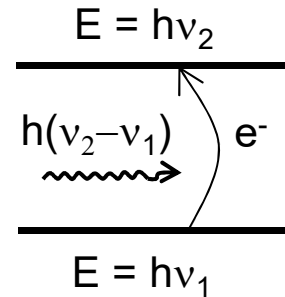
Fig. 17.3.

Sources of line broadening:

- **Natural** : $\sigma(\nu) \propto \frac{1}{(\nu - \nu_0)^2 + 1/(2\pi\tau)^2}$ $\longrightarrow$ Life-time of the state
- **Thermal** : Doppler effect due to the thermal agitation $\delta\nu/\nu \propto \sqrt{T}$
- **Collisions** (Stark effect) : Perturbation of the potential due to nearby ions and free electrons

Opacity

Bound-bound transitions



Calculation of the opacity :

Cross section : $1/\ell_\nu = \kappa_\nu \rho = \sum_{i,j} \overset{\text{Nbr} / \text{m}^3}{n_i} \overset{\text{Section efficace}}{\sigma_{\nu i,j}}$

- Cross section determination for all transitions ! ...
- Determination of the occupation of all electronic states

Contribution of the bound-bound transitions to the total opacity :

$T < 10^6 \text{ K} \longrightarrow$ up to 50%
 $T \approx 10^7 \text{ K} \longrightarrow$ less than 10%

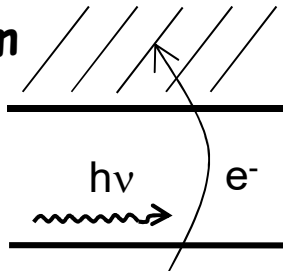
Opacity

Bound-free transitions

Continuum

$$E = h\nu_1 + \chi$$

$$E = h\nu_1$$



Absorption $\forall \nu \geq \chi/h, E_{\text{cin}}(e^-) = h\nu - \chi$

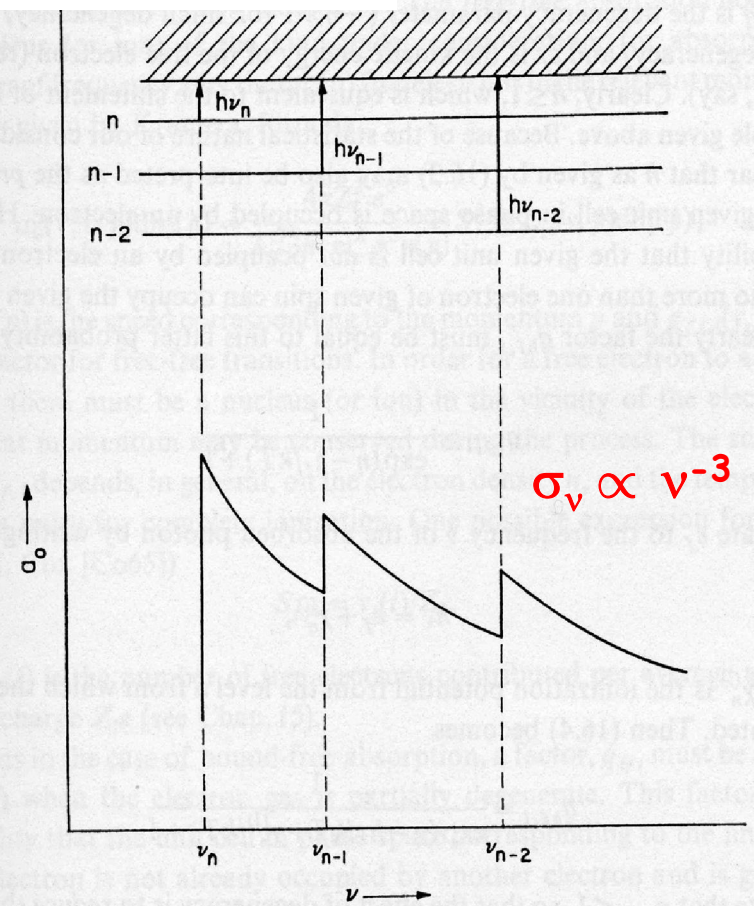
For a given transition :

$$\sigma_{\nu,i} \propto \nu^{-3}$$

For each new transition ($\nu \geq \chi_i/h$)

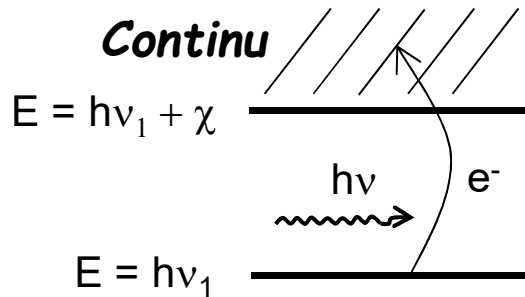
→ discontinuity

Section efficace



Opacity

Bound-free transitions



Absorption $\forall \nu \geq \chi/h, E_{\text{cin}}(e^-) = h\nu - \chi$

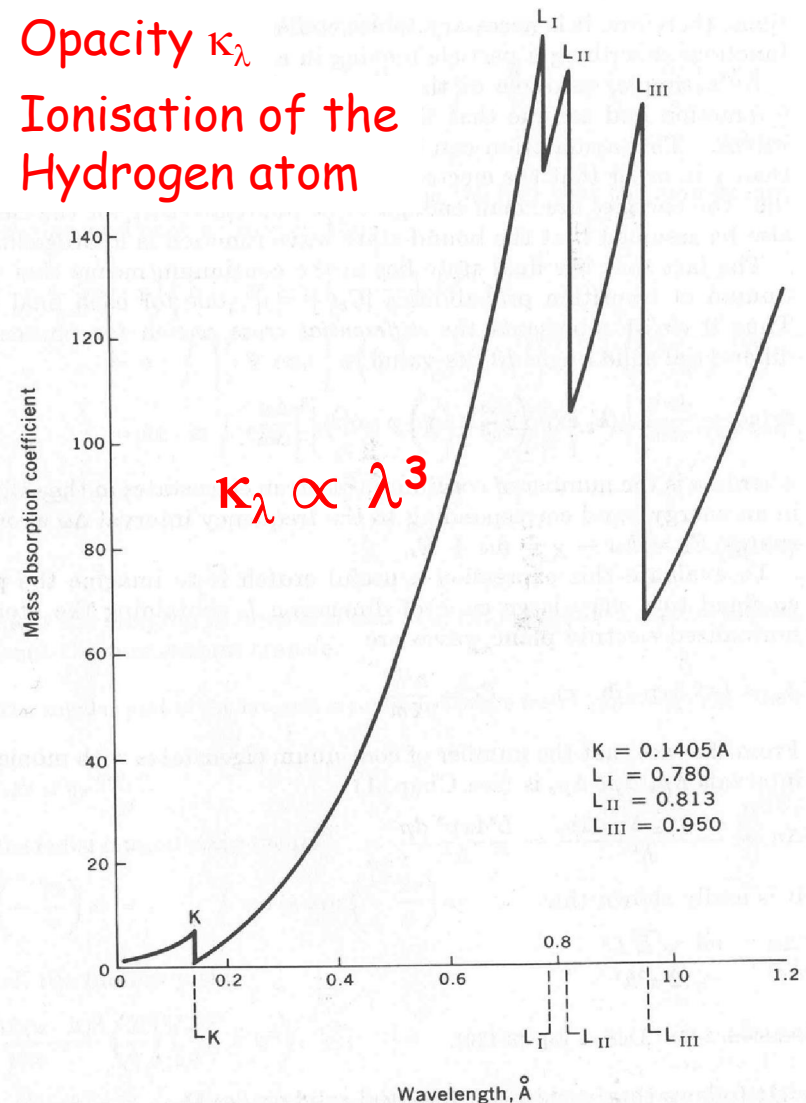
For a given transition :

$$\sigma_{\lambda,i} \propto \lambda^3$$

For each new transition ($\lambda \leq hc/\chi_i$)

→ discontinuity

Opacity κ_λ
Ionisation of the
Hydrogen atom



Opacity

Bound-free transitions

Absorption $\forall \nu \geq \chi/h, E_{\text{cin}}(e^-) = h\nu - \chi$

Opacity calculation :

- Cross section determination for all possible ionizations ...
- Determination of the occupation of all electronic states :
free electrons, states of ionisation (Saha eq.) and excitation (Boltzmann)

Grosso-modo ... At a given temperature : $h\nu \sim kT$

Transitions with $\chi_i \gg kT$ never occur because too few photons have enough energy

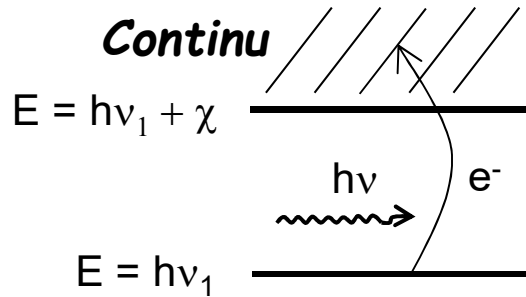
Transitions with $\chi_i \ll kT$ never occur because too few ions are at the required initial ionization state (Saha)



transitions with $\chi_i \simeq kT$ occur

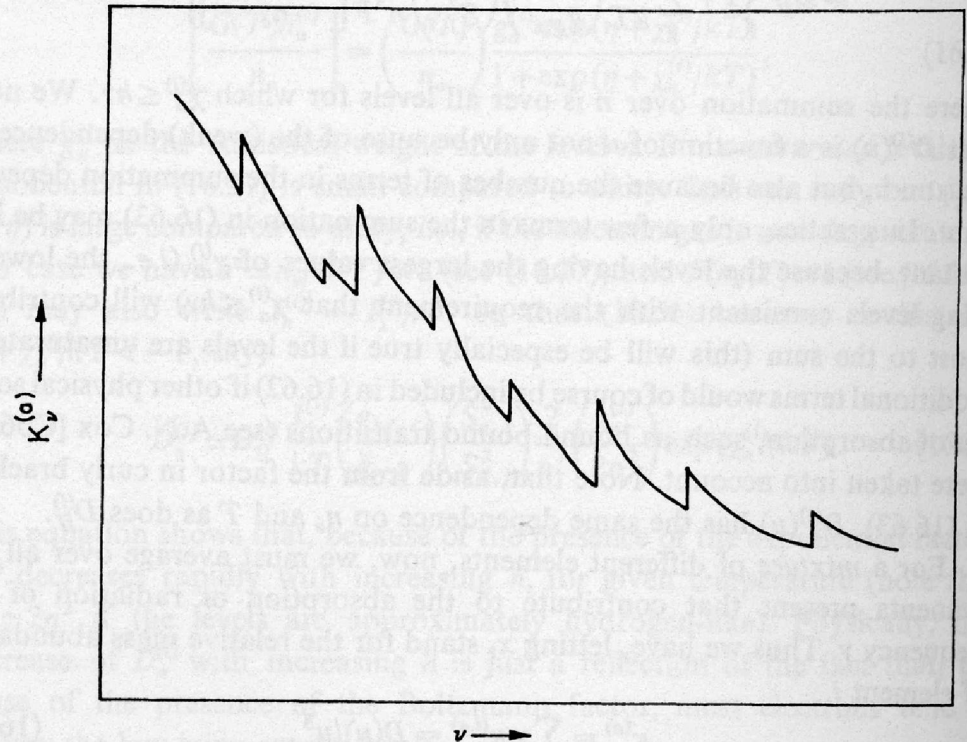
Opacity

Bound-free transitions



Absorption $\forall \nu \geq \chi/h$

Total opacity : complicated !



Opacity

Bound-free transitions

For a given transition : $\sigma_{v,i} \propto \nu^{-3}$

Dependence with respect to the temperature :

Grosso modo ... $\kappa_{\nu} \propto \nu^{-3}$, $h\nu \sim kT \rightarrow \kappa \propto T^{-3}$

Kramers law (approximative !) :

$$\kappa_{bf} \cong 4.34 \times 10^{24} Z (1 + X) \rho T^{-3.5} \text{ m}^2/\text{kg}$$

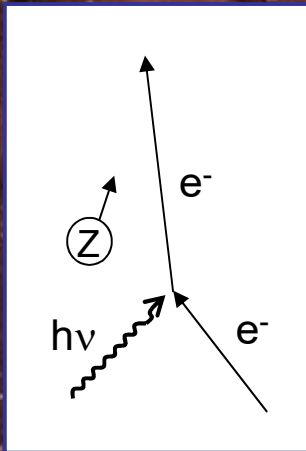
Z = mass fraction of heavy elements

Heavy elements are the main contributors because of their high ionisation potential

The initial ionization states are occupied in the high temperature layers

Opacity

Free-free absorptions



Inverse Bremsstrahlung

$$E_{\text{cin, fin}}(e^-) = E_{\text{cin, init}}(e^-) + h\nu$$

An ion is needed to ensure the momentum conservation

Kramers law(approx.)

$$n_e \simeq \frac{\rho}{m_u} \frac{1+X}{2}$$

$$\kappa_{ff} \cong 3.68 \cdot 10^{21} (X+Y) (1+X) \rho T^{-3.5} \text{ m}^2/\text{kg}$$

$$\sigma_\lambda \propto \lambda^3 / v$$

Initial velocity
of the electron

$$\lambda \sim hc/(kT)$$

$$v \sim (3kT/m_e)^{1/2}$$

$$\kappa \propto T^{-3.5}$$

Limitation: Degenerated gas

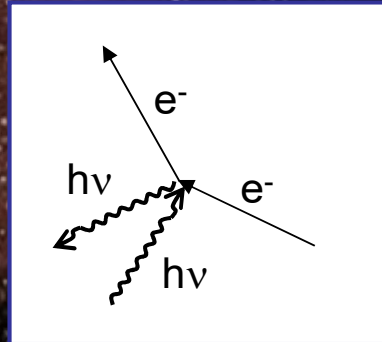
High number of impossible
transitions (many occupied states)

Smaller opacity

Opacity

Electronic scattering

Elastic collision
electron - photon



$$\sigma_T = 0.665 \times 10^{-28} \text{ m}^2$$
$$\kappa_{es} = 0.04 / \mu_e \text{ m}^2/\text{kg}$$
$$\cong 0.02 (1 + X)$$

Independent of T and ν !

Classical interpretation:

- Electromag. wave $\rightarrow$ oscillation of the electron
- $\rightarrow$ reemission in another direction

Non-relativistic case : Thomson diffusion

Relativistic case : Compton diffusion

Opacity

Approximation



Lois de Kramers

Transitions bound-bound

Transitions bound-free

$$\kappa_{bf} \cong 4.34 \cdot 10^{24} Z (1 + X) \rho T^{-3.5} \text{ m}^2/\text{kg}$$

Transitions free-free

$$\kappa_{ff} \cong 3.68 \cdot 10^{21} (X + Y) (1 + X) \rho T^{-3.5} \text{ m}^2/\text{kg}$$

Diffusion électronique

$$\kappa_{es} \cong 0.02 (1 + X) \text{ m}^2/\text{kg}$$

Dans les 3
derniers cas :

$$\kappa \propto \text{nbr. e}^-/\text{m}^3 \propto 1 + X$$

Conduction électronique

Si le gaz des électrons est
fortement **dégénéré**, voire relativiste ($\rho \nearrow$, $T \searrow$)



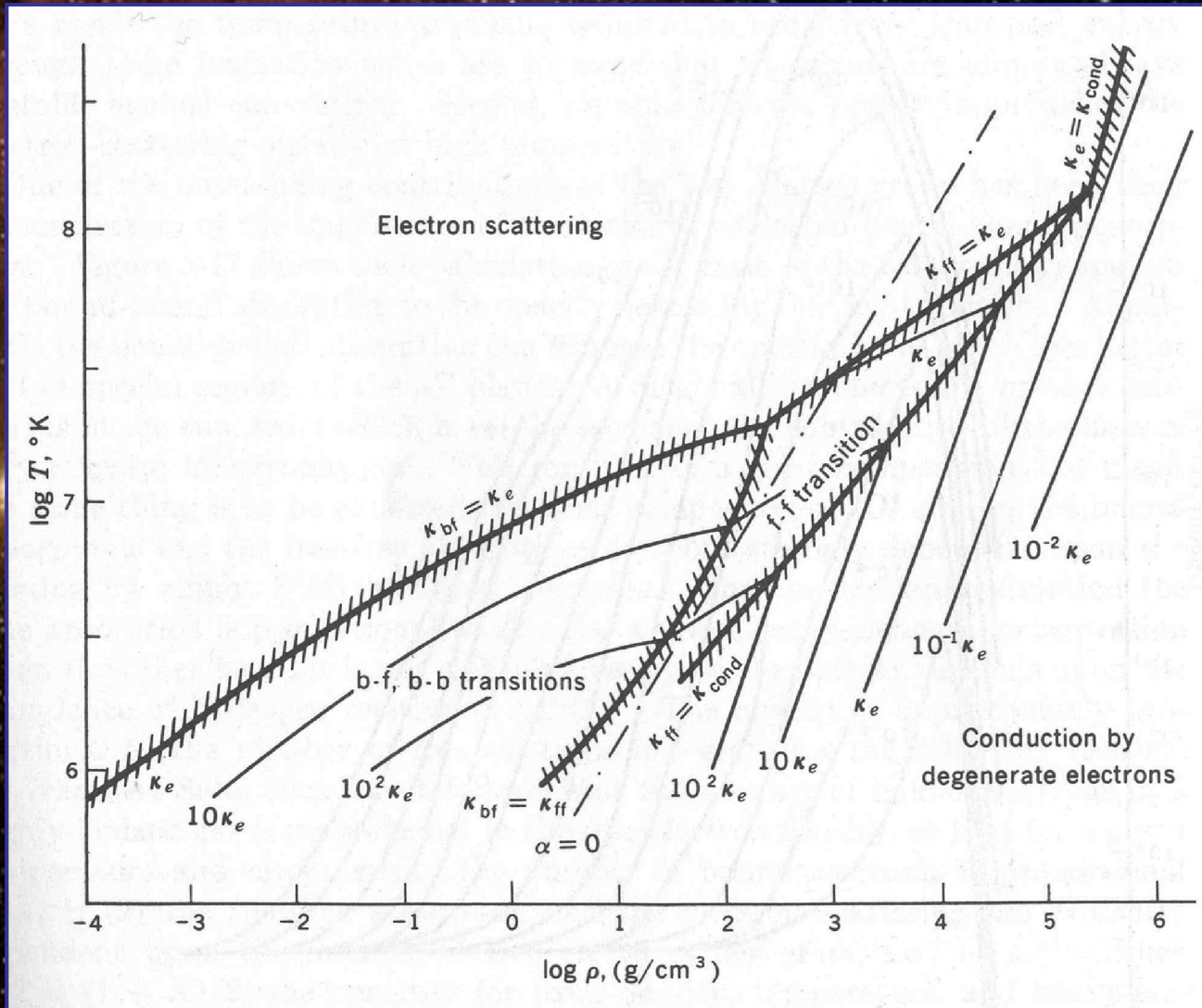
Energie cinétique des électrons très élevée



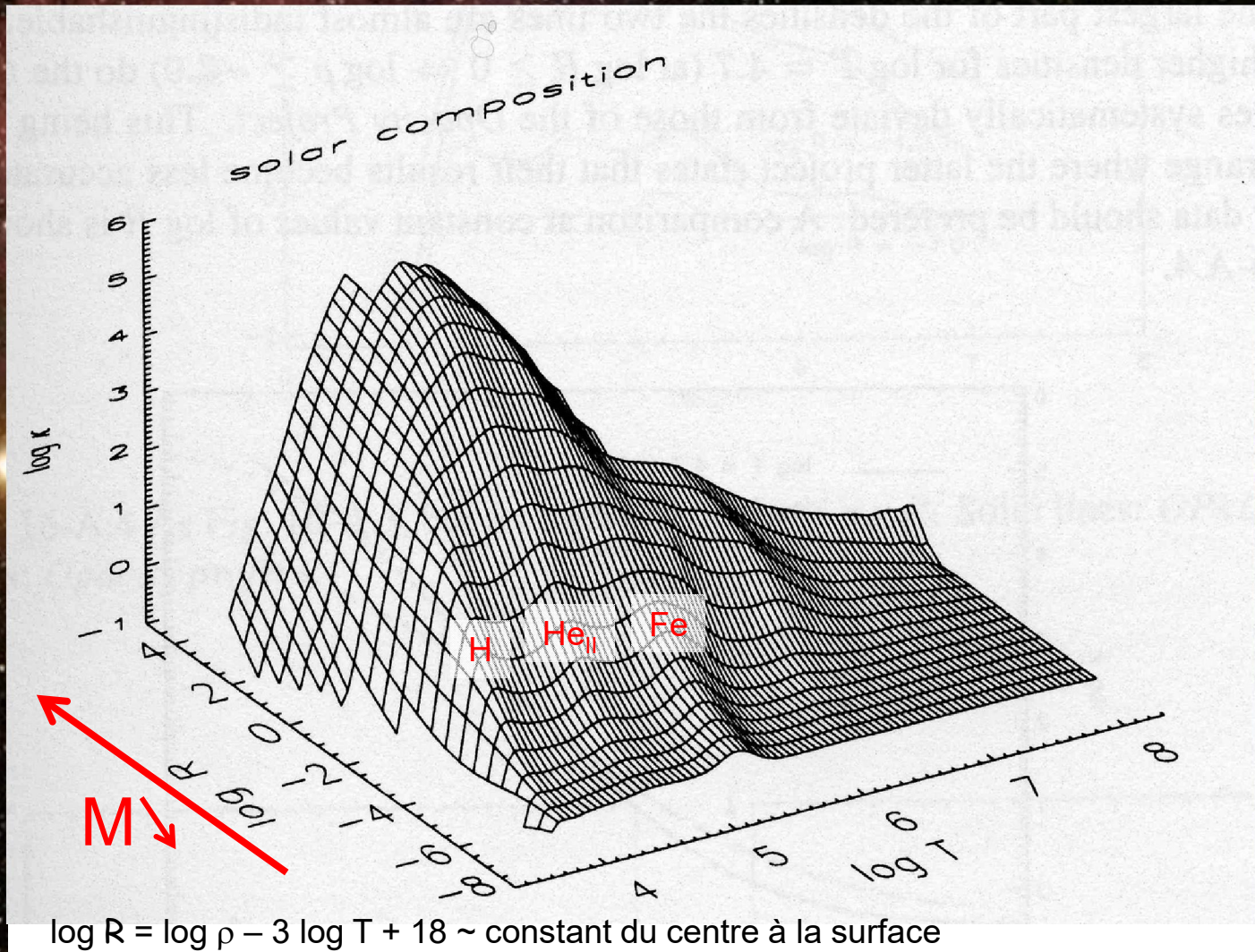
Ils transportent efficacement l'énergie par **conduction**

« Comme si l'opacité était encore plus faible »

Opacité



Opacité



Quelques modèles « stellaires » simples

Modèle à densité constante

$$\begin{aligned} P(r) &= \int_r^R \frac{Gm\rho}{r^2} dr & m(r) &= \frac{4\pi}{3} r^3 \rho = M \frac{r^3}{R^3} & \rho &= \frac{3M}{4\pi R^3} \\ &= \frac{3GM^2}{8\pi R^6} (R^2 - r^2) \end{aligned}$$

$$T(r) = \frac{P}{\rho} \frac{\mu m_u}{k} = \frac{GM}{R^3} \frac{\mu m_u}{2k} (R^2 - r^2)$$

$$\frac{T}{P} = \frac{1}{\rho} \frac{\mu m_u}{k} = \text{cst.} \quad \rightarrow \quad \frac{d \ln T}{d \ln P} = 1 \quad \gg \quad 2/5$$

Hyper instable vis-à-vis de la convection !

Un tel modèle n'est pas réaliste, la densité doit augmenter vers le centre pour que $d \ln T / d \ln P \simeq 1 - d \ln \rho / d \ln P$ soit suffisamment petit.

Quelques modèles « stellaires » simples

Résultats exercices

Exercice 1, homogène $X = 0.7$, $Y = 0.3$

$$\mu = 0.61538$$

$$R = 327\,500 \text{ km}$$

Exercice 2, homogène $Y = 1$

$$\mu = 4/3$$

$$R = 709\,600 \text{ km}$$

Exercice 3, cœur (10% de la masse): $Y = 1$, enveloppe $X = 0.7$, $Y = 0.3$

$$R/R_1 = 2.737$$

$$R_1 = 191\,000 \text{ km}$$

$$R_{\text{tot}} = 524\,000 \text{ km}$$

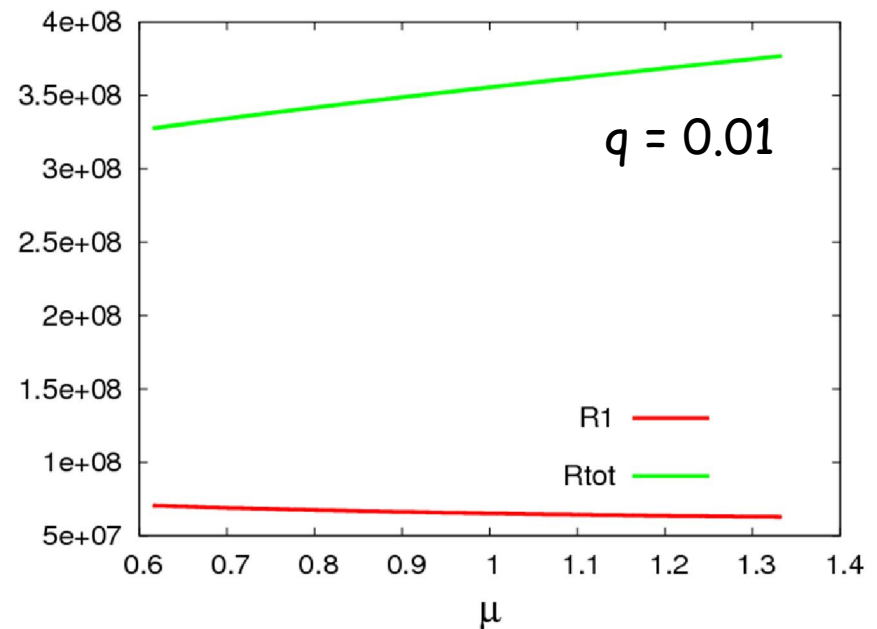
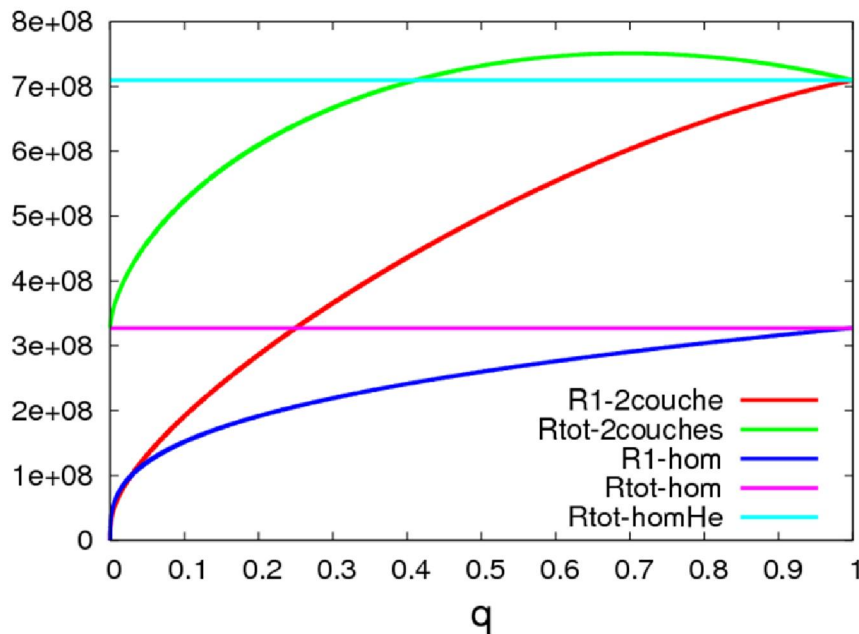
Quelques modèles « stellaires » simples

Résultats exercices

Exercice 3, cœur (10% de la masse): $Y = 1$, enveloppe $X = 0.7$, $Y = 0.3$

$$R/R_1 = 2.737 \quad R_1 = 191\,000 \text{ km}$$

$$R_{\text{tot}} = 524\,000 \text{ km}$$



Polytropic spheres

Non-relativistic degenerated gas : $P = K_1 \rho^{5/3}$

Relativistic degenerated gas : $P = K_1 \rho^{4/3}$

Polytrope (definition) : $P = K \rho^\gamma = K \rho^{1+1/n}$

Hydrostatic
equilibrium

$$\frac{dP}{dr} = -\rho g = -\rho \frac{d\phi}{dr}$$

$$\begin{aligned} \frac{d\phi}{dr} &= -K \gamma \rho^{\gamma-2} \frac{d\rho}{dr} \\ &= -K \frac{\gamma}{\gamma-1} \frac{d\rho^{\gamma-1}}{dr} \end{aligned}$$

$$\begin{aligned} \phi &= -K \frac{\gamma}{\gamma-1} \rho^{\gamma-1} = -K(n+1) \rho^{1/n} \\ \rho &= \left[\frac{-\phi}{K(n+1)} \right]^n \end{aligned}$$

Poisson
equation

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = 4\pi G \rho = 4\pi G \left[\frac{-\phi}{K(n+1)} \right]^n$$

Polytropic spheres

Poisson

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = 4\pi G \rho = 4\pi G \left[\frac{-\phi}{K(n+1)} \right]^n$$

Change of variables

↳ dimensionless variables

$$z = Ar \quad , \quad w = \frac{\phi}{\phi_c} = \left(\frac{\rho}{\rho_c} \right)^{1/n}$$
$$A^2 = \frac{4\pi G}{(n+1)^n K^n} (-\phi_c)^{n-1} = \frac{4\pi G}{(n+1)K} \rho_c^{\frac{n-1}{n}}$$

Equation of
Lane-Emden

$$\frac{1}{z^2} \frac{d}{dz} \left(z^2 \frac{dw}{dz} \right) + w^n = 0$$
$$w(0) = 1 \quad , \quad w'(0) = 0$$

Polytropic spheres

Equation de
Lane-Emden

$$\frac{1}{z^2} \frac{d}{dz} \left(z^2 \frac{dw}{dz} \right) + w^n = 0 \quad w = \frac{\phi}{\phi_c} = \left(\frac{\rho}{\rho_c} \right)^{1/n}$$

Cases with analytical
solutions :

$$n = 0 \ (\rho \text{ cst.}) : w(z) = 1 - z^2/6$$

$$n = 1 \ (\gamma = 2) : w(z) = \sin z/z$$

$$n = 5 \ (\gamma = 6/5) : w(z) = (1 + z^2/3)^{-1/2}$$

$n < 5 \ (\gamma > 6/5) \longrightarrow$ Finite radius, $n \geq 5 \longrightarrow$ Infinite radius

Other important cases:

$n = 3/2 \ (\gamma = 5/3, \text{ non-relativistic degenerated})$

$n = 3 \ (\gamma = 4/3, \text{ relativistic degenerated})$

$n = \infty \ (\gamma = 1, \text{ isothermal sphere, infinite radius})$

Polytropic spheres

Hierarchy of polytropes :

$n = 0$: constant density, $d \ln T / d \ln P = 1$ (convectively unstable)



$n = 3/2$ ($\gamma = 5/3$) : non-relativistic degenerated or convective zone

$|\nabla \rho| \nearrow$

$n = 3$ ($\gamma = 4/3$) : relativistic degenerated or radiation pressure $\nearrow$

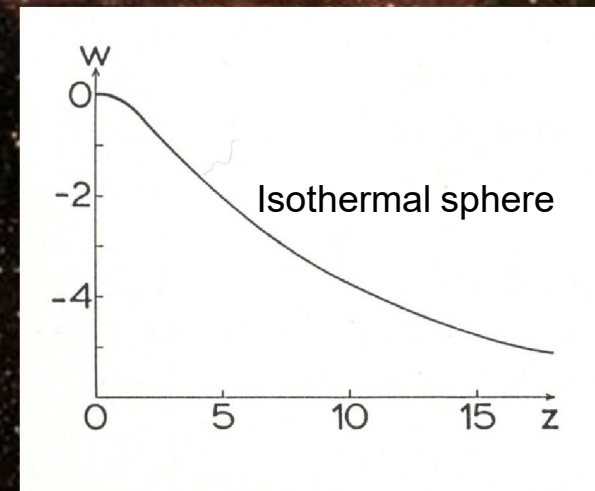
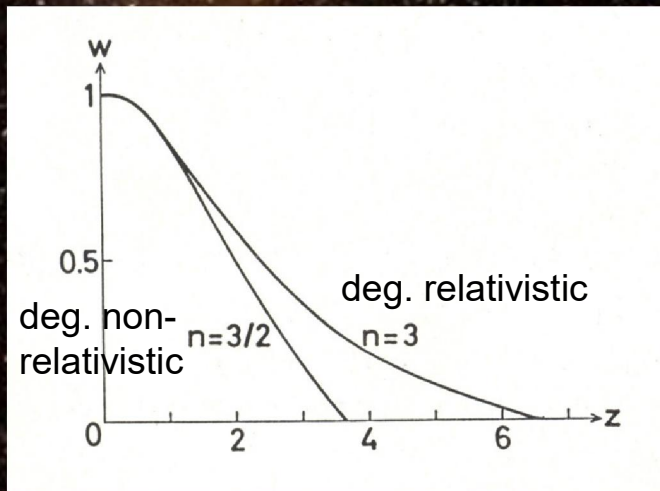
$|\nabla T| \searrow$

$n = 3.25$: $\sim$ radiative envelope



$n = 5$ ($\gamma = 6/5$) : Limite finite radius $\longleftrightarrow$ infinite

$n = \infty$ ($\gamma = 1$) : isothermal sphere, infinite radius



Un modèle simple d'enveloppe radiative

Hypothèses:

$$L = - \frac{16\pi r^2 a c T^3}{3\kappa\rho} \frac{\partial T}{\partial r}$$

- Enveloppe radiative : $\frac{dT}{dP} = \frac{3\kappa L}{64\pi\sigma G T^3 m}$

- Opacités : $\kappa = \kappa_0 P^a T^b$ Kramers : $a = 1, b = -4.5$

- L, m constants ($\varepsilon = 0, \rho$ petit)

$$\frac{dT}{dP} \propto T^{b-3} P^a \frac{L}{m} \quad \frac{d(T^{4-b})}{d(P^{a+1})} \propto \frac{L}{M} = \text{cste.}$$

$$T^{4-b} = B \frac{L}{M} (P^{a+1} + C)$$

$$T^{8.5} = B \frac{L}{M} (P^2 + C) \simeq B P^2 L/M \propto \rho^2 T^2 L/M$$

$$T^{3.25} \simeq C_0 \sqrt{L/M} \rho \quad P \simeq C_1 (L/M)^{\frac{1}{6.5}} \rho^{1+\frac{1}{3.25}}$$

Polytrophe
 $n = 3.25$

Polytropic spheres

Mass

$$\frac{\rho}{\rho_c} = \left(\frac{\phi}{\phi_c} \right)^n = w^n$$

$$m(r) = \int_0^r 4\pi r^2 \rho dr = 4\pi \frac{r^3}{z^3} \rho_c \int_0^z w^n z^2 dz = -4\pi r^3 \rho_c \frac{1}{z} \frac{dw}{dz}$$

$$R = \frac{z_n}{A} = z_n \left[\frac{(n+1)K}{4\pi G} \right]^{1/2} \rho_c^{\frac{1-n}{2n}}$$

$$A^2 = \frac{4\pi G}{(n+1)K} \rho_c^{\frac{n-1}{n}}$$

$$M = -4\pi \left[\frac{(n+1)K}{4\pi G} \right]^{3/2} z_n^2 \frac{dw}{dz}(z_n) \rho_c^{\frac{3-n}{2n}}$$

$$R \propto \rho_c^{\frac{1-n}{2n}} \propto M^{\frac{1-n}{3-n}}$$

Mass-radius relation
of polytropes

Polytropes : degenerated gas, white dwarfs

Mass - radius relation
of polytropes (fixed K !)

$$R \propto M^{\frac{1-n}{3-n}}$$

$$0 \leq n < 1 : M \nearrow \rightarrow R \nearrow$$

$$1 < n < 3 : M \nearrow \rightarrow R \searrow$$

Increasing the mass by a factor q , keeping the same radius

$$m(r) \longrightarrow m'(r) = q m(r)$$

$$\rho(r) \longrightarrow \rho'(r) = q \rho(r)$$

$$P(r) \longrightarrow P'(r) = K (\rho')^{1+1/n} = q^{1+1/n} P(r)$$

$$\text{Weight}(r) = \longrightarrow \text{Weight}'(r) = q^2 \text{Weight}(r)$$

$$\int_r^R G m \rho / r^2 dr$$

$$n < 1 \longrightarrow P'(r) > P_{\text{oid}}'(r)$$

The pressure increases more
than the weight $\longrightarrow$ dilatation

$$n > 1 \longrightarrow P'(r) < P_{\text{oid}}'(r)$$

The pressure increases less than
the weight $\longrightarrow$ contraction

Polytropes : gaz dégénéré, naines blanches

Relation Rayon - Masse
des polytropes (K fixé)

$$R \propto M^{\frac{1-n}{3-n}}$$

Gaz dégénéré non-relativiste, $n = 3/2$:

$$R \propto M^{-1/3}$$

Quand la dégénérescence devient complète non-relativiste:

$R \sim \text{constant}$: Refroidissement à rayon constant des naines blanches

Gaz dégénéré relativiste, $n = 3$:

$M = \text{Constante} !$

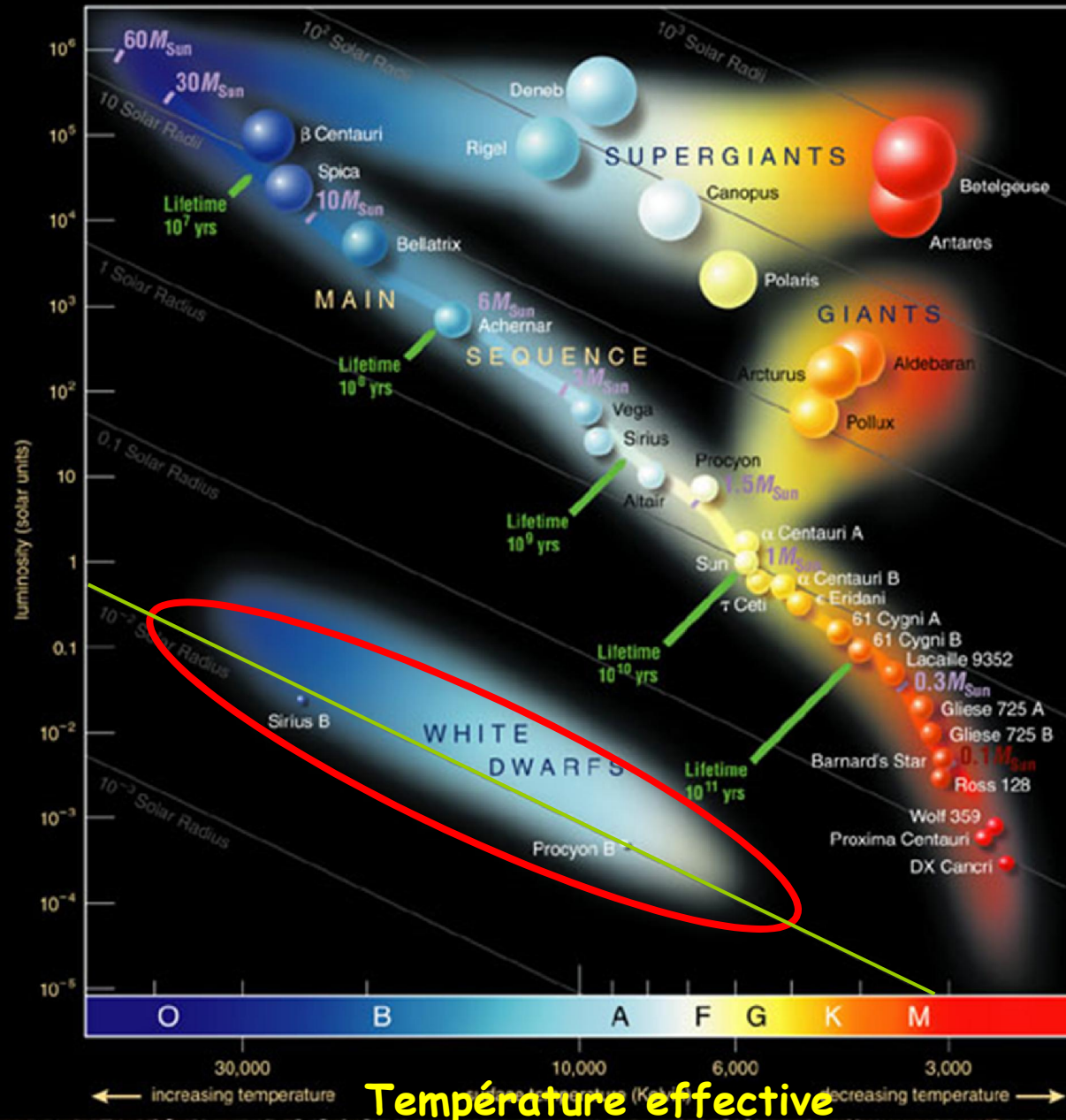
$$\begin{aligned} M &= -4\pi \left[\frac{K}{\pi G} \right]^{3/2} z_3^2 \frac{dw}{dz}(z_3) \\ &= \left(\frac{2}{\mu_e} \right)^2 1.459 M_{\odot} \end{aligned}$$

$$K = \frac{1.2435 \times 10^{15}}{\mu_e^{4/3}} \quad (cgs)$$

Masse limite de Chandrasekhar

Refroidissement à rayon constant des naines blanches

Luminosité



Température effective

Polytropes : gaz dégénéré relativiste

Masse limite de Chandrasekhar

$$M_{ch} = \left(\frac{2}{\mu_e}\right)^2 1.459 M_{\odot}$$

$$R \propto \rho_c^{\frac{1-n}{2n}} \propto \rho_c^{-1/3}$$
$$M \propto \rho_c R^3 = \text{cste.}$$

M est fixe, R vaut ce qu'on veut !

Je multiplie par un facteur $\times$ constant la distance au centre de chaque couche :

$$r(m) \longrightarrow r'(m) = \times r(m)$$

$$\rho(m) \longrightarrow \rho'(m) = \rho(m)/\times^3$$

$$P(m) \longrightarrow P'(m) = K (\rho')^{4/3} = K \rho^{4/3}/\times^4 = P(m)/\times^4$$

$$\text{Poids}(m) \longrightarrow \text{Poids}'(m) = \text{Poids}(m)/\times^4$$

$$P(m) = \text{Poids}(m) \longrightarrow P'(m) = \text{Poids}'(m)$$

On garde une solution quel que soit le facteur de dilatation $\times$

Polytropes : gaz dégénéré relativiste

Masse limite de Chandrasekhar

Que se passe-t-il si $M > M_{ch}$?

$$M_{ch} = \left(\frac{2}{\mu_e} \right)^2 1.459 M_{\odot}$$

$P(m) < \text{Poids}(m) \longrightarrow \text{contraction}$

$$r(m) \longrightarrow r'(m) = x r(m)$$

$$\rho(m) \longrightarrow \rho'(m) = \rho(m)/x^3$$

$$P(m) \longrightarrow P'(m) = P(m)/x^4$$

$$\text{Poids}(m) \longrightarrow \text{Poids}'(m) = \text{Poids}(m)/x^4$$

$P'(m) < \text{Poid}'(m) \longrightarrow \text{L'effondrement continue}$

On n'a jamais
de solution

Critère de stabilité dynamique

Considérons un modèle d'étoile tout à fait général à l'équilibre hydrostatique

Pas besoin d'être un polytrophe

Au départ : $P(m) = \text{Poids}(m)$

Contraction homologue adiabatique du gaz $\longrightarrow P'/P = (\rho'/\rho)^{\Gamma_1}$

$$r(m) \longrightarrow r'(m) = x r(m)$$

$$\rho(m) \longrightarrow \rho'(m) = \rho(m)/x^3$$

$$P(m) \longrightarrow P'(m) = P(m) (\rho'/\rho)^{\Gamma_1} = P(m)/x^{3\Gamma_1}$$

$$\text{Poids}(m) \longrightarrow \text{Poids}'(m) = \text{Poids}(m)/x^4$$

$$\Gamma_1 \equiv \left. \frac{\partial \ln P}{\partial \ln \rho} \right|_S$$

Si $\Gamma_1 > 4/3$ ($x < 1$),
 $P'(m) > \text{Poids}'(m)$,
l'étoile revient à l'équilibre :
Dynamiquement **STABLE**

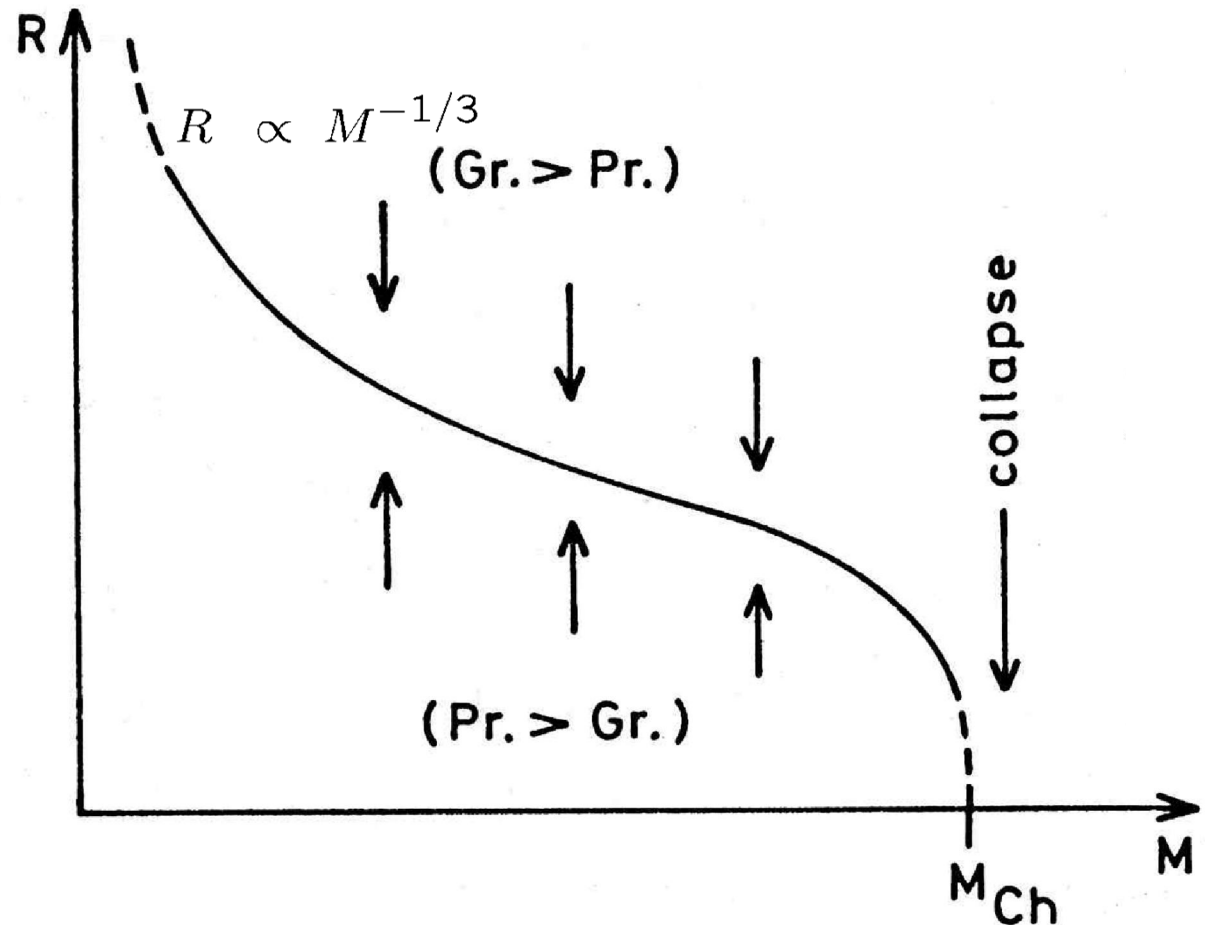
Si $\Gamma_1 < 4/3$ ($x < 1$),
 $P'(m) < \text{Poids}'(m)$,
l'étoile s'effondre :
Dynamiquement **INSTABLE**

Polytropes : gaz dégénéré relativiste

Masse limite de Chandrasekhar

$$M_{ch} = \left(\frac{2}{\mu_e} \right)^2 1.459 M_{\odot}$$

La loi
Masse - Rayon



Que se passe-t-il quand la masse de Chandrasekhar est atteinte ?

1^{er} scénario (le plus fréquent): Une naine blanche de C-O dans un système binaire accrète lentement de la matière de son compagnon géante rouge

2^e scénario : Pendant les combustions d'H et He en couche, le coeur de C-O croît en masse

Le coeur de C-O atteint la masse de Chandrasekhar !

L'étoile se contracte jusqu'à des densités extrêmes

Réactions nucléaires dans milieu dégénéré → emballement thermique

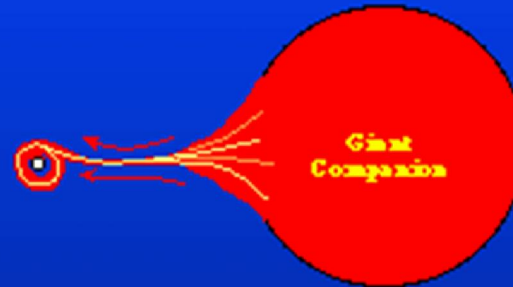
→ l'étoile entière s'embrase en quelques secondes



L'étoile explose ! **Supernova de type Ia** $L \sim 10^{11} L_{\odot}$

Le Phénomène de Nova

Thin hydrogen surface layer accumulated on white dwarf through accretion ring



Ignition of surface layer under degenerate conditions



Thermonuclear runaway until degeneracy lifted



Explosive Burning of Hydrogen Shell

- Accretion rate $< 6 \times 10^{-6}$ solar masses per year
- $10^{-5} - 10^{-4}$ solar mass layer ignites and burns degenerately

- Runaway until degeneracy is lifted, leading to explosive hydrogen burning
- $T = 2-3 \times 10^8$ K; $\rho = 10^3 - 10^4$ g/cm³
- Time: 100 - 1000 seconds to eject layer

Que se passe-t-il quand la masse de Chandrasekhar est atteinte ?

Etoiles très massives

Scénario principal :

Fusion centrale du Si, contraction,
fusion du Si en couche, contraction, ...

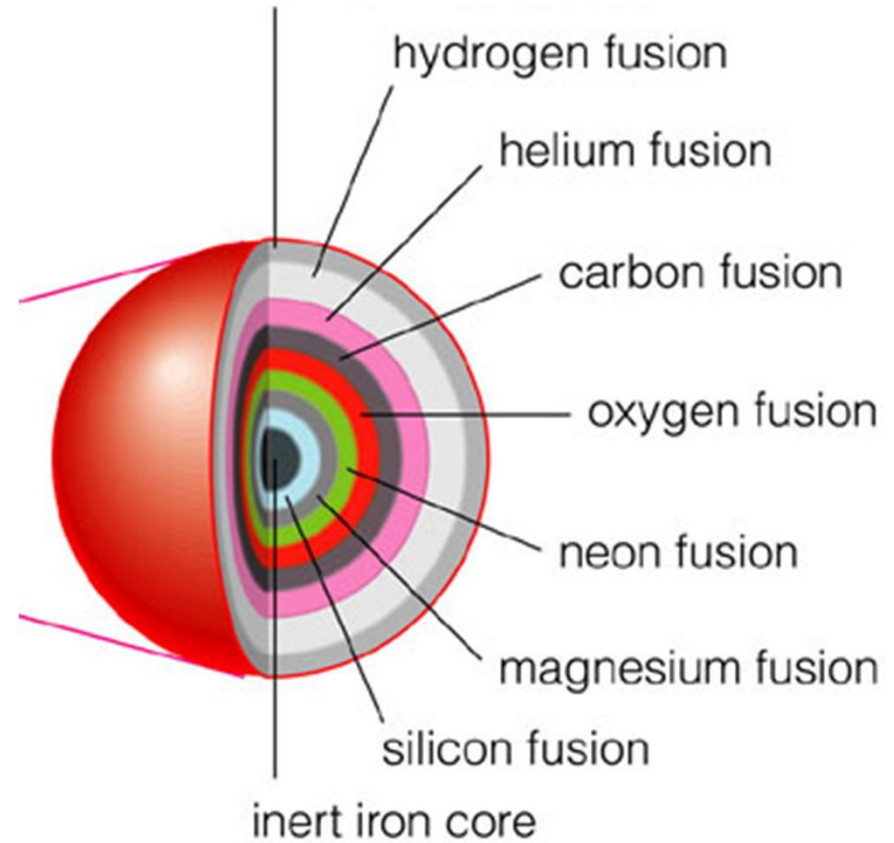
⇒ Le cœur de Fe se contracte,
dégénère et croît en masse.

Quand $M_{\text{Fe}} \simeq$ masse de Chandrasekhar

⇒ **collapse** jusqu'à ce que les noyaux
se touchent ⇒ rebond, onde de choc

⇒ Explosion, **Supernova de type II**

Structure en pelure d'oignon



Les réactions nucléaires

Rôle

Principale **source d'énergie**
au cours de la vie d'une étoile



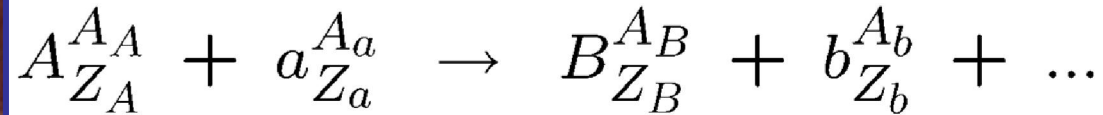
Calcul du taux de production
d'énergie nucléaire ϵ
/ u. de masse et de temps

Nucléosynthèse stellaire :
Création des différents
éléments chimiques
de l'univers :



Calcul de **l'évolution
temporelle** des
abondances au cours
de la vie des étoiles

Les réactions nucléaires



Lois de conservation

- Conservation du nombre de nucléons

$$A_A + A_a = A_B + A_b + \dots$$

- Conservation des charges

$$Z_A + Z_a = Z_B + Z_b + \dots$$

- Quantité de mouvement

(Problème à 2 corps, non-relativiste)

$$\begin{aligned} m_1(v_1 - V) &= -m_2(v_2 - V) \\ &= \frac{m_1 m_2}{m_1 + m_2} (v_1 - v_2) = m_\mu v \end{aligned}$$

Masse réduite

Eq. de mouvement de la "particule relative" dans un potentiel fixe :

Classique (Newton): $m_\mu \frac{dv}{dt} = -\nabla V(r) \quad (r = r_1 - r_2)$

Quantique (Schroedinger): $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_\mu} \nabla^2 \psi + V \psi$

Les réactions nucléaires

Lois de conservation

- Moment cinétique total

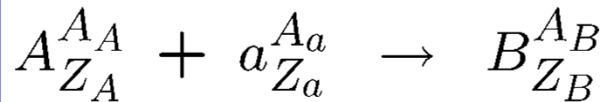
$$J = \sum L + \sum S = \text{Cst.}$$

Moments ang. orb.

Spins

- Nombre leptonique

- Energie



Chaleur fournie
par la réaction

$$Q = (M_A + M_a - M_B)c^2$$

Excès de "masse"
= - énergie de liaison

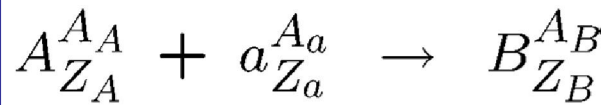
$$\Delta\mathcal{M}_a = (M_a - A_a m_u)c^2 = 931.478 (M_a/m_u - A_a) \text{ Mev}$$

Energie de liaison par nucléon

$$f_a = -\Delta\mathcal{M}_a / A_a$$

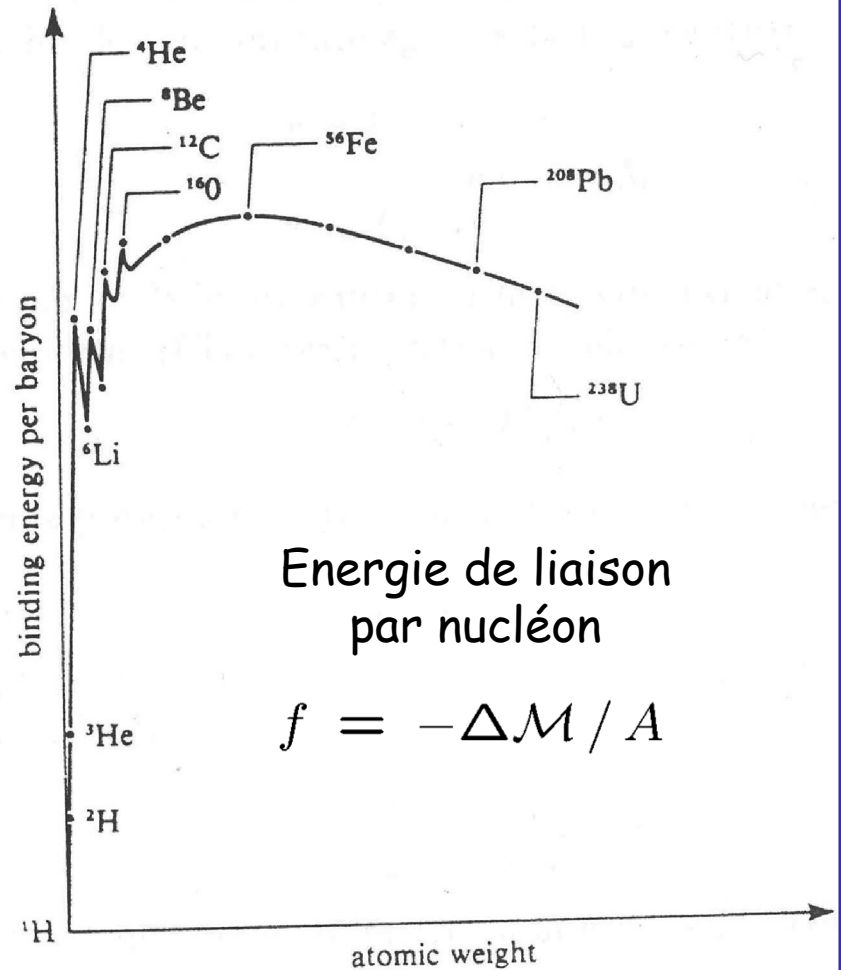
$$Q = \Delta\mathcal{M}_A + \Delta\mathcal{M}_a - \Delta\mathcal{M}_B = A_B \left[f_B - \left(\frac{A_A}{A_B} f_A + \frac{A_a}{A_B} f_a \right) \right]$$

• Energie



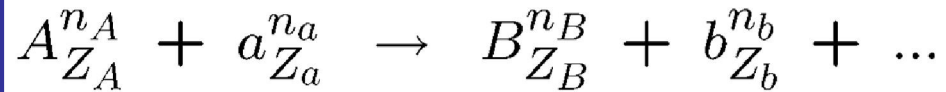
Chaleur fournie
par la réaction:

$$Q = (M_A + M_a - M_B)c^2$$



$$Q = \Delta\mathcal{M}_A + \Delta\mathcal{M}_a - \Delta\mathcal{M}_B = A_B \left[f_B - \left(\frac{A_A}{A_B} f_A + \frac{A_a}{A_B} f_a \right) \right]$$

Les réactions nucléaires



$$r_v = \sigma(v) v N_A N_a$$

Taux de réaction
= nbre de réactions
/ u. de temps
et de volume

Section
efficace

Vitesse
relative

Nbre de couples
de particules
/ u. de volume²

Distribution de vitesses →

$$r = \langle \sigma(v) v \rangle N_A N_a$$

Taux de production d'énergie
 ϵ = énergie produite
/ u. de temps et de masse

$$\epsilon_k = \frac{Q_k}{M_A M_a} \rho X_A X_a \langle \sigma v \rangle_k$$

Sommer sur toutes les réactions

Les réactions nucléaires

$$r = \langle \sigma(v) v \rangle N_A N_a$$

$$\langle \sigma(v) v \rangle = \int_0^\infty \sigma(E) v(E) f(E) dE$$

Energie cinétique
totale dans le repère
du centre de masse :

$$\begin{aligned} E &= \frac{1}{2} \frac{m_A m_a}{m_A + m_a} v^2 \\ &= \frac{1}{2} m_\mu v^2 \end{aligned}$$

Distribution statistique des
énergies cinétiques E :
(Boltzmann, Fermi-Dirac, ...)

$$f(E) = \frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(kT)^{3/2}} e^{-E/kT}$$

Toute la difficulté du
problème se ramène au calcul
de la section efficace $\sigma(E)$

Les réactions nucléaires

Détermination de la section efficace $\sigma(E)$

Différents facteurs interviennent

1) Facteur géométrique P_λ

Extension quantique du noyau :

$$P_\lambda = \lambda^2 / (4\pi) = (h / (m_\mu v))^2 / (4\pi) \sim 1/E$$

 De Broglie

C'est la section efficace
dans l'exp. de diffusion
quantique,
moment cin. orb. nul ($l=0$)

2) Facteur de Gamov P_G

Franchissement de la barrière coulombienne

Les réactions nucléaires

Détermination de la section efficace $\sigma(E)$

2) Facteur de Gamov P_G

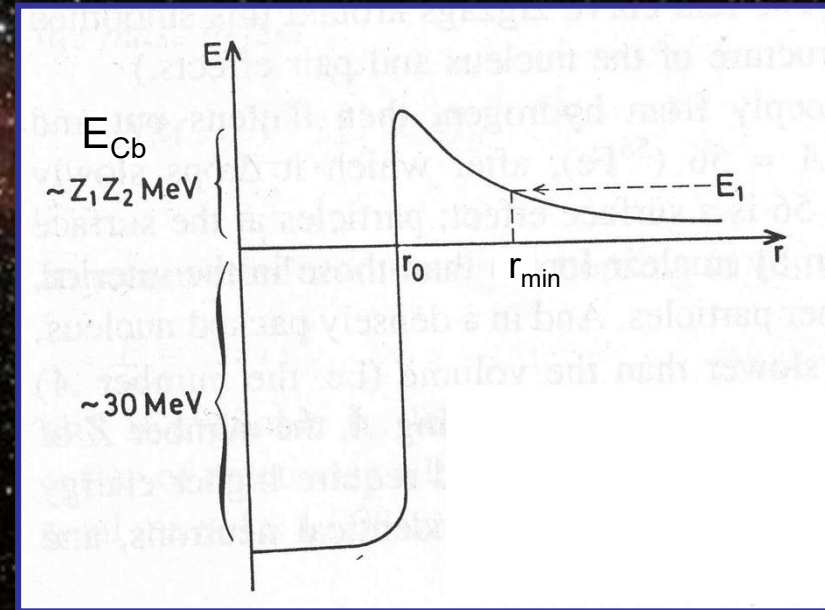
La barrière coulombienne

$$E_{Cb} = \frac{Z_A Z_a e^2}{r_0} \simeq Z_A Z_a \text{ Mev}$$

r_0 : rayon nucléaire

$$\langle E_{Cin} \rangle = \frac{3}{2} kT \simeq 1 \text{ kev}$$

pour $T \sim 10^7 \text{ K}$



→ Plus de $1000 \times$ trop petit

→ Facteur e^{-1000} dans Max-Boltz. !

→ Tout à fait impossible

Les réactions nucléaires

Détermination de la section efficace $\sigma(E)$

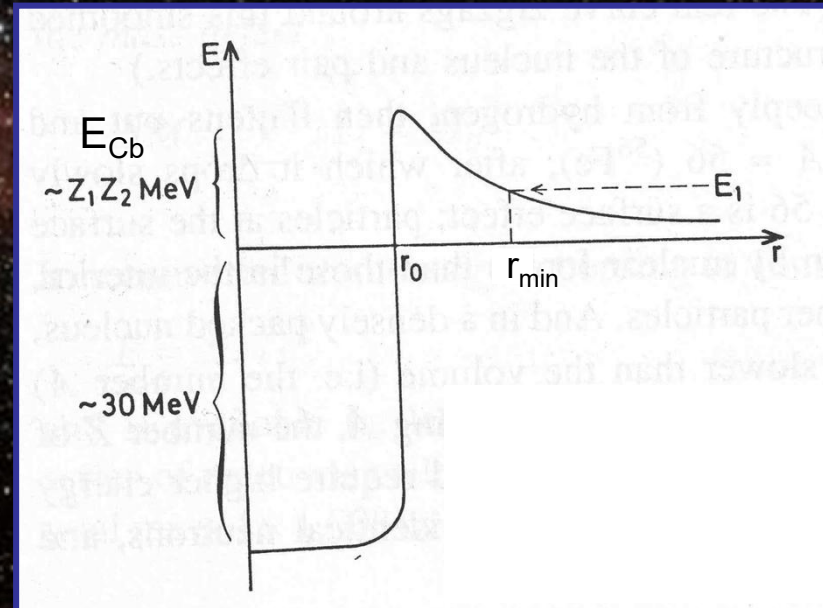
2) Facteur de Gamov P_G

La barrière coulombienne

Heureusement ...

Des noyaux peuvent franchir
la barrière de potentiel par

l'Effet tunnel



Les réactions nucléaires

Détermination de la section efficace $\sigma(E)$

2) Facteur de Gamov P_G

L'Effet tunnel

Barrière carrée

Eq. Schroedinger :
$$\frac{\hbar^2}{2m_\mu} \frac{d^2\psi}{dr^2} + (E - V)\psi = 0$$

Fonction d'onde: $\psi(r) \propto \exp(-kr) \quad k = (1/\hbar)\sqrt{2m_\mu(V - E)}$

Probabilité de franchissement de la barrière :
$$= \exp(-2k\Delta r)$$

Les réactions nucléaires

Détermination de la section efficace $\sigma(E)$

2) Facteur de Gamov P_G

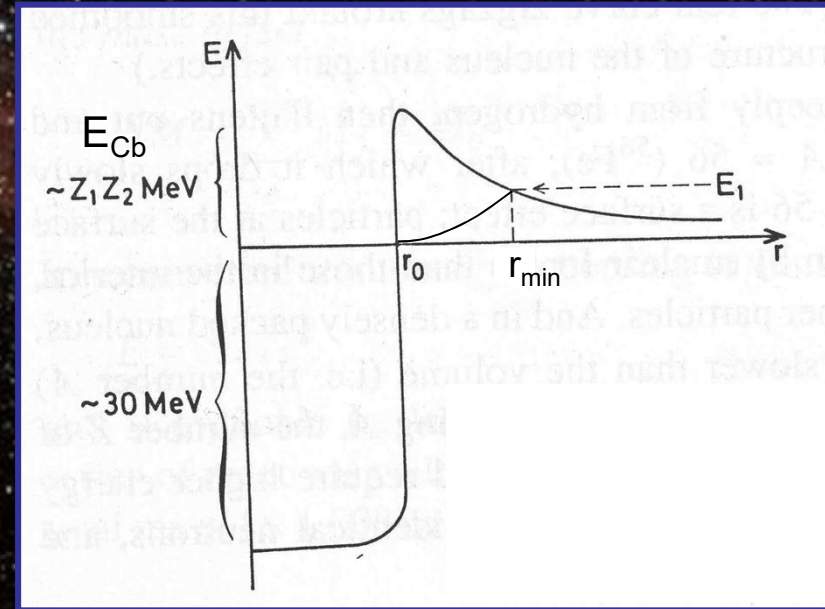
L'Effet tunnel

Potentiel coulombien

Probabilité de franchissement de la barrière (approximation WKB) :

$$\begin{aligned} P_G &\simeq \exp\left(-2 \int_{r_0}^{r_{min}} k \, dr\right) \\ &\simeq \exp\left(-(\pi/\hbar) \sqrt{2m_\mu E} \, r_{min}\right) \\ &= \exp(-2\pi^2 r_{min}/\lambda) \end{aligned}$$

$$\lambda = h/(m_\mu v)$$



$$\begin{aligned} k(r) &= (1/\hbar) \sqrt{2m_\mu (E_{Cb}(r_0/r) - E)} \\ E &\ll E_{Cb} \end{aligned}$$

$$E = \frac{1}{2} m_\mu v^2 = \frac{h^2}{2m_\mu \lambda^2}$$

Les réactions nucléaires

Détermination de la section efficace $\sigma(E)$

2) Facteur de Gamov P_G

L'Effet tunnel

Probabilité de franchissement de la barrière :

$$P_G \simeq \exp(-2\pi^2 r_{min}/\lambda)$$

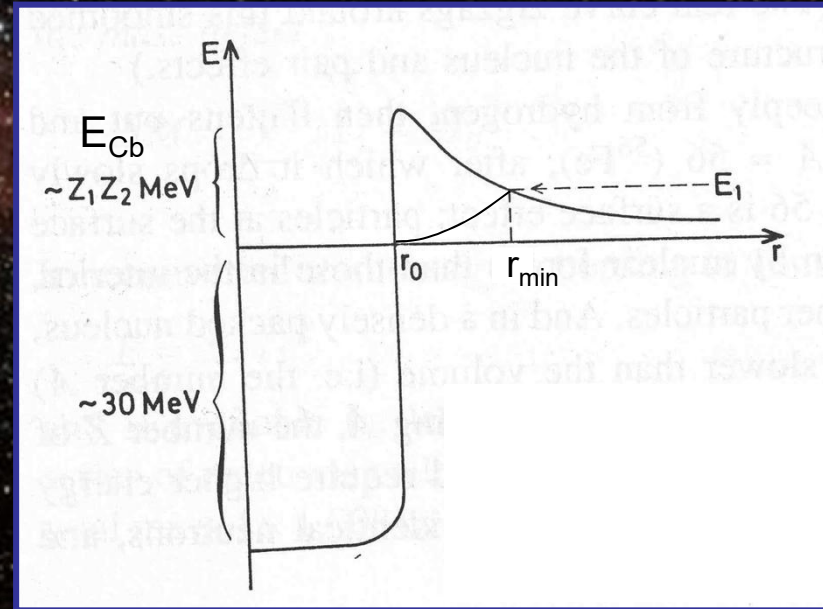
$$\lambda = h/(m_\mu v)$$

$$\frac{Z_A Z_a e^2}{r_{min}} = \frac{1}{2} m_\mu v^2 \Rightarrow r_{min} = \frac{2 Z_A Z_a e^2}{m_\mu v^2}$$

$$P_G \approx \exp(-2\pi Z_A Z_a e^2 / (\hbar v))$$

$$= \exp(-b E^{-1/2})$$

$$b = 31.29 Z_A Z_a A_\mu^{1/2} \quad (\text{keV})^{1/2}$$



$$A_\mu = \frac{A_A A_a}{A_A + A_a}$$

Les réactions nucléaires

Détermination de la section efficace $\sigma(E)$

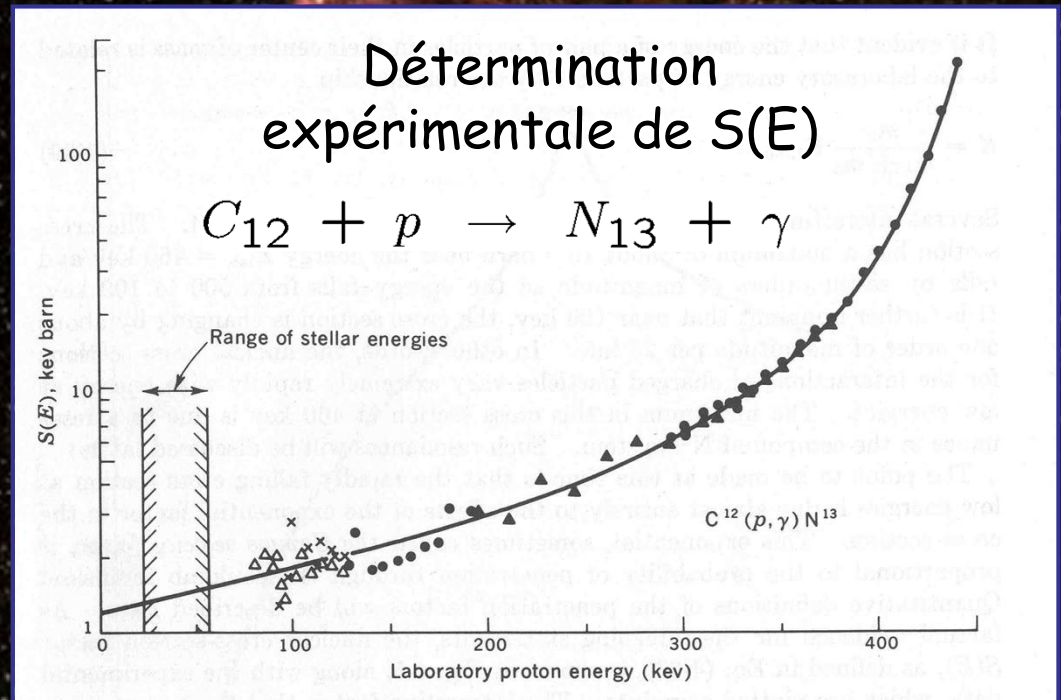
3) Facteur nucléaire $S(E)$

$$\sigma(E) = (1/E) P_G(E) S(E)$$

Propriétés nucléaires du noyau $A + a$,
interaction forte
(et parfois faible)

$S(E)$ varie
lentement avec E

Sauf tout près
d'une résonance



Les réactions nucléaires

Détermination de la section efficace $\sigma(E)$

$$\sigma(E) = (1/E) P_G S(E) = \frac{S(E)}{E} \exp(-b E^{-1/2})$$

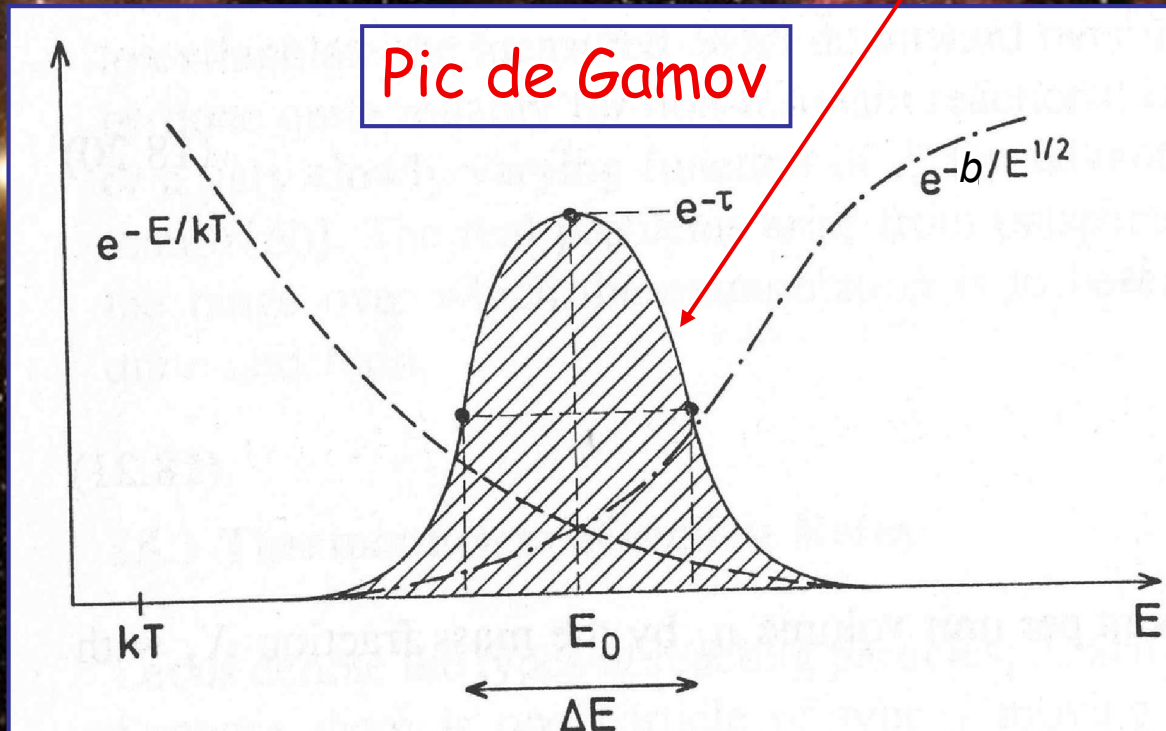


$$\begin{aligned} \langle \sigma v \rangle &= \int_0^\infty \sigma(E) v(E) f(E) dE \\ &= \int_0^\infty \frac{S(E)}{E} \exp(-b E^{-1/2}) \sqrt{\frac{2E}{m_\mu}} \frac{2}{\sqrt{\pi}} \frac{E}{kT} \exp\left(-\frac{E}{kT}\right) \frac{dE}{(kT E)^{3/2}} \\ &= \left(\frac{8}{m_\mu \pi}\right)^{1/2} \frac{1}{(kT)^{3/2}} \int_0^\infty S(E) \exp\left(-\frac{E}{kT} - b E^{-1/2}\right) dE \end{aligned}$$

Les réactions nucléaires

Détermination de la section efficace $\sigma(E)$

$$\langle \sigma v \rangle = \left(\frac{8}{m_{\mu\pi}} \right)^{\frac{1}{2}} \frac{1}{(kT)^{\frac{3}{2}}} \int_0^{\infty} S(E) \exp \left(-\frac{E}{kT} - bE^{-\frac{1}{2}} \right) dE$$

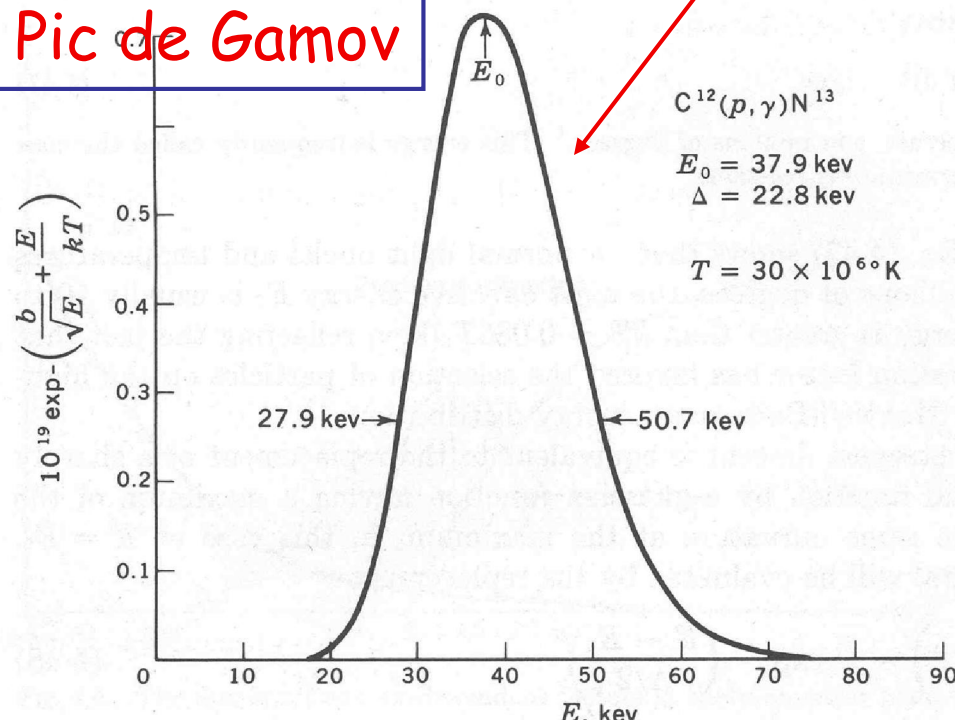


Les réactions nucléaires

Détermination de la section efficace $\sigma(E)$

$$\langle \sigma v \rangle = \left(\frac{8}{m_{\mu\pi}} \right)^{\frac{1}{2}} \frac{1}{(kT)^{\frac{3}{2}}} \int_0^{\infty} S(E) \exp \left(-\frac{E}{kT} - bE^{-\frac{1}{2}} \right) dE$$

Pic de Gamov



Les réactions nucléaires

$$\langle \sigma v \rangle = \left(\frac{8}{m_{\mu\pi}} \right)^{\frac{1}{2}} \frac{S_0}{(kT)^{\frac{3}{2}}} \int_0^{\infty} \exp \left(-\frac{E}{kT} - \frac{b}{\sqrt{E}} \right) dE$$

Approx. par une gaussienne

$$\begin{aligned} \exp \left(-\frac{E}{kT} - \frac{b}{\sqrt{E}} \right) &= \exp(-f(E)) \\ &\approx \exp(-f(E_0)) \exp \left(-\left(\frac{E - E_0}{\Delta E/2} \right)^2 \right) \end{aligned}$$

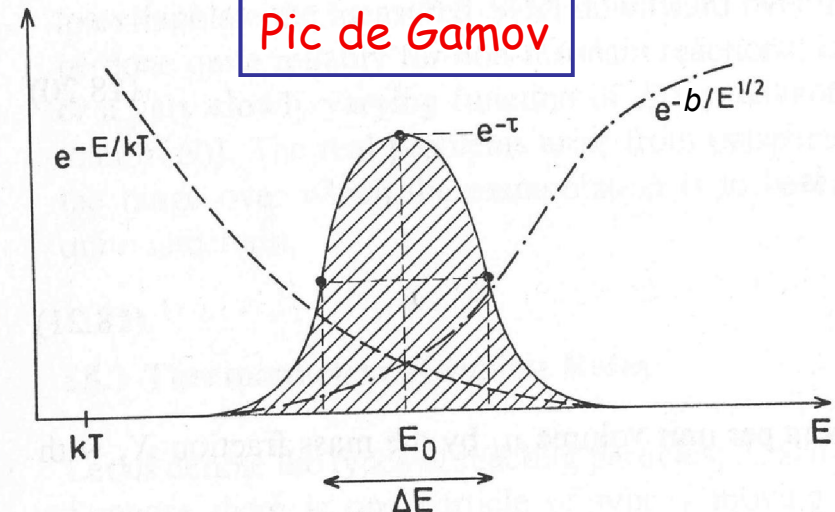
$$\begin{aligned} \langle \sigma v \rangle &\propto T^{-3/2} \Delta E e^{-f(E_0)} \\ &\propto T^{-3/2} T^{5/6} e^{-\tau} \\ &\propto \tau^2 e^{-\tau} \end{aligned}$$

$$f'(E_0) = 0 \Rightarrow E_0 = \left(\frac{bkT}{2} \right)^{2/3}$$

$$\tau = f(E_0) = 3E_0/(kT) \propto T^{-1/3}$$

$$\Delta E = \sqrt{8/f''} = 4\sqrt{E_0 kT/3} \propto T^{5/6}$$

Pic de Gamov



Les réactions nucléaires

$$\langle \sigma v \rangle \propto \tau^2 e^{-\tau}$$

$$T_7 = T(\text{K}) / 10^7$$

$$W = Z_j^2 Z_k^2 A = Z_j^2 Z_k^2 \frac{A_j A_k}{A_j + A_k},$$

$$\tau = 19.721 W^{1/3} T_7^{-1/3},$$

$$E_0 = 5.665 \text{ keV} \cdot W^{1/3} T_7^{2/3},$$

$$\frac{E_0}{kT} = \frac{\tau}{3} = 6.574 W^{1/3} T_7^{-1/3},$$

$$\Delta E = 4.249 \text{ keV} \cdot W^{1/6} T_7^{5/6},$$

$$\nu = \frac{\partial \ln \langle \sigma v \rangle}{\partial \ln T} = \frac{\tau}{3} - \frac{2}{3}$$
$$\simeq 6.574 W^{1/3} T_7^{-1/3}$$

Noyaux légers :

$$\nu \sim 6 \quad (\langle \sigma v \rangle \propto T^6)$$

Noyaux plus lourds :

$$\nu \sim 20-30 !! \quad (\langle \sigma v \rangle \propto T^{30})$$



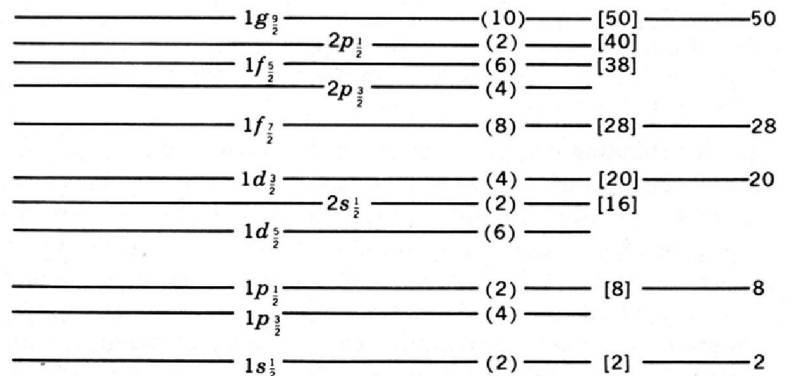
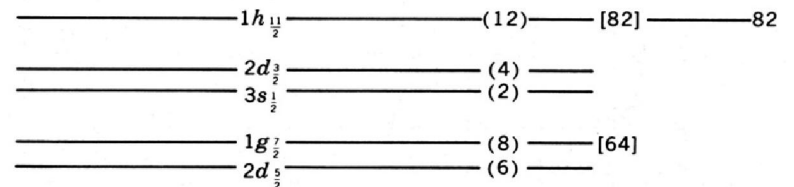
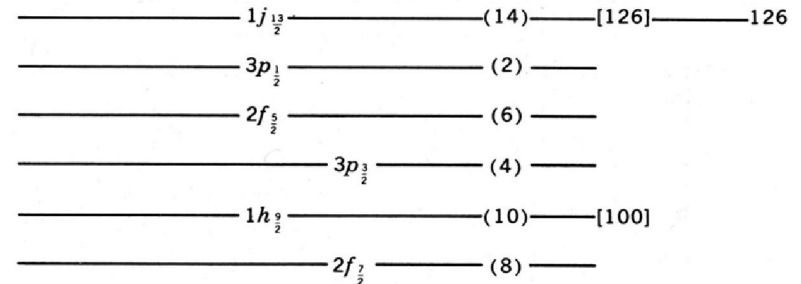
Extrêmement sensible
à la température !!

Les réactions nucléaires

Réactions résonnantes

Les noyaux ont des niveaux discrets d'énergie ~ atomes

Approx. en couches (shell-model) :
Chaque nucléon est plongé dans le potentiel fixe créé par les autres



Les réactions nucléaires

Réactions résonnantes

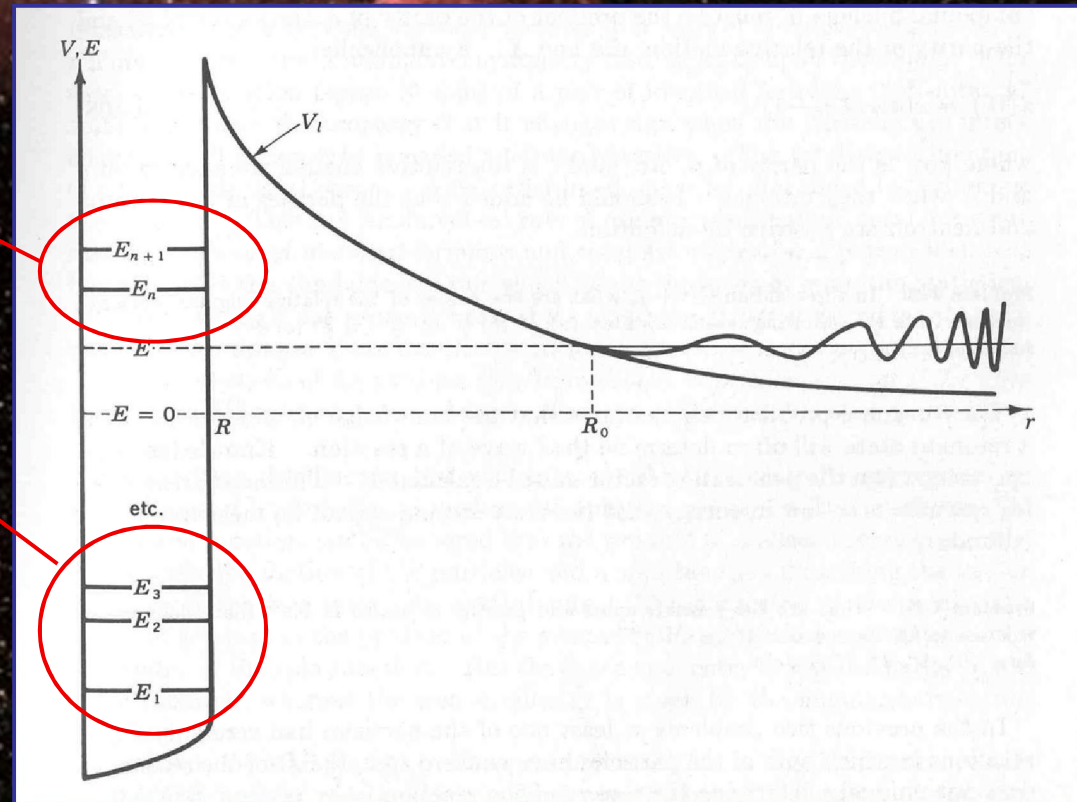
Les noyaux ont des niveaux discrets d'énergie ~ atomes

Niveaux quasi-stationnaires

Niveaux stationnaires

Si E proche de E_n

↓
Réaction résonnante



Les réactions nucléaires

$$\Gamma = \hbar/\tau$$

Réactions résonnantes

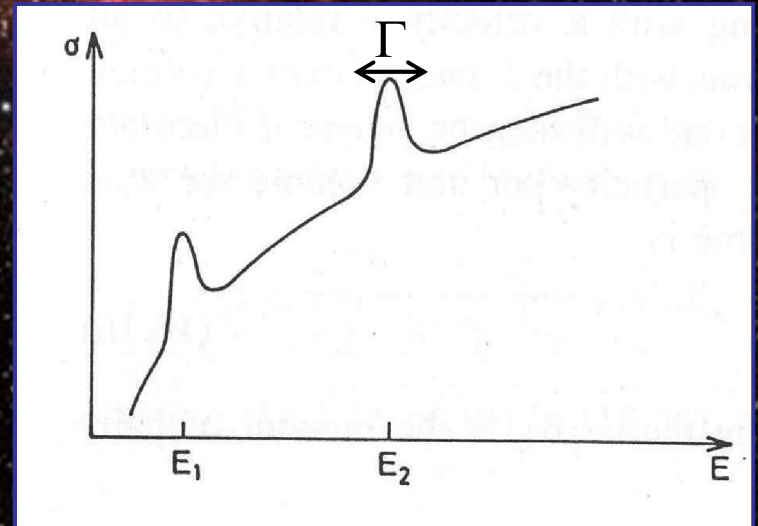
$S(E)$ non-constant $\longrightarrow$ Ne pas le sortir de l'intégrale

$$\langle \sigma v \rangle \propto e^{-E_{res}/(kT)}$$

Encore plus sensible
à la température
ex : réaction 3α : $\varepsilon \sim T^{40}$!!!

Autour de la résonance :

$$\sigma(E) \propto \frac{1}{(E - E_{res})^2 + (\Gamma/2)^2}$$



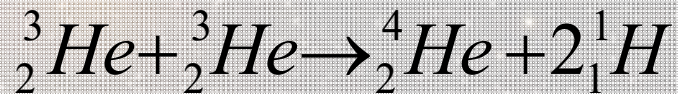
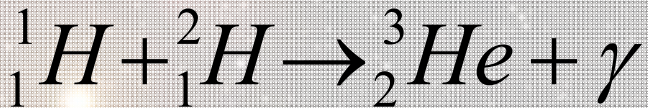
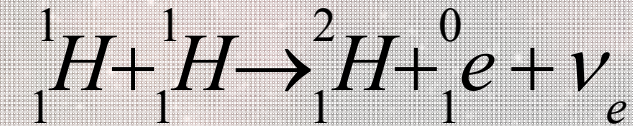
σ très grand pour les noyaux entrant en collision avec un moment cinétique et une énergie proches de la résonance

$$\text{Pour } E \sim E_{res}, \sigma \sim \sigma_{max} = (2l+1) \lambda^2 / (4\pi) = (2l+1) \pi (\hbar / (m_{\mu} v))^2$$

Combustion de l'hydrogène

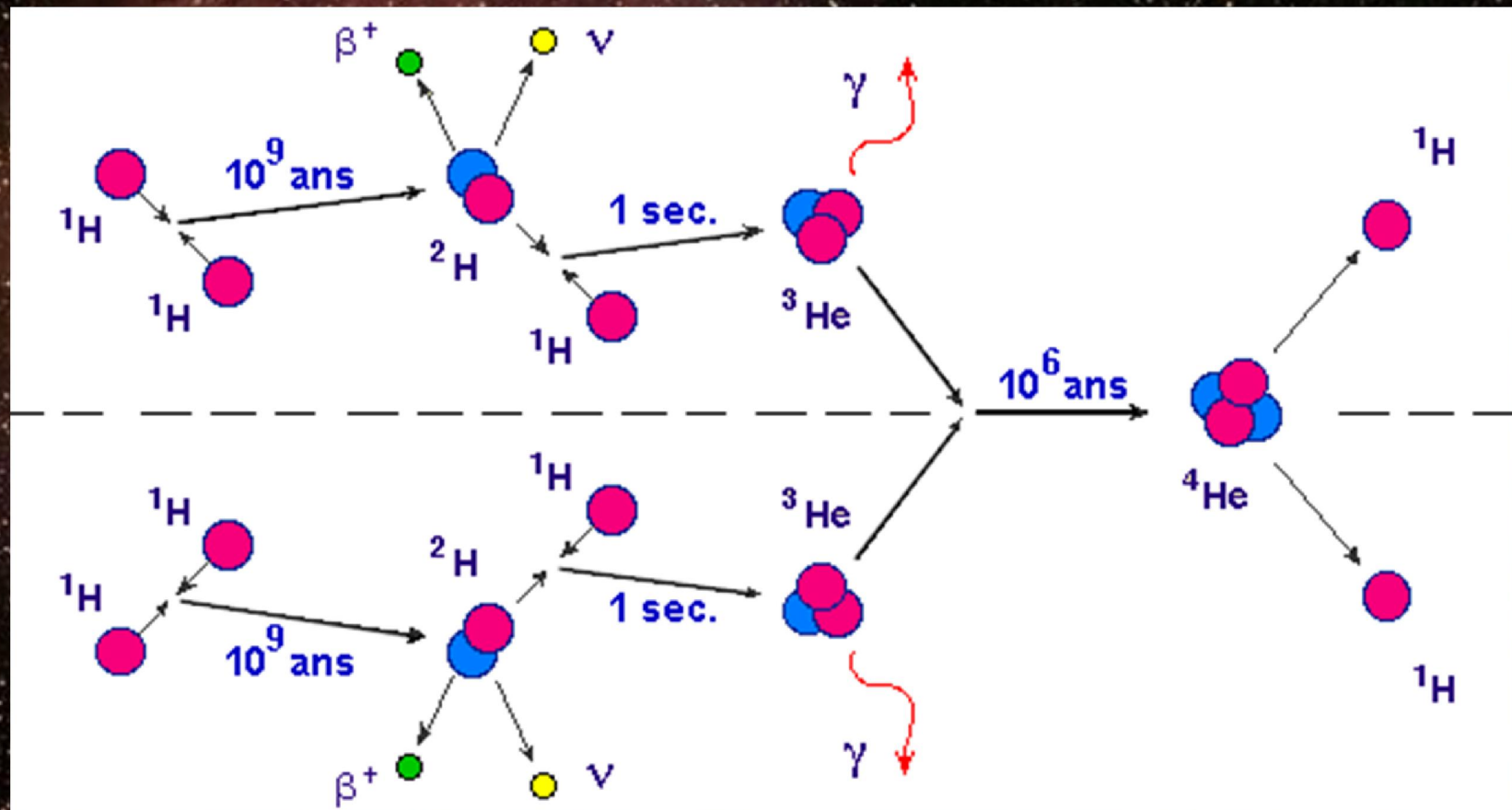
1. Chaîne proton-proton ($T > 10^7$ K)

PPI



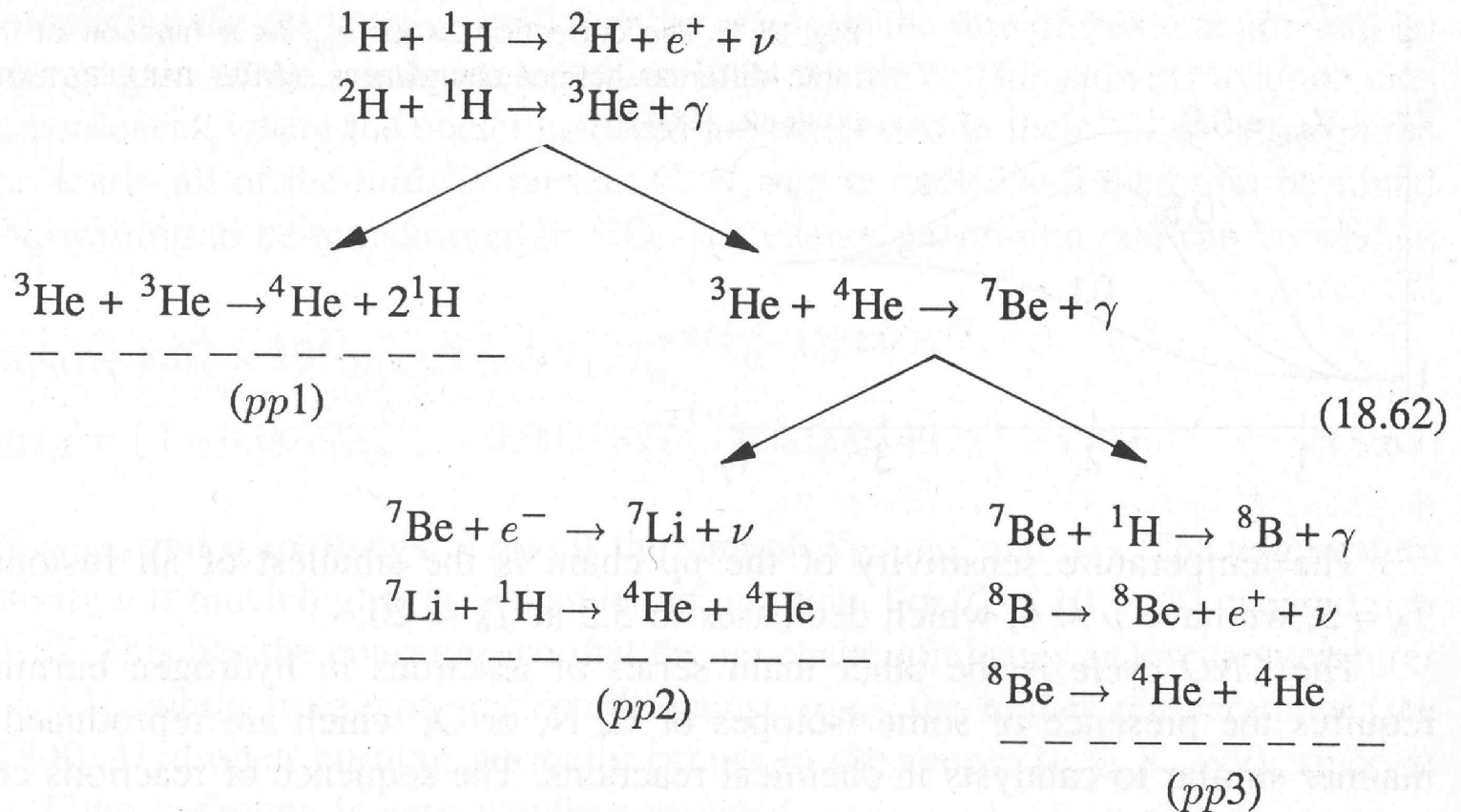
$$\begin{aligned} Q_{glob} &= 4\Delta\mathcal{M}_H - \Delta\mathcal{M}_{\text{He}^4} \\ &= 26.73 \text{ Mev} \end{aligned}$$

Chaîne proton-proton



Combustion de l'hydrogène

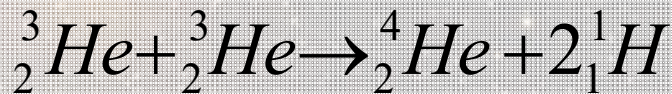
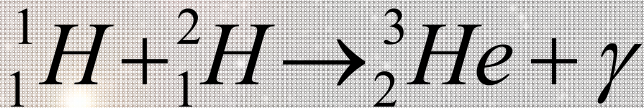
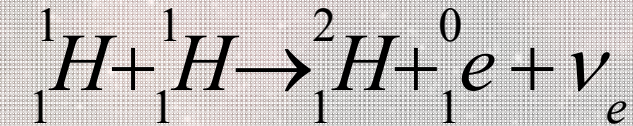
1. Chaîne proton-proton ($T > 10^7$ K)



Combustion de l'hydrogène

1. Chaîne proton-proton ($T > 10^7$ K)

PPI



$$\begin{aligned} Q_{glob} &= 4\Delta\mathcal{M}_H - \Delta\mathcal{M}_{He^4} \\ &= 26.73 \text{ Mev} \end{aligned}$$

Mais :

- Pertes par les neutrinos !
- Hors équilibre !

Energie du neutrino

$$\epsilon \neq \frac{Q_{glob}}{4m_u^2} \rho X^2 < \sigma v >_{pp}$$

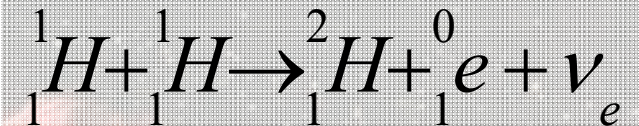
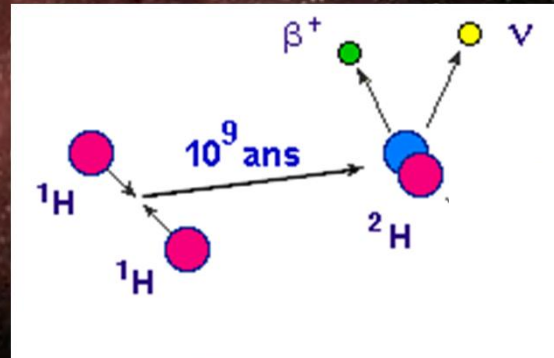
$$\epsilon = \sum_k \frac{Q_k - Q_{\nu,k}}{M_{1,k} M_{2,k}} \rho X_{1,k} X_{2,k} < \sigma v >_k$$

Somme sur les réactions

Combustion de l'hydrogène

1. Chaîne proton-proton ($T > 10^7$ K)

Réaction p-p



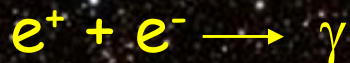
Franchissement de la barrière coulombienne suivi immédiatement de
 $p \longrightarrow n + e^+ + \nu$ (interaction faible)

Changement de spin du proton
 transformé en neutron $\longrightarrow$

Transition faible de
 Gamow-Teller

Probabilité très faible car mauvais
 recouvrement des fonctions d'onde
 $(\int \psi_f^* \psi_i dV \searrow)$ $\longrightarrow$

Temps de réaction très long :
 $\tau_H \sim 10^9$ ans



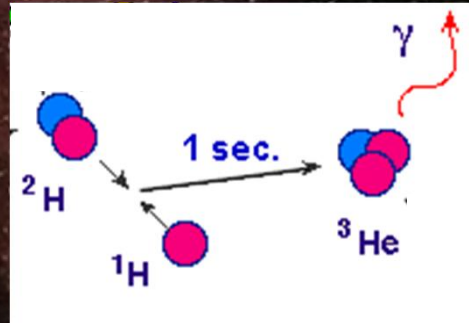
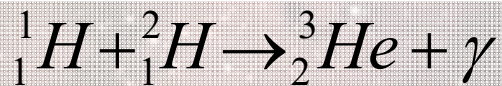
$E_{\text{neutrino}} \simeq 0.262$ Mev perdue $\longrightarrow$

$$\begin{aligned} Q &= 2\Delta\mathcal{M}_H - \Delta\mathcal{M}_D - E_{\text{neutrino}} \\ &= 1.442 - 0.262 = 1.180 \text{ Mev} \end{aligned}$$

Combustion de l'hydrogène

1. Chaîne proton-proton ($T > 10^7$ K)

Combustion du deuterium



Temps de réaction
très court :
 $\tau_D \sim 1$ sec.

$$0 \simeq \frac{dD}{dt} = \langle \sigma v \rangle_{pp} \frac{H^2}{2} - \langle \sigma v \rangle_{pd} HD$$
$$\rightarrow \left(\frac{D}{H} \right)_{eq} = \frac{\langle \sigma v \rangle_{pp}}{2 \langle \sigma v \rangle_{pd}} = \frac{\tau_D}{2\tau_H} \approx 10^{-17}$$

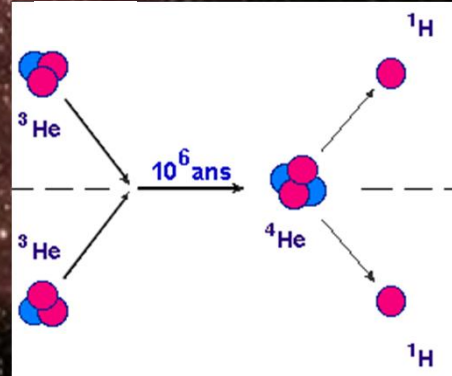
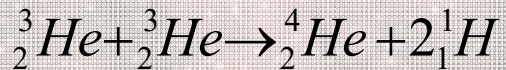
Sur terre :
 $D/H = 1.5 \times 10^{-4}$
(spallation)

Première phase de combustion du Deuterium
initial, puis équilibre.

Combustion de l'hydrogène

1. Chaîne proton-proton ($T > 10^7$ K)

Combustion $\text{He}_3 + \text{He}_3$



Deuterium à l'équilibre

$$\frac{d\text{He}^3}{dt} = \langle \sigma v \rangle_{pp} \frac{H^2}{2} - \langle \sigma v \rangle_{33} (\text{He}^3)^2$$

Au début, réaction très lente !!

$$\left(\frac{\text{He}^3}{H} \right)_{eq} = \sqrt{\frac{\langle \sigma v \rangle_{pp}}{2 \langle \sigma v \rangle_{33}}}$$

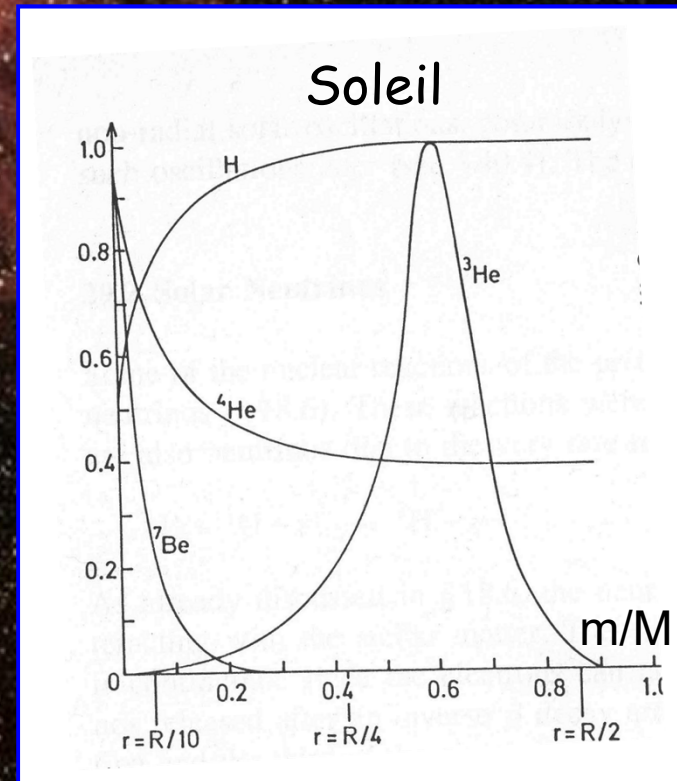
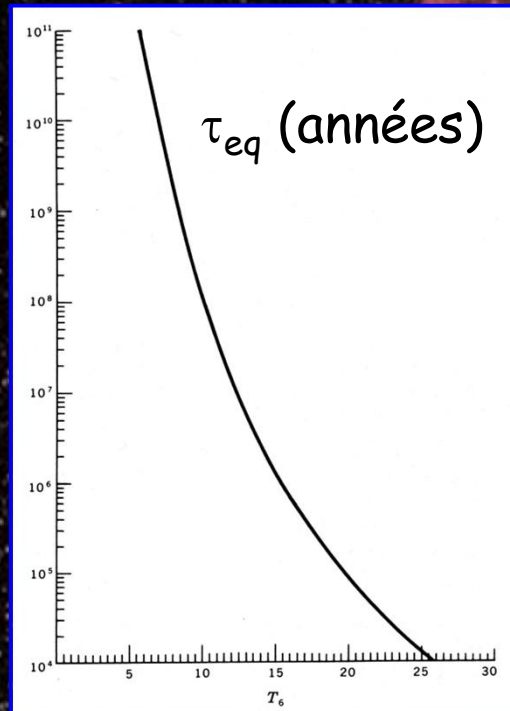
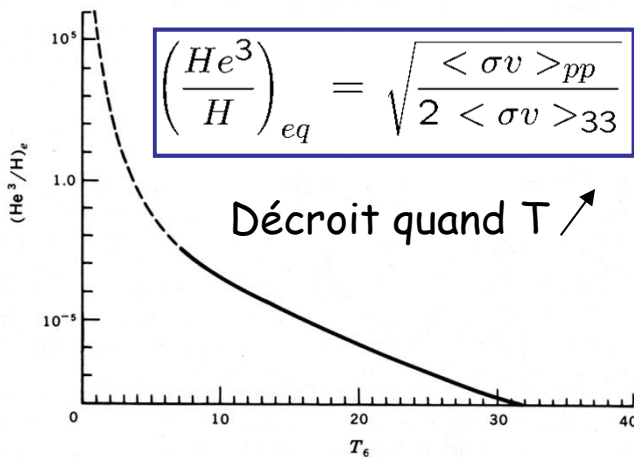
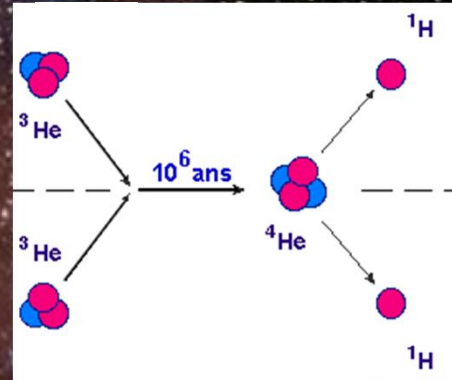
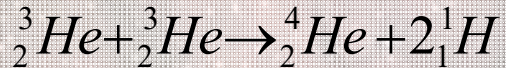
Décroît quand $T \nearrow$

Pour T trop petit, He^3 n'a jamais le temps d'atteindre sa valeur d'équilibre, donc bosse

Evolution de la composition chimique

1. Chaîne proton-proton

Combustion $\text{He}_3 + \text{He}_3$



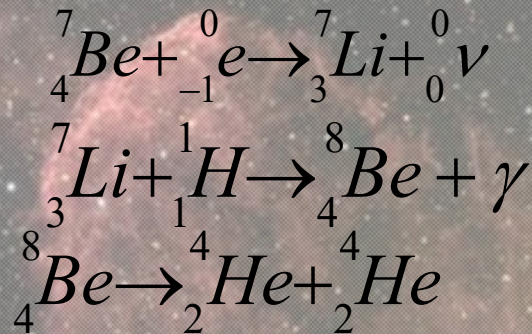
Combustion de l'hydrogène

1. Chaîne proton-proton ($T > 10^7$ K)

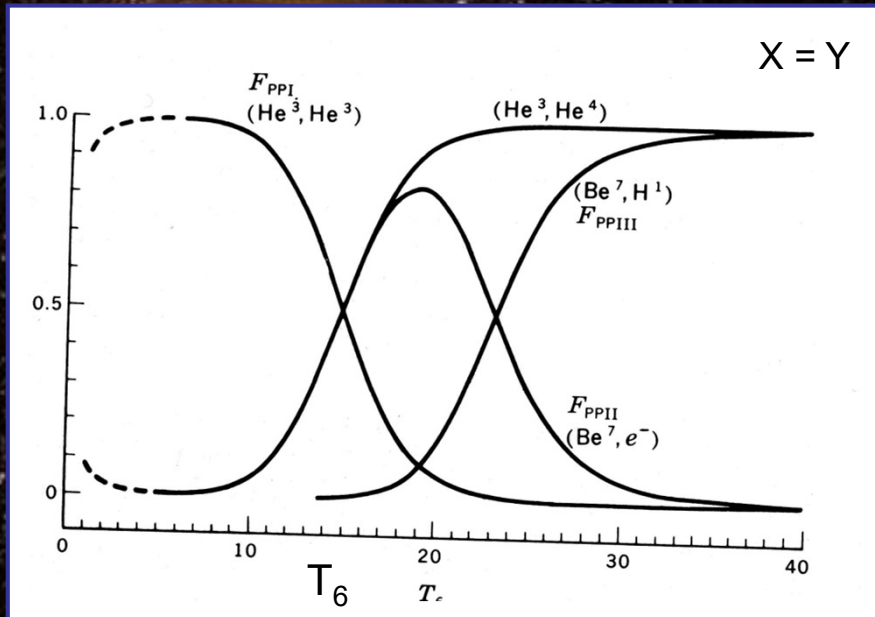
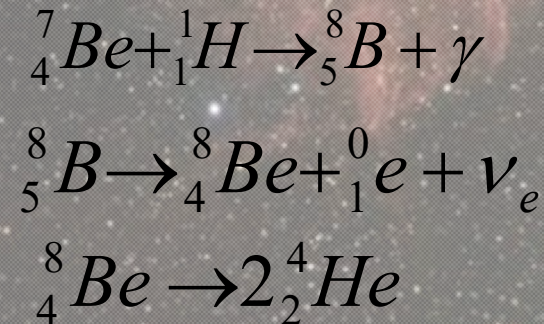
Chaînes PPII et PPIII



PPII



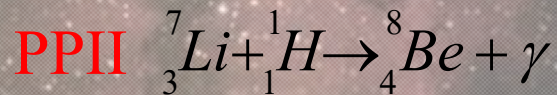
PPIII



Evolution de la composition chimique

1. Chaîne proton-proton

Problème du Lithium



$$T_{\text{Li}} \sim 2 \cdot 10^6 \text{ K}$$

L'abondance de Lithium à la surface du Soleil est 140 * plus petite que sa valeur initiale

L'étude d'amas jeunes (Hyades, ...) montre que l'abondance en surface diminue au cours du temps



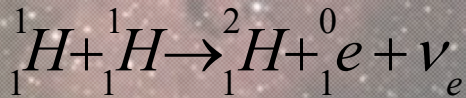
Mécanismes hydrodynamiques non-standards de transport des régions profondes vers la surface : diffusion, circulation méridienne, **ondes internes**, ...

Combustion de l'hydrogène

1. Chaîne proton-proton ($T > 10^7$ K)

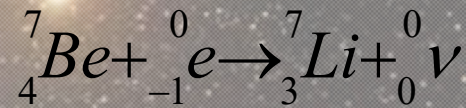
Différentes énergies perdues
par les neutrinos :

PPI



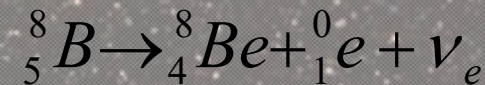
$$E_\nu = 0.263 \text{ MeV} \quad \text{Perte : } \frac{2 \times 0.263}{26.73} = 2 \%$$

PPII

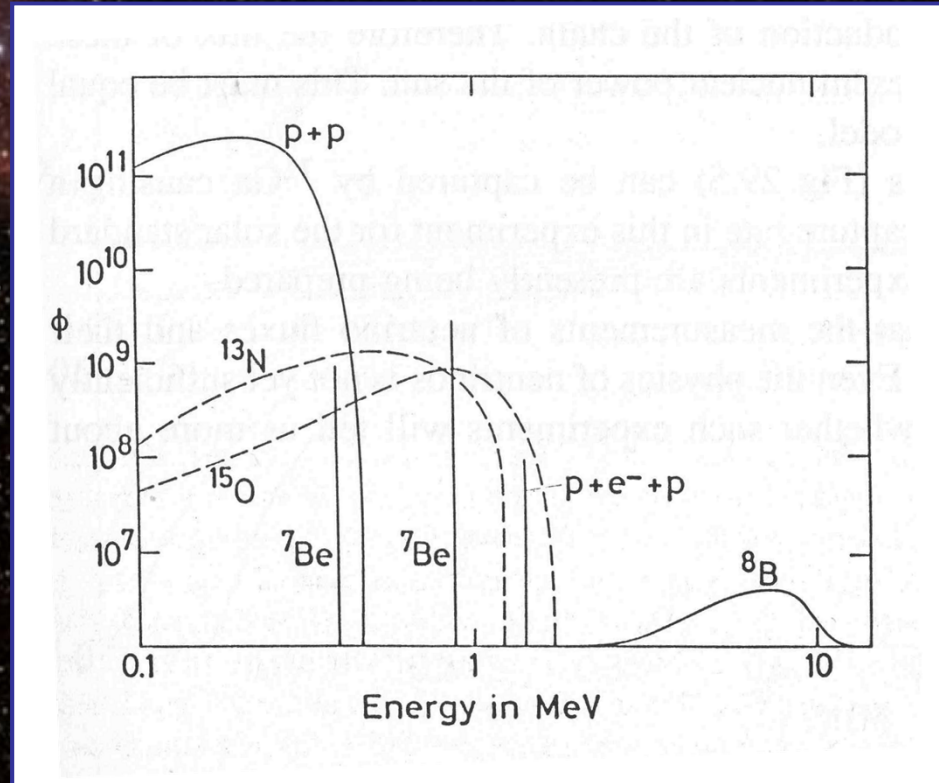


$$E_\nu = 0.8 \text{ MeV} \quad \text{Perte : } \frac{0.263 + 0.8}{26.73} = 4 \%$$

PPIII

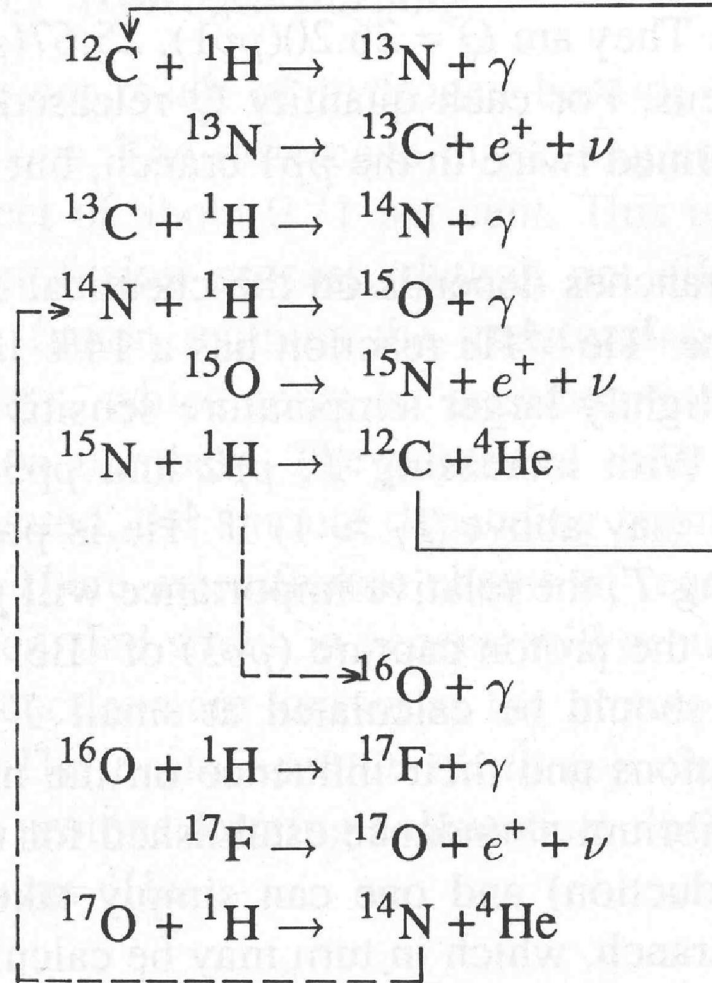


$$E_\nu = 7.2 \text{ MeV} \quad \text{Perte : } \frac{0.263 + 7.2}{26.73} = 27.9 \%$$

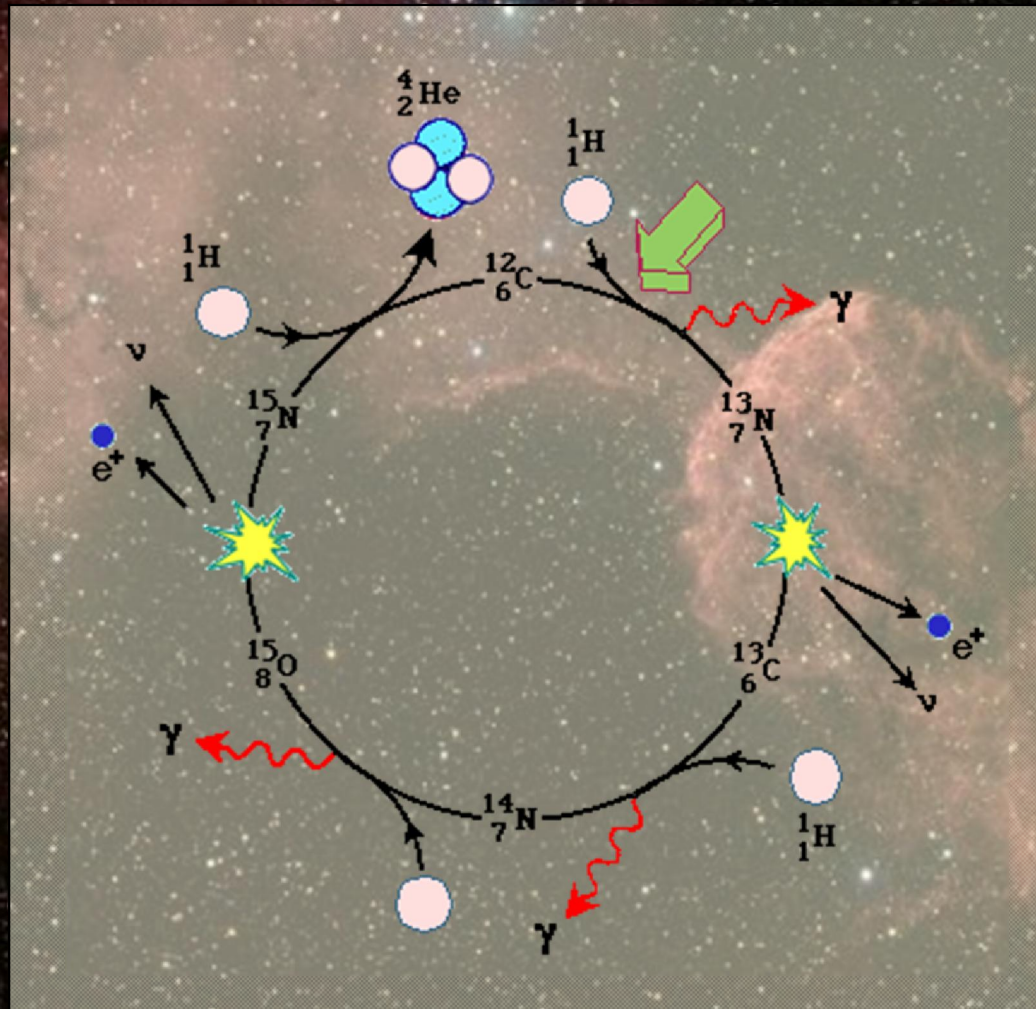


Combustion de l'hydrogène

2. Cycle CNO $T > 17 \times 10^6 \text{ K}$



Cycle CNO

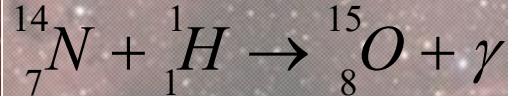


Combustion de l'hydrogène

2. Cycle CNO $T > 17 \times 10^6 \text{ K}$

Boucle principale :

Réaction la plus lente :



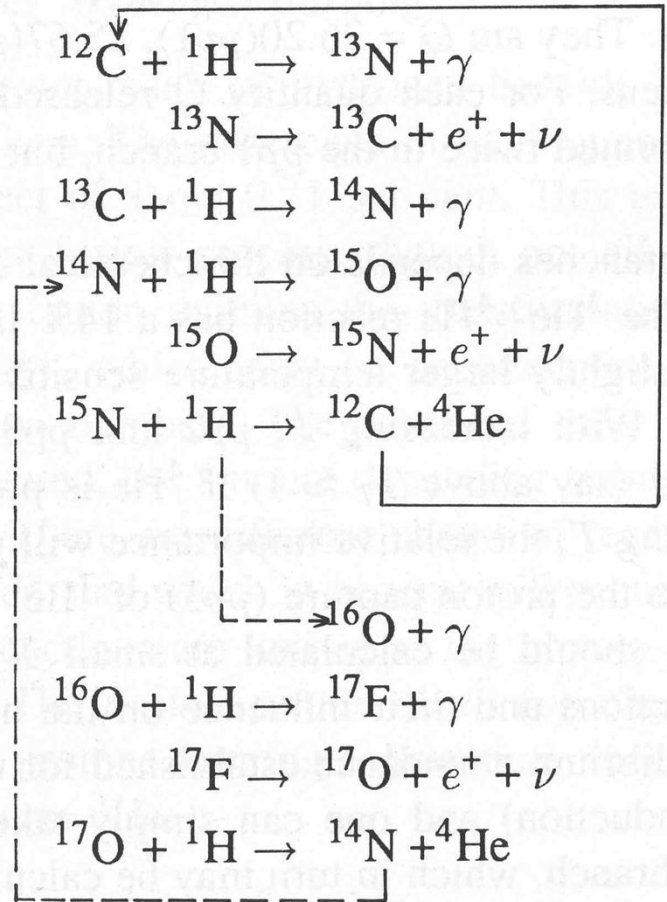
A l'équilibre, il ne reste presque que N^{14} (goulot)

$$\begin{bmatrix} C^{12} \\ C^{13} \\ N^{14} \end{bmatrix} = N \begin{bmatrix} 0.0122 \\ 0.00305 \\ 0.985 \end{bmatrix}$$

Boucle secondaire:

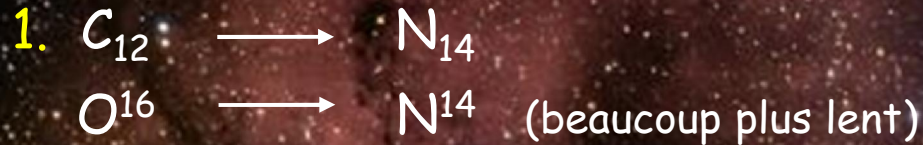
Rapport de branchement
très bas : $\sim 4 \times 10^{-4}$

Rôle principal:
conversion $O^{16} \longrightarrow N^{14}$



Evolution de la composition chimique

2. Cycle CNO $T > 15 \cdot 10^6 \text{ K}$



Au départ :

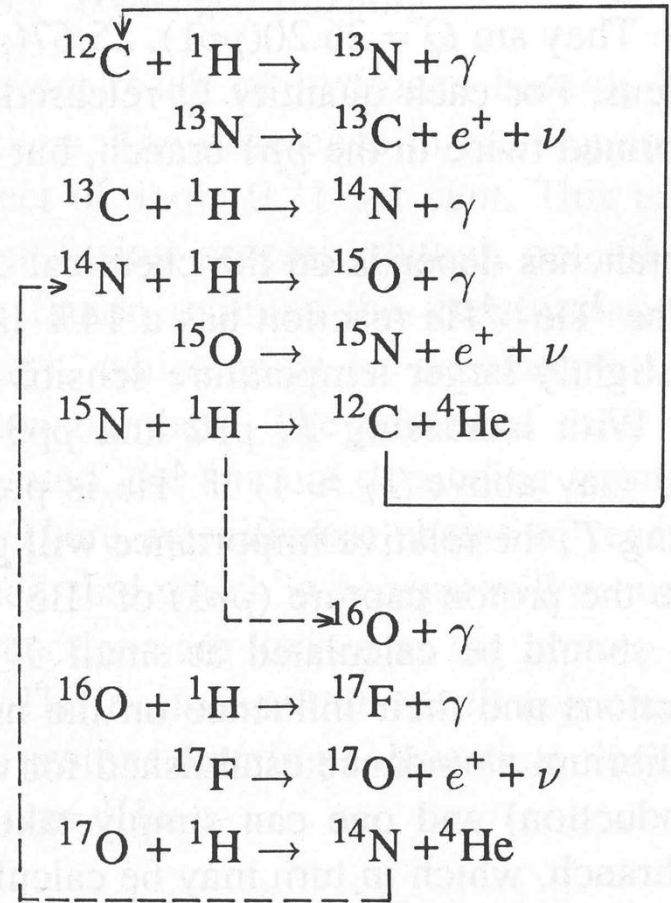
$$\text{C}^{12} : \text{N}^{14} : \text{O}^{16} = 5.5 : 1 : 9.5$$

A l'équilibre (typiquement) :

$$\text{C}^{12} : \text{N}^{14} : \text{O}^{16} = 0.15 : 15 : 0.3$$

2. Evolution complexe car :

- Présence d'un coeur convectif qui homogénise et dont la taille change
- Certains constituants n'atteignent jamais la valeur d'équilibre



Combustion de l'hydrogène

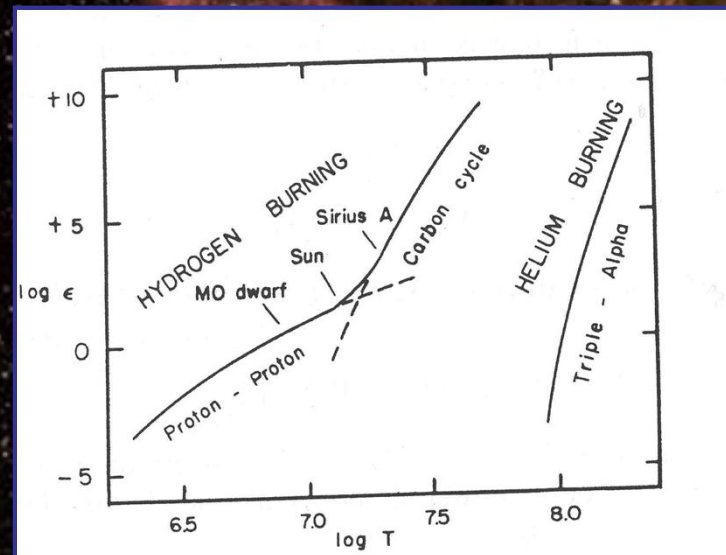
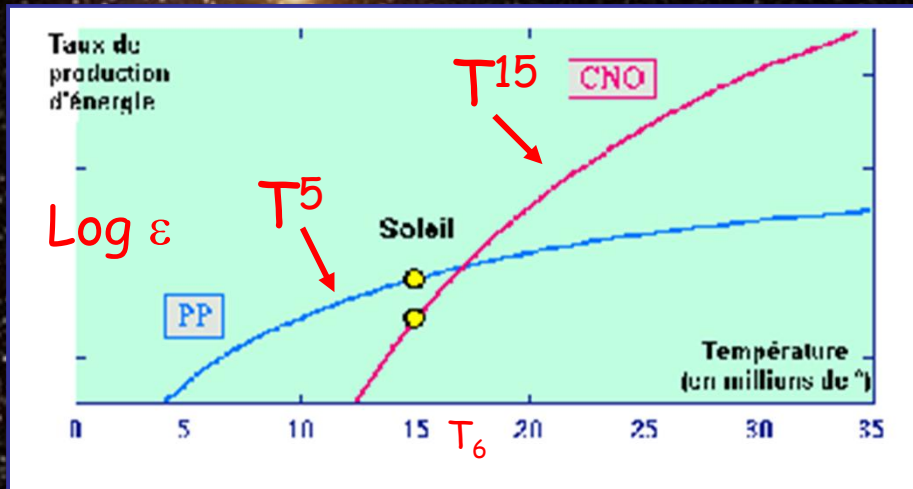
Cycle CNO versus chaîne P-P

$$\varepsilon_{pp} \propto T^5 \quad \varepsilon_{CNO} \propto T^{15}$$

Le cycle CNO domine
pour $T > 17 \times 10^6 \text{ K}$
→ Étoiles massives

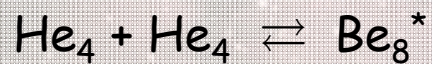
$$W = Z_j^2 Z_k^2 A = Z_j^2 Z_k^2 \frac{A_j A_k}{A_j + A_k}$$

$$\nu = \frac{\partial \ln \langle \sigma v \rangle}{\partial \ln T} \simeq 6.574 W^{1/3} T_7^{-1/3}$$

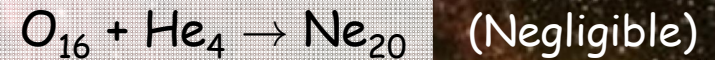
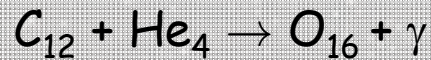
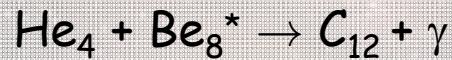


Helium burning:

3α reaction ($T \sim 10^8$ K)



Life time $\sim 10^{-16}$ s ($\text{Be}_8^*/\text{He}_4$) $\sim 10^{-9}$

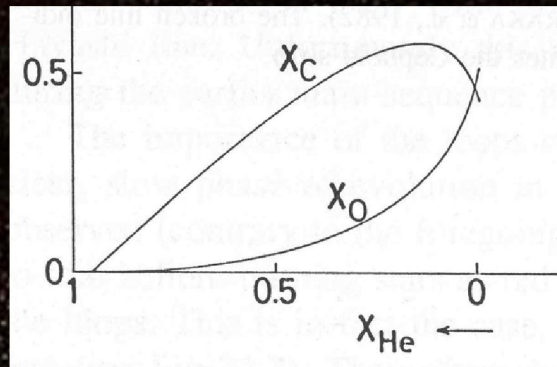


Resonant reaction $\rightarrow$ Very sensitive to T $\epsilon \propto \rho^2 T_8^{-3} e^{-44/T_8} \propto \rho^2 T^{40} (T = 10^8 \text{ K})$

$$E_{3\alpha} = 3 \Delta M_{\text{He}_4} - \Delta M_{\text{C}_{12}}$$

$$= 7.375 \text{ Mev}$$

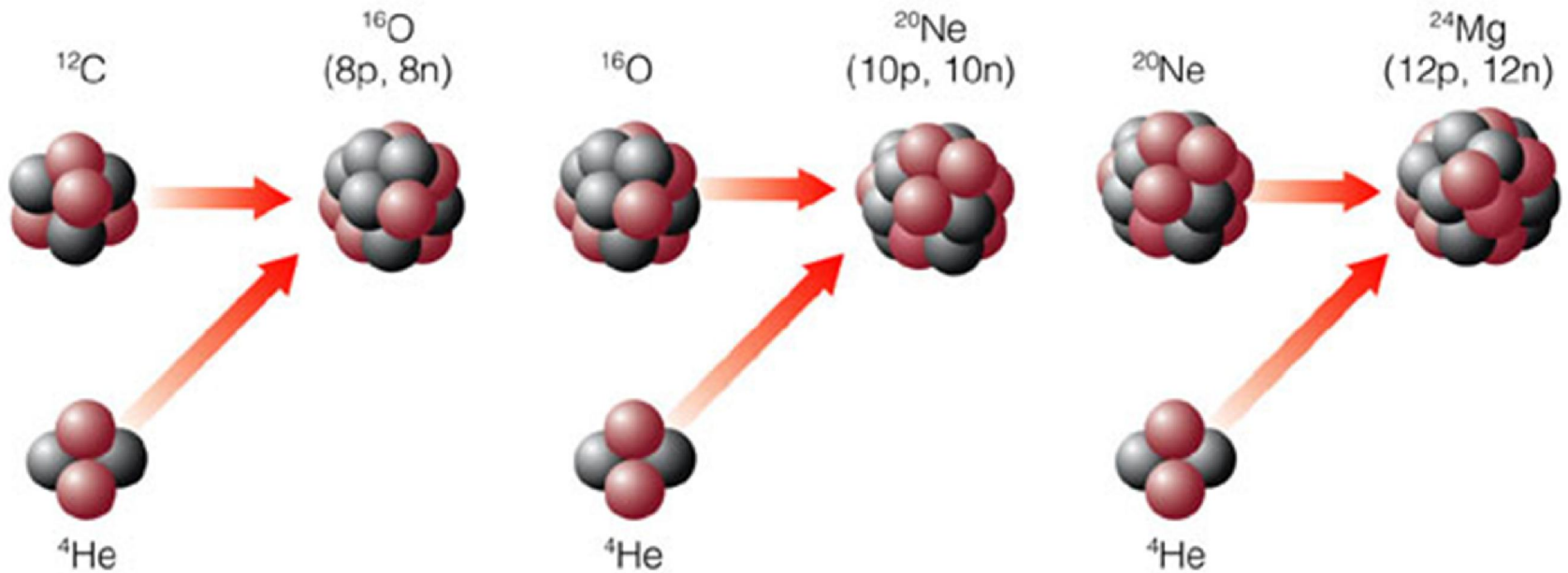
$$\ll 26.7 \text{ Mev (pp)}$$



Finally : $\text{C}_{12} \approx \text{O}_{16}$

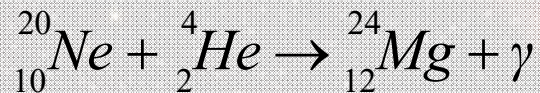
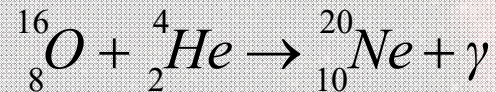
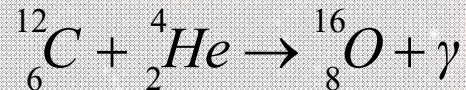
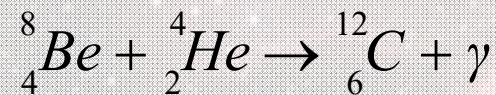
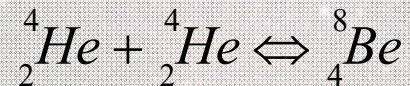
α Captures

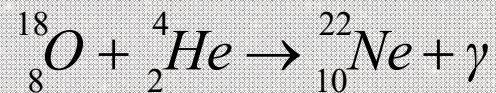
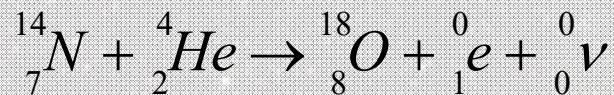
Helium-capture reactions



Helium burning

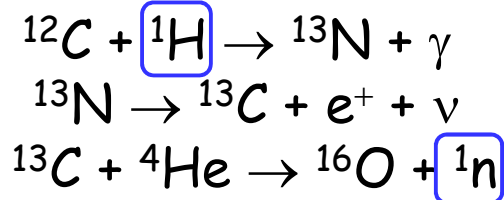
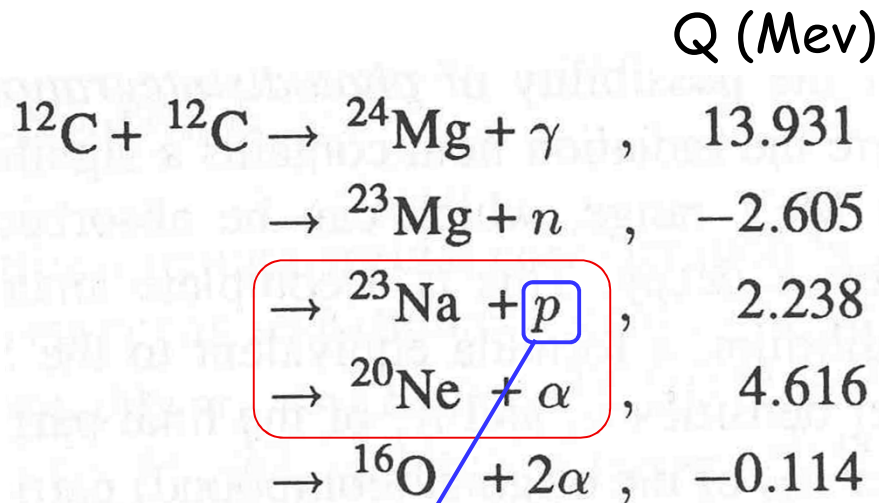
3 α process and α captures ($T > 100 \cdot 10^6$ K)



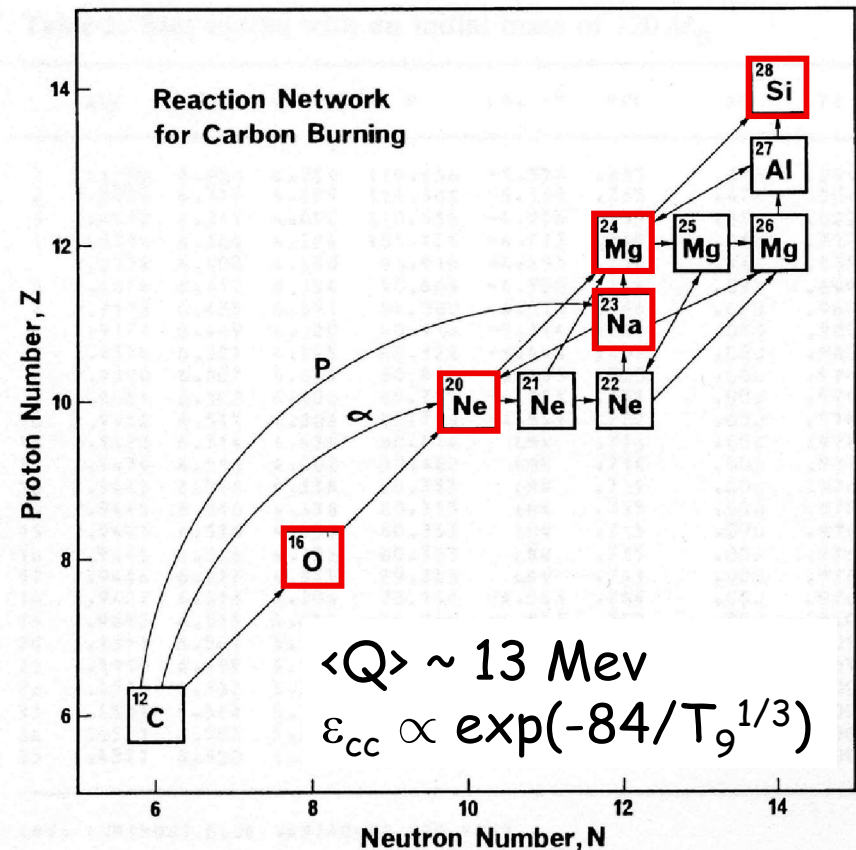


Advanced phases of nuclear reactions in high mass stars

Carbon burning ($T \sim (0.5-1) \times 10^9$ K)



Source of neutrons



Advanced phases of nuclear reactions in high mass stars

Photo-disintegration of neon ($T \sim 1.5 \times 10^9$ K)



« Thermodynamic » equilibrium ${}_8^{16}\text{O} + {}_2^4\text{He} \rightleftharpoons {}_{10}^{20}\text{Ne}$

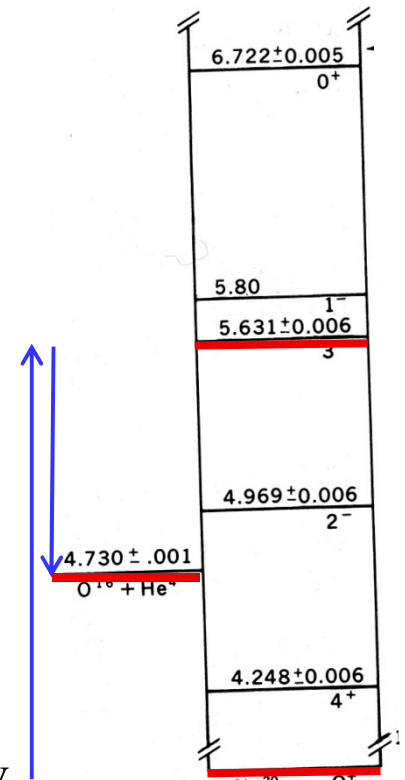
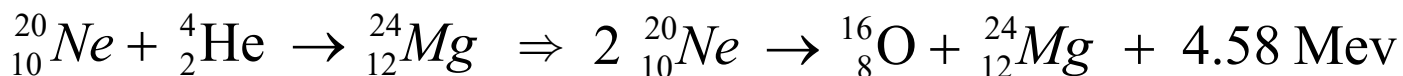
$$\left(\frac{n_1 n_2}{n_{12}} \right)_e = \frac{(2\pi\mu kT)^{\frac{3}{2}}}{h^3} \frac{G_1 G_2}{G_{12}} \exp - \frac{Q}{kT} \quad \sim \text{Saha}$$

$T \nearrow \Rightarrow$ equilibrium towards more de photo-disintegration

As much reactions in each sense $\rightarrow$ reaction rate :

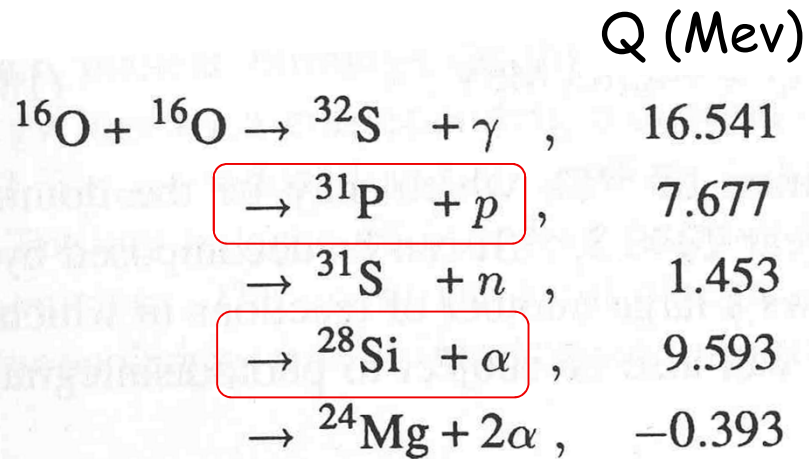
$$\lambda_\gamma = \frac{(2\pi\mu kT)^{\frac{3}{2}}}{h^3} \frac{G_1 G_2}{G_{12}} \exp \left(- \frac{Q}{kT} \right) \lambda_{12} \quad \log \lambda_\gamma (s^{-1}) = 12.7 - \frac{28.4}{T_9}$$

The liberated α nuclei are immediately captured:



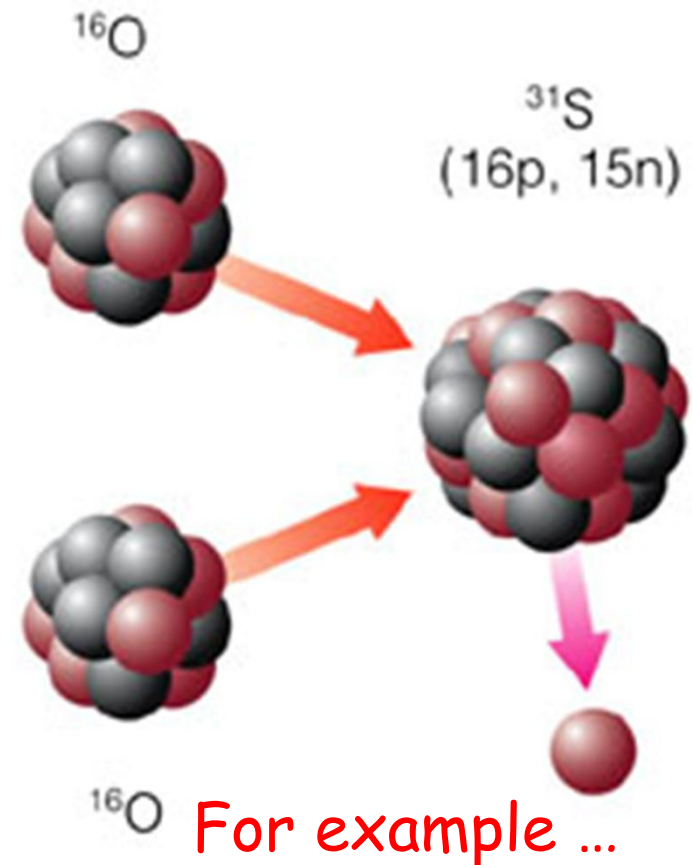
Advanced phases of nuclear reactions in high mass stars

Oxygen burning ($T \sim 2 \times 10^9$ K)



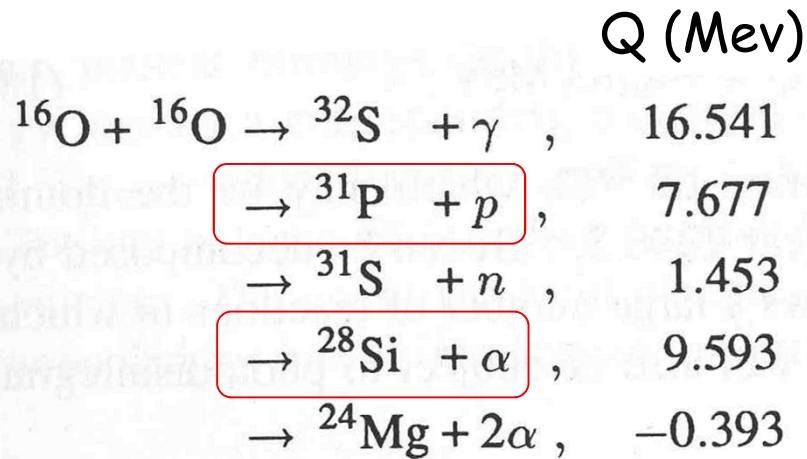
$\langle Q \rangle \sim 16$ Mev

Final product:
2/3 of ^{28}Si for 1/3 of ^{32}S



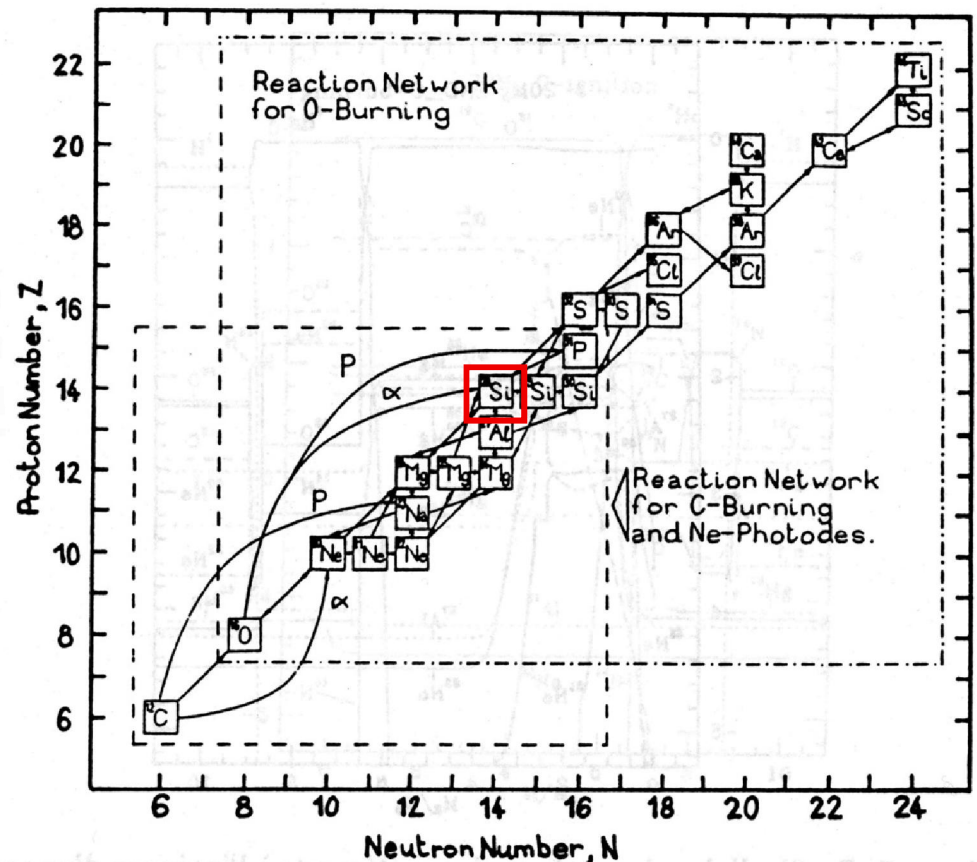
Advanced phases of nuclear reactions in high mass stars

Oxygen burning ($T \sim 2 \times 10^9$ K)



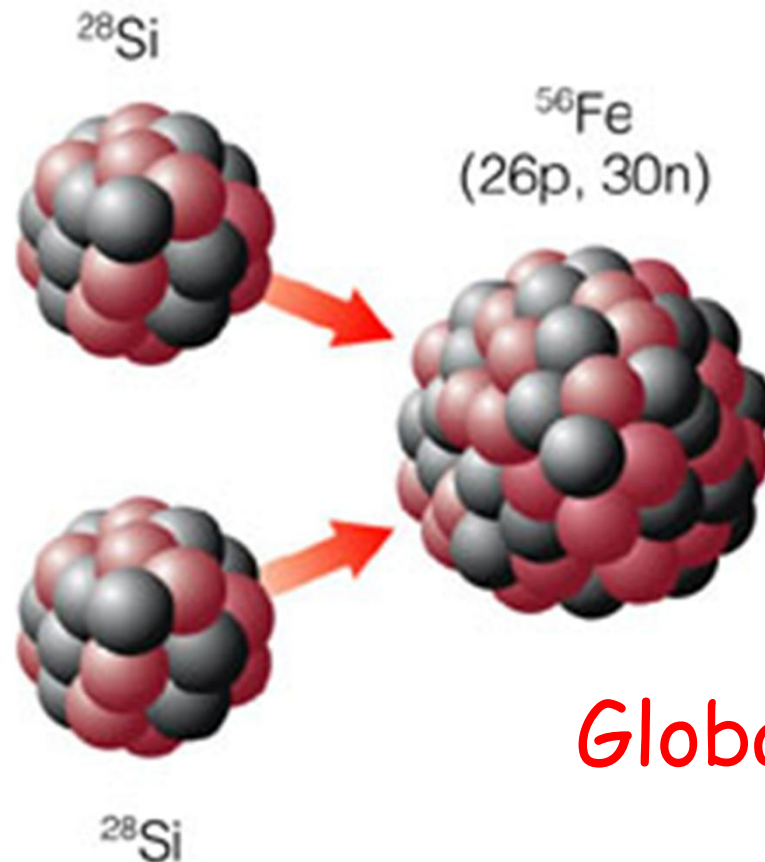
$\langle Q \rangle \sim 16$ Mev

Final product:
2/3 of ^{28}Si for 1/3 of ^{32}S



Advanced phases of nuclear reactions in high mass stars

Silicon « burning » ($T \sim 3 \times 10^9$ K)



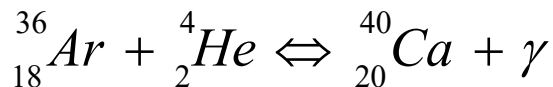
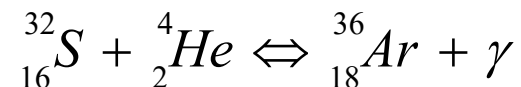
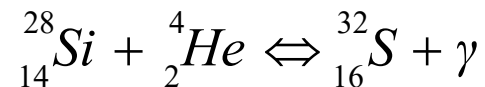
Global result ...

Advanced phases of nuclear reactions in high mass stars

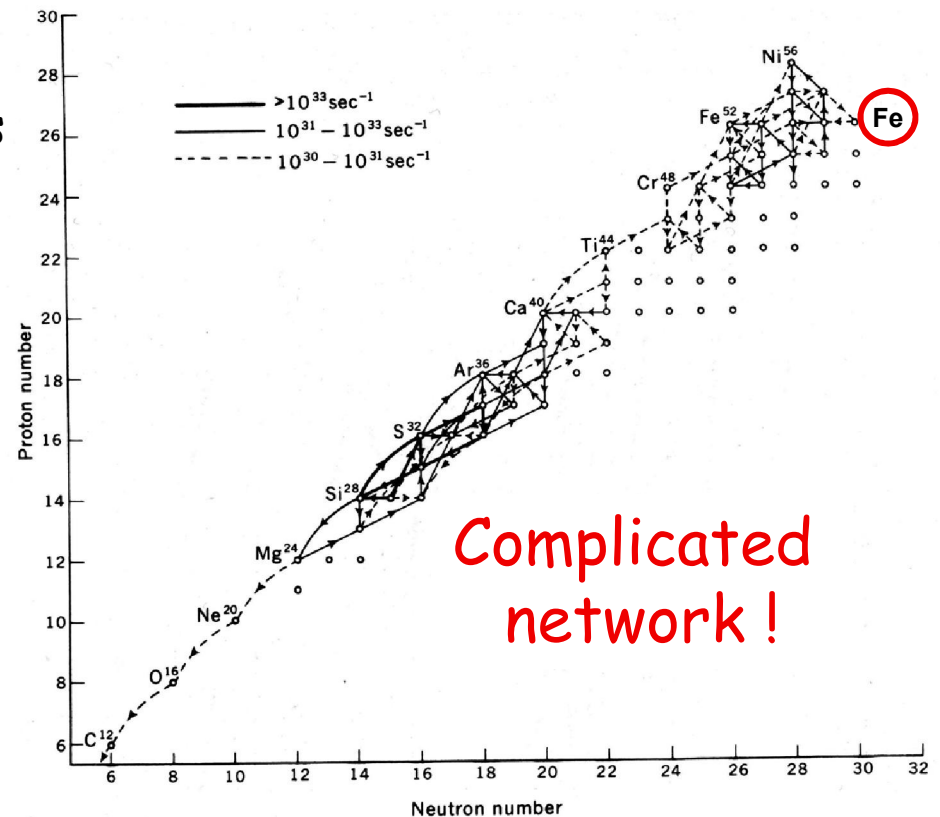
Silicon « burning » ($T \sim 3.3 \times 10^9$ K)

$kT \ll Q$ for the weakly linked nuclei
 $\Rightarrow$ photodisintegrations and captures
 towards more stable nuclei
 $\Rightarrow$ rearrangement: Fe, Ni $\nearrow$, ...

α captures:



⋮



Advanced phases of nuclear reactions in high mass stars

Nuclear thermodynamic equilibrium ($T \sim 7 \times 10^9$ K)

$kT \ll Q$ for most reactions $\Rightarrow$ nuclear thermodynamic equilibrium

$\Rightarrow$ Equal rates of direct and inverse reactions

$\Rightarrow$ Abundance ratios given by « Saha-like » equations:

$$\frac{n_n N_{A-1,Z}}{N_{A,Z}} \propto T^{3/2} \exp\left(-\frac{Q_n}{kT}\right)$$


$$N_{A,Z} = G_{A,Z} A^{3/2} \frac{n_p^Z n_n^{A-Z}}{2^A} \left(\frac{(2\pi M_{A,Z} kT)^{3/2}}{h^3} \right)^{1-A} \exp\left(\frac{Q_{A,Z}}{kT}\right)$$

$$Q_{A,Z} = c^2 [Z M_H + (A - Z) M_n - M(A, Z)]$$

All abundances are obtained if n_p and n_n are given $\Rightarrow$ 2 additional constraints are required :

$$\rho = m_u \sum_i \mu_i N_i \quad \frac{\bar{Z}}{\bar{N}} = \frac{\sum Z N_{A,Z} + n_p}{\sum (A - Z) N_{A,Z} + n_n} \quad \begin{array}{l} \text{Fixed because equilibrium well} \\ \text{before the } \beta \text{ disintegrations} \end{array}$$

Advanced phases of nuclear reactions in high mass stars

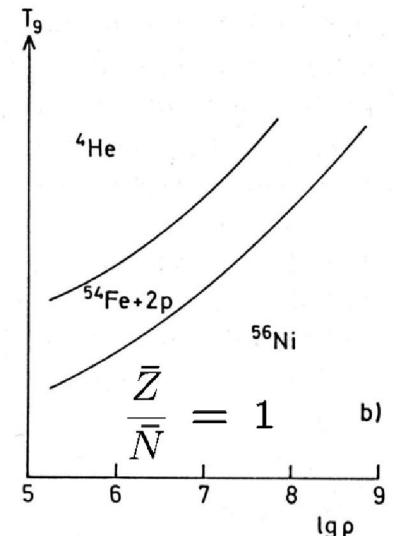
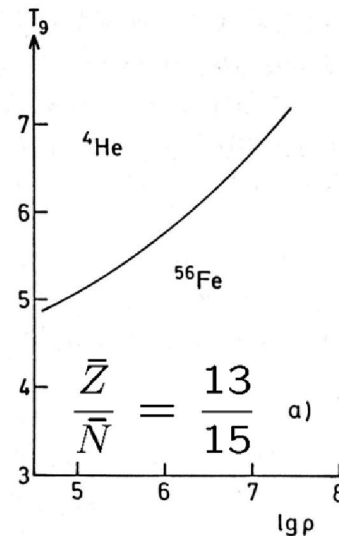
Nuclear thermodynamic equilibrium ($T \sim 7 \times 10^9$ K)

$$N_{A,Z} = G_{A,Z} A^{3/2} \frac{n_p^Z n_n^{A-Z}}{2^A} \left(\frac{(2\pi M_{A,Z} kT)^{3/2}}{h^3} \right)^{1-A} \exp \left(\frac{Q_{A,Z}}{kT} \right)$$

1) $kT \ll \text{not too large} \Rightarrow$ most abundant elements :
maximum of $f_{A,Z} = Q_{A,Z} / A \Rightarrow$ mainly Ni, next Fe

2) If enough time, β disintegrations
 $\Rightarrow Z/N \searrow (dN_i = -N_i dt/\tau_i)$
 $\Rightarrow$ mainly ^{56}Fe

3) $kT > Q \Rightarrow$ photodisintegrations
 $\Rightarrow ^4\text{He}, \dots \nearrow$



Core of a pre-supernova

Duration of ultimate phase ($15 M_{\odot}$) :

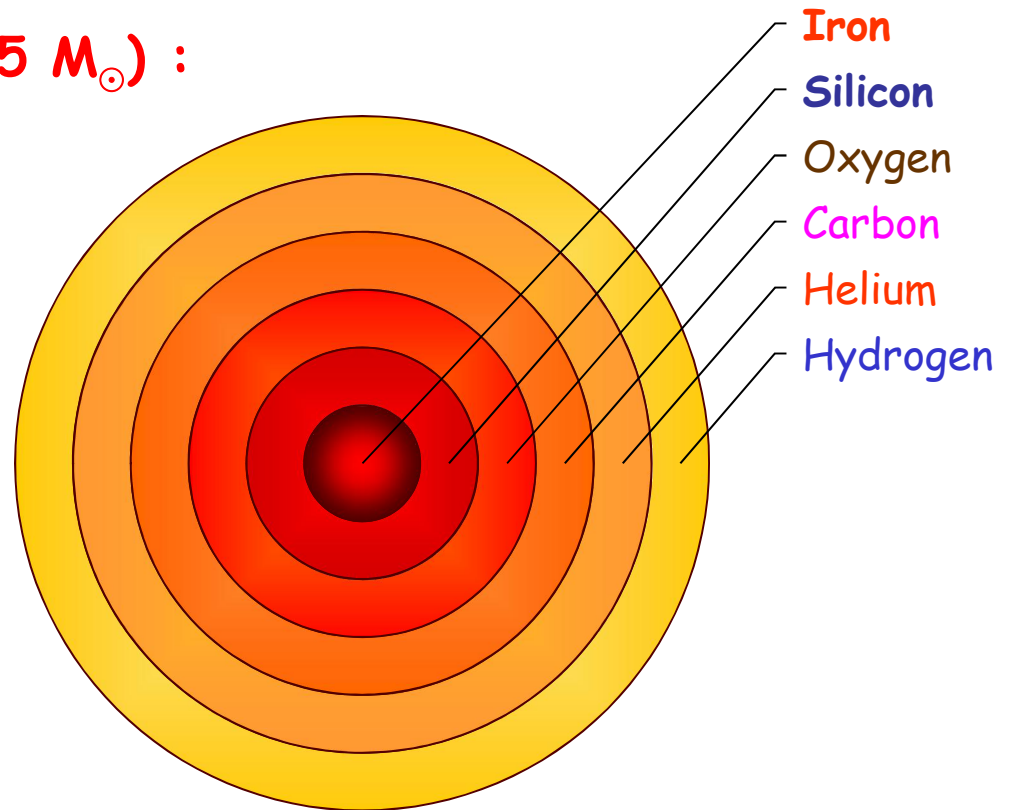
Burning of

Carbon : ~ 6000 ans

Neon : ~ 7 ans

Oxygen : ~ 2 ans

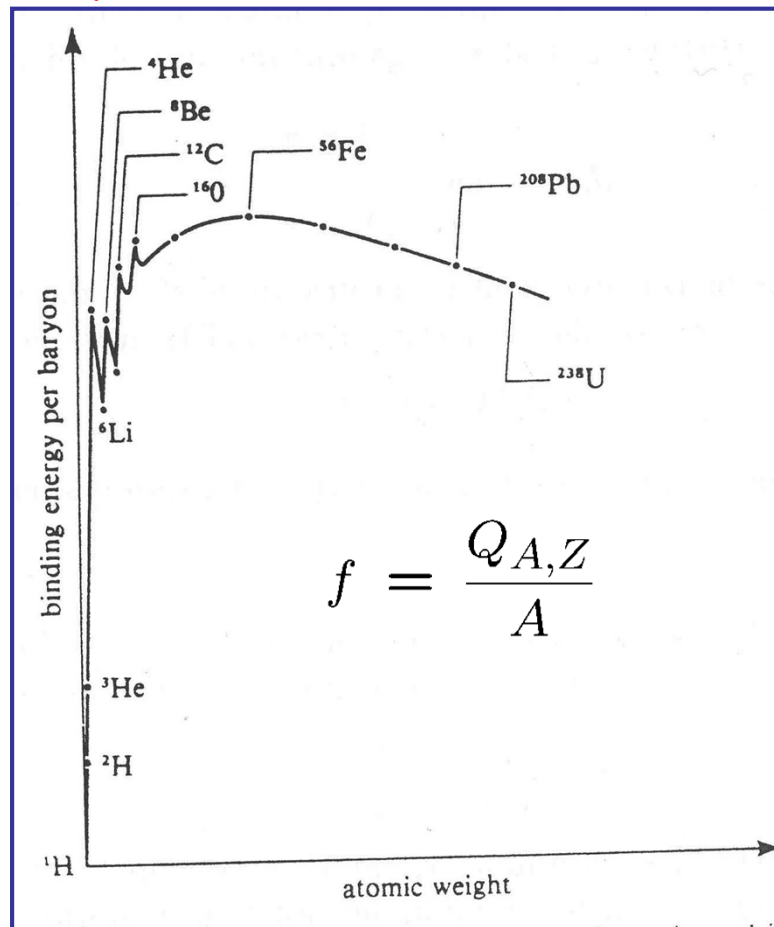
Silicon : ~ 6 jours



Onion-like structure

Advanced phases of nuclear reactions in high mass stars

Nuclear thermodynamic equilibrium ($T \sim 7 \times 10^9$ K)



Equations de structure

Conservation
de la masse

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi \rho r^2}$$

$$\rho = \rho(P, T, X_i)$$

$$\kappa = \kappa(P, T, X_i)$$

$$\varepsilon = \varepsilon(P, T, X_i)$$

Equilibre
hydrostatique

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}$$

Transfert
d'énergie

Radiation



Convection

$$\frac{\partial T}{\partial m} = -\frac{3\kappa L}{64\pi^2 ac r^4 T^3}$$

$$\frac{\partial T}{\partial m} = -\nabla_{\text{ad}} \frac{Gm}{4\pi r^4} \frac{T}{P}$$

Conservation
de l'énergie

$$\frac{\partial L}{\partial m} = \varepsilon$$

+ 4 conditions
aux limites

Méthode de résolution numérique

Calcul d'un modèle statique: composition chimique fixée en chaque couche

Posons: $\bar{Y} = (r, P, T, L)$

Non-linéaire !

Système à résoudre: $d\bar{Y}/dm = \bar{f}(\bar{Y})$ (+ 4 conditions aux limites)

Méthode des différences finies $\Rightarrow$ N couches: $i=1, \dots, N$

$$\frac{\bar{Y}_{i+1} - \bar{Y}_i}{m_{i+1} - m_i} = \bar{f}\left(\frac{\bar{Y}_{i+1} + \bar{Y}_i}{2}\right)$$

Algorithme de Newton-Raphson

$\Rightarrow$ Linéarisation du système autour de l'estimation à l'étape k:

$$\frac{\delta \bar{Y}_{i+1} - \delta \bar{Y}_i}{m_{i+1} - m_i} - J_{\bar{f}}(\delta \bar{Y}_{i+1} + \delta \bar{Y}_i)/2 = \bar{f}\left(\frac{\bar{Y}_{k,i+1} + \bar{Y}_{k,i}}{2}\right) - \frac{\bar{Y}_{k,i+1} - \bar{Y}_{k,i}}{m_{i+1} - m_i}$$

Jacobienne

$$\bar{Y}_{k+1,i} \leftarrow \bar{Y}_{k,i} + \delta \bar{Y}_i$$

Système de $4N-2$ équations linéaires

Méthode de résolution numérique

Calcul d'un modèle statique: composition

Posons: $\bar{Y} = (r, P, T, L)$

Système à résoudre: $d\bar{Y}/dm = \bar{f}(\bar{Y})$

Méthode des différences finies $\Rightarrow$ N

$$\frac{\bar{Y}_{i+1} - \bar{Y}_i}{m_{i+1} - m_i} = \bar{f}\left(\frac{\bar{Y}_{i+1} + \bar{Y}_i}{2}\right)$$

Algorithme de Newton-Raphson

$\Rightarrow$ Linéarisation du système autour de l'e

$$\frac{\delta \bar{Y}_{i+1} - \delta \bar{Y}_i}{m_{i+1} - m_i} - J_{\bar{f}}(\delta \bar{Y}_{i+1} + \delta \bar{Y}_i)/2 = \bar{f}\left(\frac{\bar{Y}_{k,i+1} + \bar{Y}_{k,i}}{2}\right) - \frac{\bar{Y}_{k,i+1} - \bar{Y}_{k,i}}{m_{i+1} - m_i}$$

Jacobienn

$$\bar{Y}_{k+1,i} \leftarrow \bar{Y}_{k,i} + \delta \bar{Y}_i$$

Système de $4N-2$ équations linéaires

(fitmass) VARIABLES (center)

j=1 j=2 j=3 j=4

$v_1^1 \ v_2^1 \ v_3^1 \ v_4^1 \ v_1^2 \ v_2^2 \ v_3^2 \ v_4^2 \ v_1^3 \ v_2^3 \ v_3^3 \ v_4^3 \ v_2^4 \ v_3^4$

EQUATIONS

CL 1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
CL 2	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
1-2	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
1-2	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
1-2	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
1-2	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
2-3					$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
2-3					$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
2-3					$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
2-3					$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
3-4									$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
3-4									$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
3-4									$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
3-4									$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Méthode de résolution numérique

Calcul de l'évolution temporelle d'un modèle

L'évolution temporelle découle essentiellement de la variation de composition chimique due aux réactions nucléaires, phénomènes de transport (diffusion, ...)

$\Rightarrow \mu \text{ change} \Rightarrow \text{équation d'état change} \Rightarrow \text{structure change}$

Modèle au temps $t_k \rightarrow$ Modèle au temps $t_{k+1} = t_k + \delta t$

1) Calcul de la nouvelle composition: $X_i(m, t_k) \rightarrow X_i(m, t_{k+1}) = X_i(m, t_k) + \delta X_i$

Cinétique des réactions nucléaires:

Attention aux échelles de temps !

Certaines réactions sont très rapides
(combustion du deuterium, ...)

$\Rightarrow$ forcer la mise à l'équilibre,
ou utiliser une méthode implicite

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho} \left(\sum_j r_{ji} - \sum_k r_{ik} \right) \simeq \frac{\delta X_i}{\delta t}$$



$$r_{ac} = \langle \sigma v \rangle_{abc} X_A X_B$$

2) Avec la nouvelle composition chimique, le modèle en t_k n'est plus solution des équations
 $\Rightarrow$ on itère (méthode de Newton-Raphson) jusqu'à la convergence
On obtient ainsi le modèle en t_{k+1}

Méthode de résolution numérique

Calcul de l'évolution temporelle d'un modèle

L'évolution temporelle découle essentiellement de la variation de composition chimique due aux réactions nucléaires, phénomènes de transport (diffusion, ...)

$\Rightarrow \mu \text{ change} \Rightarrow \text{équation d'état change} \Rightarrow \text{structure change}$

Modèle au temps $t_k \rightarrow$ Modèle au temps $t_{k+1} = t_k + \delta t$

1) Calcul de la nouvelle composition: $X_i(m, t_k) \rightarrow X_i(m, t_{k+1}) = X_i(m, t_k) + \delta X_i$

Cinétique des réactions nucléaires:

Exemple du deuterium

Attention aux échelles de temps !

Certaines réactions sont très rapides (combustion du deuterium, ...)

$\Rightarrow$ forcer la mise à l'équilibre, ou utiliser une méthode implicite



$$\frac{dD}{dt} = \langle \sigma v \rangle_{pp} \frac{H^2}{2} - \langle \sigma v \rangle_{pd} HD$$

$$\rightarrow \left(\frac{D}{H} \right)_{eq} = \frac{\langle \sigma v \rangle_{pp}}{2 \langle \sigma v \rangle_{pd}} \approx 10^{-17}$$

2) Avec la nouvelle composition chimique, le modèle en t_k n'est plus solution des équations

$\Rightarrow$ on itère (méthode de Newton-Raphson) jusqu'à la convergence

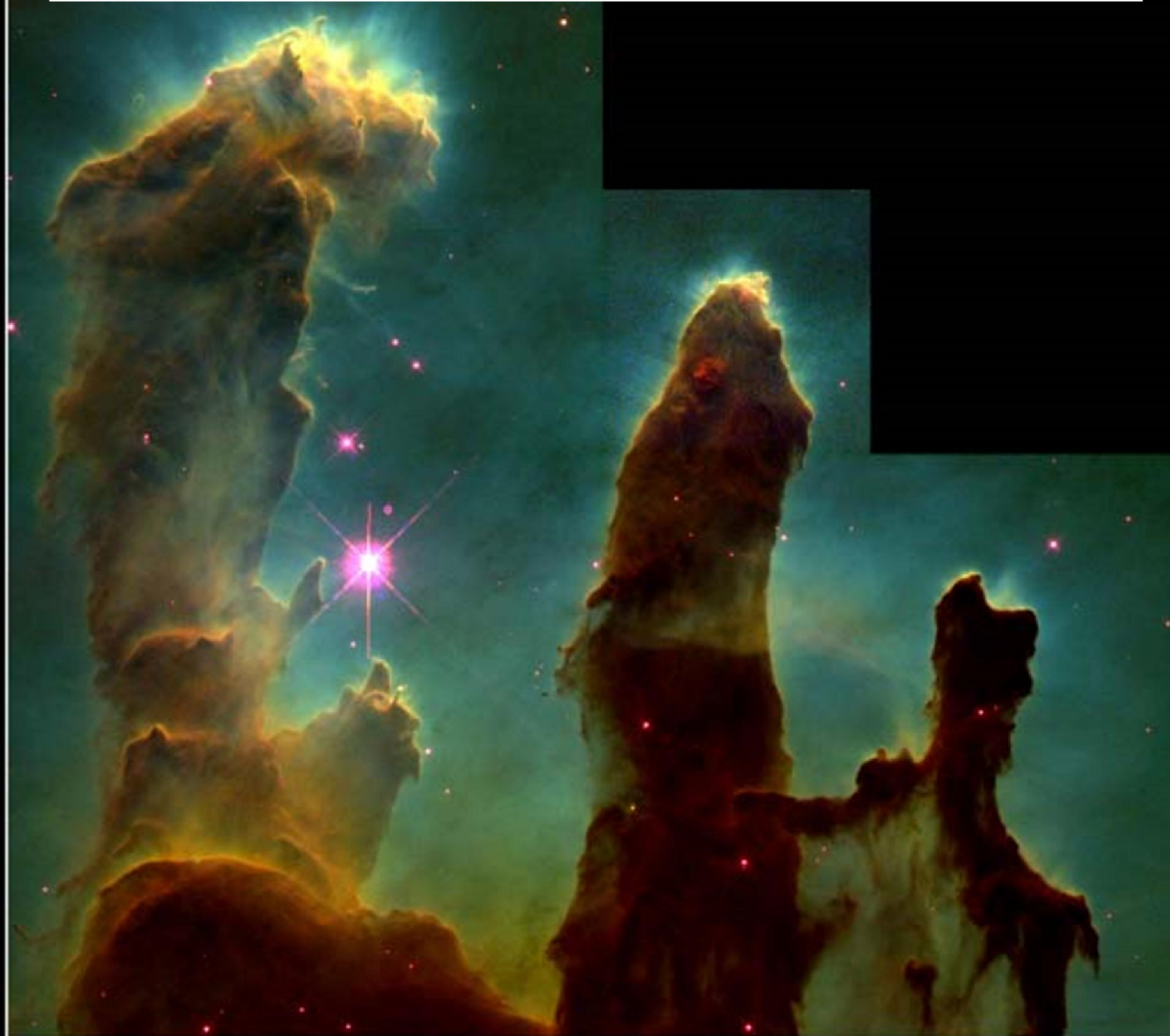
On obtient ainsi le modèle en t_{k+1}

Contraction gravifique

Formation des étoiles



Formation des étoiles



Gaseous Pillars • M16

HST • WFPC2

PRC95-44a • ST ScI OPO • November 2, 1995
J. Hester and P. Scowen (AZ State Univ.), NASA

Contraction gravifique

Instabilité du nuage proto-stellaire (H_2)

Critère de Jeans : le nuage est instable si

Longueur d'onde de la perturbation

$$\lambda \geq \lambda_J = \left(\frac{\pi}{G\rho} \right)^{1/2} v_s \approx \tau_{ff} v_s$$

Vitesse du son isotherme

$$v_s^2 = P/\rho = \mathcal{R} T/\mu$$

Si $\tau_{ff} v_s < \lambda$, le milieu n'a pas le temps de contrebalancer la perturbation sur l'échelle de temps de chute libre $\tau_{ff} \rightarrow$ effondrement


$$M \geq M_J = \left(\frac{\pi \mathcal{R}}{G\mu} \right)^{3/2} T^{3/2} \rho^{-1/2}$$

$$\rho \sim 10^{-24} \text{ g/cm}^3, T \sim 10 \text{ K}, \mu=2 \longrightarrow M_J \sim 1000 M_\odot$$

Contraction gravifique

Etape 1 : Effondrement gravifique ~ isotherme

Temps de chute libre $\sim (G\rho)^{-1/2} \sim 10^8$ ans $\gg$ temps d'ajustement thermique
 $\sim \lambda/c$ (milieu optiquement mince)

Contraction isotherme

$$M_J \sim \rho^{-1/2}$$

Fragmentations

Contraction gravifique

Etape 1 : Effondrement gravifique ~ isotherme

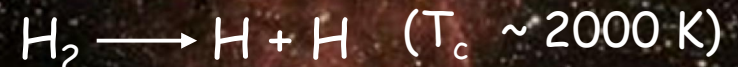
Fragmentations car $M_J \searrow$

Etape 2 : Effondrement gravifique ~ adiabatique

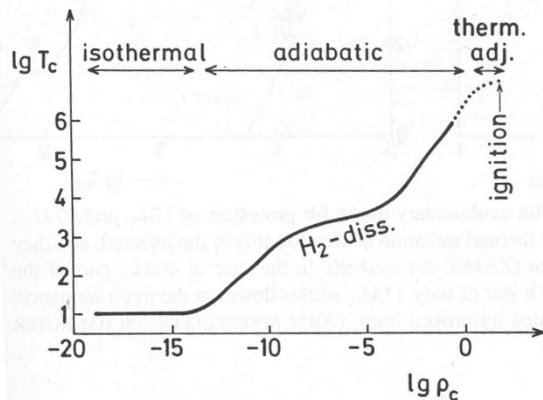
Quand $M \sim 1 M_\odot$, temps de chute libre ~ temps d'ajustement thermique :

Contraction adiabatique $\longrightarrow T_c, P_c \nearrow$

- Apparition d'un 1^{er} front d'onde de choc



- Apparition d'un 2^{er} front d'onde de choc
- Equilibre hydrostatique



Etape 3 : Contraction gravifique tout en maintenant l'équilibre hydrostatique

Théorème du Viriel

Equilibre hydrostatique

$$\frac{\partial P}{\partial m} = - \frac{G m}{4\pi r^4}$$

Masse

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}$$

Intégration par partie

$$\int_0^M 4\pi r^3 \frac{\partial P}{\partial m} dm = \left[4\pi r^3 P \right]_0^M - \int_0^M 12\pi r^2 \frac{\partial r}{\partial m} P dm ,$$

$$\int_0^M \frac{Gm}{r} dm = 3 \int_0^M \frac{P}{\rho} dm$$

Gaz parfait ionisé dégenéré ou non, non-relativiste

$$E_g := - \int_0^M \frac{Gm}{r} dm$$

$$\frac{P}{\rho} = \frac{2}{3} u$$

Energie potentielle gravifique totale

$$E_g = - 2 E_i$$

$$E_i = \int_0^M u dm$$

Etape 3 : Contraction gravifique tout en maintenant l'équilibre hydrostatique

Théorème du Viriel

$$\int_0^M \frac{Gm}{r} dm = 3 \int_0^M \frac{P}{\rho} dm$$

$$E_g = -2 E_i \quad (\text{Gaz parfait})$$

Energie totale : $E_{\text{tot}} = E_i + E_g$

Conservation de l'énergie globale
sans production interne



Pas de réactions nucléaires !

$$\begin{aligned} L &= -dE_{\text{tot}}/dt = -dE_i/dt - dE_g/dt \\ &= dE_i/dt = -(1/2) dE_g/dt \end{aligned}$$

Si réactions nucléaires, remplacer
L par $L - \int \varepsilon dm$

La moitié de l'énergie potentielle gravifique libérée lors de la contraction de l'étoile conduit à une augmentation de son énergie interne, l'autre moitié est rayonnée.

$$\text{Temps d'Helmoltz - Kelvin : } \tau_{\text{HK}} = GM^2 / (2 R L)$$

Naissance ou non des réactions nucléaires ?

Rôle de la dégénérescence

Question : lors de la contraction gravifique, T_c augmente-t-elle toujours ?

Theorème du Viriel $\Rightarrow P/\rho$ augmente

Pour un gaz parfait : $P/\rho \propto T \rightarrow$ **oui** pour un gaz parfait non-dégénéré

Mais ... ce n'est plus le cas pour un gaz dégénéré.

Contraction homologue

$$r \longrightarrow r' = x r = r + dr = (1+dr/r) r \quad dr/r = x-1 = \text{cst}$$

$$\rho \longrightarrow \rho' = \rho/x^3 \longrightarrow d\rho/\rho = \rho'(m)/\rho(m) - 1 \\ = 1/(1+dr/r)^3 - 1 = -3 dr/r$$

$$P=\text{Poids} \longrightarrow P' = \text{Poids}' \longrightarrow dP/P = P'(m)/P(m) - 1 \\ = P/x^4 = 1/(1+dr/r)^4 - 1 = -4 dr/r$$

Naissance ou non des réactions nucléaires ?

Rôle de la dégénérescence

Contraction homologue

$$r \longrightarrow r' = x r = r + dr = (1+dr/r) r \quad dr/r = x-1 = \text{cst}$$

$$\rho \longrightarrow \rho' = \rho/x^3 \longrightarrow d\rho/\rho = \rho'(m)/\rho(m) - 1 \\ = 1/(1+dr/r)^3 - 1 = -3 dr/r$$

$$P \longrightarrow P' = P/x^4 \longrightarrow dP/P = P'(m)/P(m) - 1 \\ = 1/(1+dr/r)^4 - 1 = -4 dr/r$$

$$\longrightarrow dP/P = (4/3) d\rho/\rho$$

Equation d'état générale : $d\rho/\rho = \alpha dP/P - \delta dT/T$

$$\longrightarrow dT/T = (\alpha/\delta) dP/P - (1/\delta) d\rho/\rho \\ = (4\alpha-3)/(3\delta) d\rho/\rho$$

Naissance ou non des réactions nucléaires ?

Rôle de la dégénérescence

$$d\rho/\rho = \alpha dP/P - \delta dT/T$$

$$dT/T = (4\alpha - 3)/(3\delta) d\rho/\rho$$

Pour un gaz parfait : $\alpha = 1$, $\delta = 1$ $\longrightarrow$ $dT/T = (1/3) d\rho/\rho$
non-dégénéré $T \nearrow$

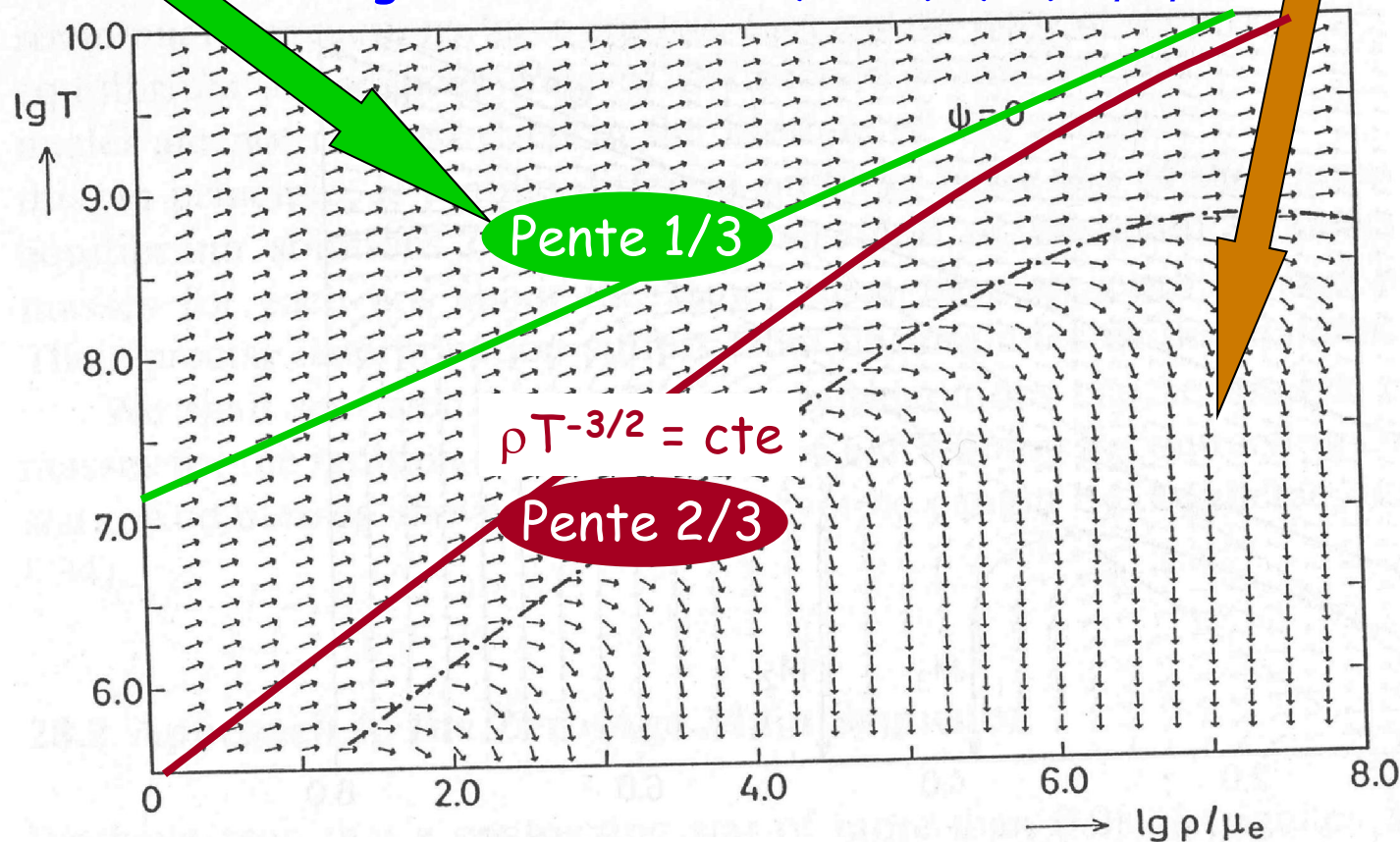
Pour un gaz dégénéré : $\alpha = 3/5$, $\delta = 0$ $\longrightarrow$ $dT/T = -\infty d\rho/\rho$
non-relativiste $T \searrow$

L'énergie gravifique libérée lors de la contraction n'est pas suffisante pour accélérer les électrons dégénérés (énergie de Fermi $\nearrow$).
L'énergie manquante est puisée dans l'énergie cinétique des ions non-dégénérés de sorte que la température diminue.

Pour un gaz parfait :
 $dT/T = (1/3) dp/\rho$

Pour un gaz dégénéré : $dT/T = -\infty dp/\rho$
 non-relativiste

Cas général : $dT/T = (4\alpha-3)/(3\delta) dp/\rho$



ψ : paramètre de dégénérescence

Naissance ou non des réactions nucléaires ?

Dilatation homologue avec changement de masse

$$r \longrightarrow r' = x r = r + dr = (1+dr/r) r$$

$$m(r) \longrightarrow m'(r') = q m(r) = (1+dm/m) m(r)$$

$$\rho = dm/(4\pi r^2 dr) \longrightarrow \rho' = dm'/dr'/(4\pi r'^2) = (q/x) dm/dr / (4\pi x^2 r^2) = (q/x^3) \rho$$

$$\longrightarrow d\rho/\rho = (1+dm/m)/(1+dr/r)^3 - 1 = dm/m - 3 dr/r$$

$$P = \int_m^M Gm/(4\pi r^4) dm \longrightarrow P' = \int_m^M Gm'/(4\pi r'^4) dm' = (q^2/x^4) P$$

$$\longrightarrow dP/P = (1+dm/m)^2/(1+dr/r)^4 - 1 = 2 dm/m - 4 dr/r$$

$$\text{A densité fixée : } d\rho/\rho = 0 \longrightarrow dr/r = 1/3 dm/m \longrightarrow dP/P = 2/3 dm/m$$

$$\text{Pour un gaz parfait: } dT/T = dP/P - d\rho/\rho = dP/P = 2/3 dm/m$$

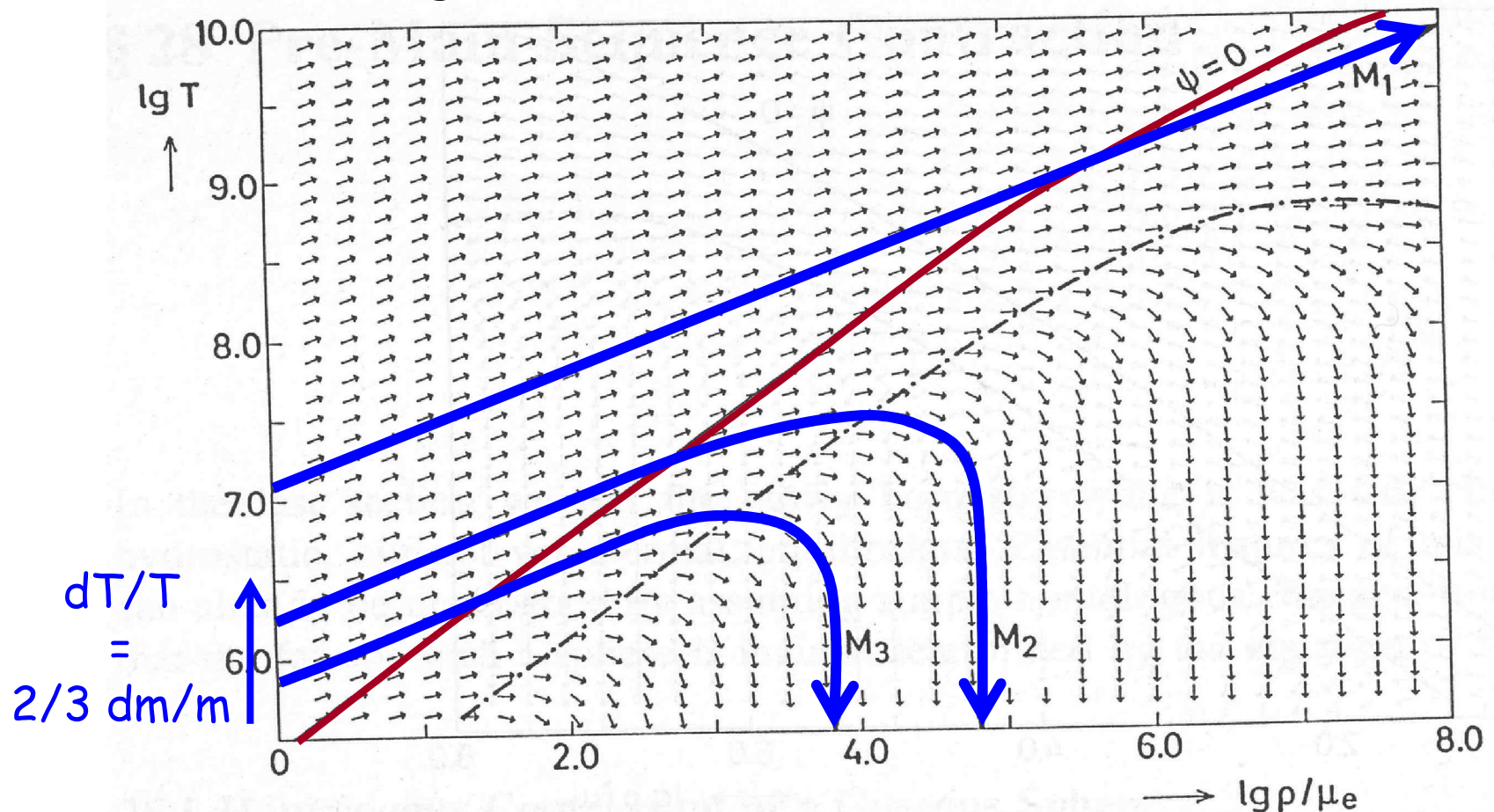
Plus la masse est élevée, plus la température centrale est élevée
(à densité fixée)

Evolution de T_c en fonction de ρ_c

Pour un gaz parfait :
 $dT/T = (1/3) dp/\rho$

Pour un gaz dégénéré : $dT/T = -\infty dp/\rho$
 non-relativiste

Cas général : $dT/T = (4\alpha - 3)/(3\delta) dp/\rho$



Naissance ou non des réactions nucléaires ?

Effet de la masse sur la trajectoire (ρ_c , T_c)

Température requise pour la combustion de l'hydrogène :

$$T_c \sim 10^7 \text{ K}$$

Masse critique :

$$M = 0.08 M_{\odot}$$

Si $M < 0.08 M_{\odot}$, l'étoile n'atteint jamais la température requise pour initier la combustion de l'hydrogène (à cause de la dégénérescence)



Naine brune

Refroidissement des naines brunes et blanches

Théorème du Viriel

$$\int_0^M \frac{Gm}{r} dm = 3 \int_0^M \frac{P}{\rho} dm$$

$$P_e = \frac{8\pi}{15h^3} \frac{p_F^5}{m_e}$$

$$E_g := - \int_0^M \frac{Gm}{r} dm$$

Energie potentielle
gravifique totale

$$u_e = \frac{1}{\rho} \int_0^{p_F} E(p) \frac{8\pi p^2}{h^3} dp = \frac{1}{\rho} \frac{8\pi p_F^5}{10h^3 m_e} = \frac{3}{2} \frac{P_e}{\rho}$$

$$E_{i, \text{tot}} = \int_0^M u dm = \frac{3}{2} \int_0^M (P/\rho) dm$$

Energie interne **totale**

$$E_g = - 2 E_{i, \text{tot}}$$

$$-1/2 dE_g/dt = L = dE_i/dt$$

L'énergie interne totale augmente lors de la contraction
d'un gaz dégénéré (naine blanche) aussi !!

Les naines blanche : refroidissement

Théorème du Viriel

$$-1/2 \delta E_g = \delta E_{i,tot}$$

$E_{i,tot}$: Energie interne totale (ions + électrons)

E_g : Energie potentielle gravifique

E_e : Energie interne des électrons

$$E_e = \frac{3}{2} \int_0^M \frac{P_e}{\rho} dm = \frac{3}{2} K \int_0^M \rho^{2/3} dm$$

$$\begin{aligned} \frac{\delta \rho}{\rho} &= \text{cst.} \rightarrow \frac{\delta E_e}{E_e} = \frac{2}{3} \frac{\delta \rho}{\rho} \\ &= -3 \frac{\delta r}{r} \end{aligned}$$

$$E_g := - \int_0^M \frac{Gm}{r} dm$$

$$\rightarrow \frac{\delta E_g}{E_g} = - \frac{\delta r}{r} = \frac{1}{3} \frac{\delta \rho}{\rho}$$

$$\rightarrow \frac{\delta E_e}{E_e} = 2 \frac{\delta E_g}{E_g}$$

$$E_g = -2 E_{i,tot}$$

$$\rightarrow \delta E_e = - \frac{E_e}{E_{i,tot}} \delta E_g \simeq - \delta E_g$$

$$\leftarrow E_{\text{ions}} \ll E_e$$

Toute l'énergie gravifique libérée augmente l'énergie cinétique des électrons

Les naines blanches : refroidissement

$$dE_e/dt \simeq -dE_g/dt$$

Toute l'énergie gravifique libérée augmente l'énergie cinétique des électrons

$$\begin{aligned} L &= -dE_{\text{tot}}/dt = -dE_{\text{ions}}/dt - dE_e/dt - dE_g/dt \\ &= -dE_{\text{ions}}/dt \end{aligned}$$

Toute l'énergie rayonnée est puisée dans l'énergie cinétique des ions

→ La température diminue

Découplage

Hydrostatique, gravité :
Gaz d'électrons

Thermique :
Gaz d'ions

Les naines blanches : refroidissement

Toute l'énergie rayonnée est puisée dans l'énergie cinétique des ions

Cœur quasi isotherme
(conduction électronique très efficace)

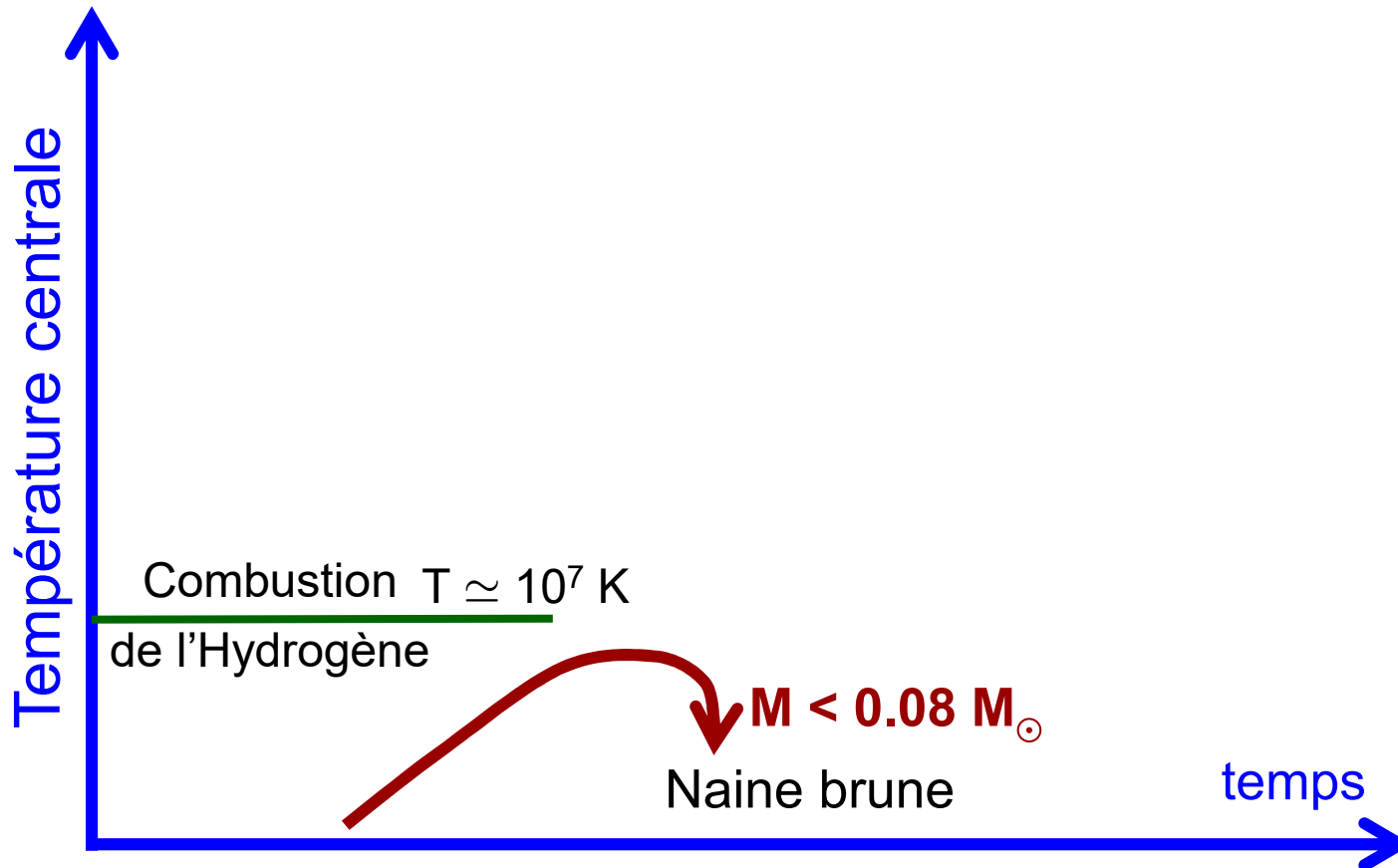
$$E_{ions} = \int_0^M u_{ions} dm = c_v T_0 M$$

$$L = - d E_{ions} / dt = - c_v M dT_0 / dt$$

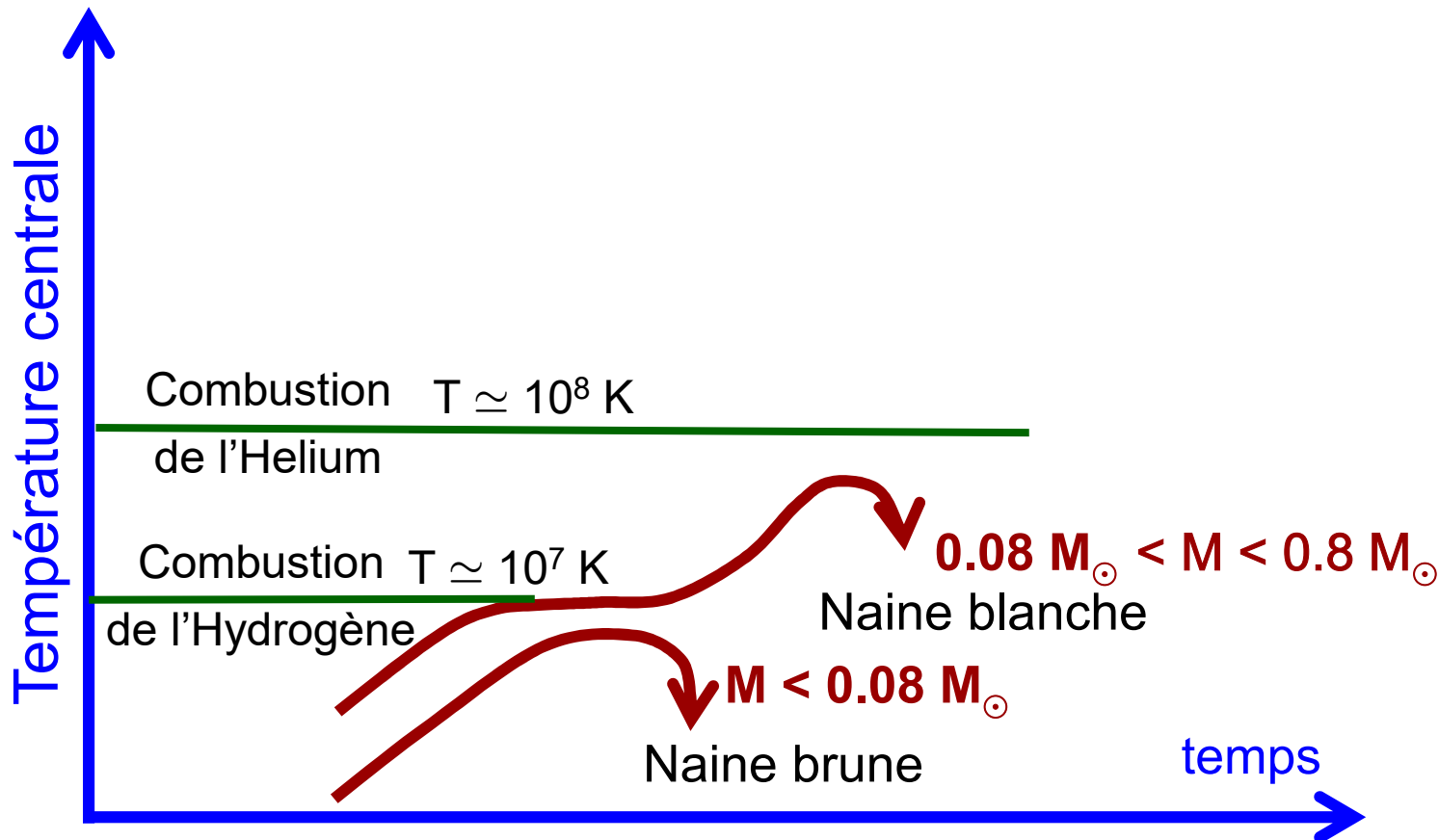


La température
diminue lentement

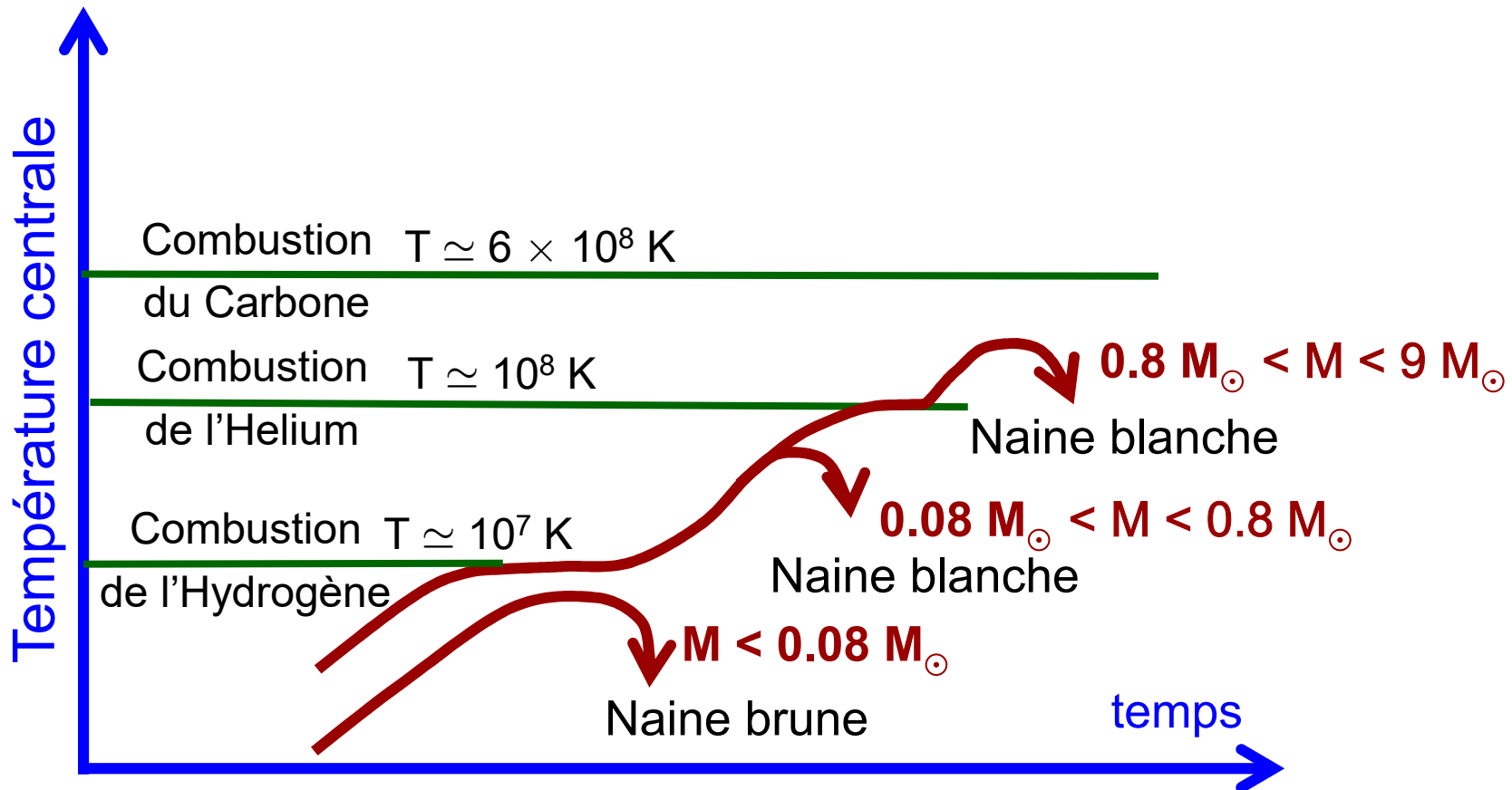
Panorama des scénarios possibles d'évolution des étoiles



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